A split exact sequence of Mackey functors.

Comment. Math. Helvetici 66 (1991) 34–69 0010–2571/91/010034–36$1.50 + 0.20/0

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1. The main theorems

We prove a theorem which allows a Mackey functor on a finite group to be computed in terms of its values on certain subgroups in a quite explicit way. The ingredients which make up the statement are an action of the group on a simplicial complex, and projectivity of the Mackey functor relative to certain subgroups. The result may be regarded as a refinement of the result of Dress [7] which shows that relative projectivity of a Mackey functor with respect to a set of subgroups implies computability in terms of those subgroups. Viewed differently, one can also see the theorem as an extension of work of K. S. Brown [3] which gives the cohomology of a group in terms of its action on the simplicial complex of p-subgroups.

THEOREM A. *Let G be a finite group, M a Mackey functor, $\mathscr{X}$ and $\mathscr{Y}$ classes of subgroups of G which are closed under taking subgroups and conjugation, and Δ a G-simplicial complex of dimension d. Suppose that*

(i) *For every simplex $\sigma \in \Delta$ the vertices of σ lie in distinct G-orbits.*

(ii) *For every subgroup $H \in \mathscr{X} - \mathscr{Y}$, Δ^H is contractible.*

(iii) *M is projective relative to $\mathscr{X}$.*

(iv) *For every $Y \in \mathscr{Y}$, $M(Y) = 0$.*

Then there are split exact sequences

$$0 \to M(G) \to \bigoplus_{\sigma \in G\backslash\Gamma_0(\Delta)} M(G_\sigma) \xrightarrow{(\phi_{\tau\sigma})} \bigoplus_{\tau \in G\backslash\Gamma_1(\Delta)} M(G_\tau) \to \cdots \to \bigoplus_{\sigma \in G\backslash\Gamma_d(\Delta)} M(G_\sigma) \to 0$$

and

$$0 \leftarrow M(G) \leftarrow \bigoplus_{\sigma \in G\backslash\Gamma_0(\Delta)} M(G_\sigma) \xleftarrow{(\psi_{\sigma\tau})} \bigoplus_{\tau \in G\backslash\Gamma_1(\Delta)} M(G_\tau) \leftarrow \cdots \leftarrow \bigoplus_{\sigma \in G\backslash\Gamma_d(\Delta)} M(G_\sigma) \leftarrow 0$$

where if σ is a face of $g\tau$ then

$$\phi_{\tau\sigma} = (\sigma \mid g\tau) \cdot c_{g^{-1}} \cdot \mathrm{res}^{G_\sigma}_{G_{g\tau}}$$

$$\psi_{\sigma\tau} = (\sigma \mid g\tau) \cdot \mathrm{ind}^{G_\sigma}_{G_{g\tau}} \cdot c_g$$

and otherwise $\phi_{\tau\sigma}$ *and* $\psi_{\sigma\tau}$ *are zero. Here* $(\sigma \mid g\tau)$ *is* $+1$ *or* -1 *according to the orientation of the embedding of* σ *in* $g\tau$, *and* $\Gamma_i(\Delta)$ *is the set of* i*-simplices in* Δ.

In this theorem, all of the mappings are given in the way we have indicated between the 0 and 1 terms. The other mappings were left unlabelled to save on notation. By a *G-simplicial complex* we mean a simplicial complex on which G acts simplicially, that is G permutes the simplices amongst themselves preserving the face relationships and sending simplices of a given dimension to simplices of the same dimension. We use the term *orientation of the embedding* to mean the following. It is possible to put a G-invariant partial order on the vertices of Δ in such a fashion that the vertices of every simplex are totally ordered. If τ is an r-simplex of Δ with vertices $(v_0, \ldots, v_r)$ taken in order, and if $\rho = (v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_r)$ is a face of τ of dimension $r-1$ then we put $(\rho \mid \tau) = (-1)^j$. In the statement of the theorem we are composing maps on the left and this will generally be our convention.

The sequences in the theorem are in fact natural with respect to G and this gives rise to a stronger but more abstract statement, which we now give. In the notation of Dress [7], we may regard Mackey functors as being defined on G-sets rather than on subgroups of G. Thus we may write $M(G/H)$ instead of $M(H)$ to denote the value of the Mackey functor M at the subgroup H. If Ω is a G-set we let M_Ω be the Mackey functor with $M_\Omega(\Psi) = M(\Psi \times \Omega)$.

THEOREM B. *In the situation of Theorem A we have split exact sequences of Mackey functors*

$$0 \to M \to M_{\Gamma_0(\Delta)} \to M_{\Gamma_1(\Delta)} \to \cdots \to M_{\Gamma_d(\Delta)} \to 0$$

and

$$0 \leftarrow M \leftarrow M_{\Gamma_0(\Delta)} \leftarrow M_{\Gamma_1(\Delta)} \leftarrow \cdots \leftarrow M_{\Gamma_d(\Delta)} \leftarrow 0.$$

The mappings in this theorem are described as follows. We give the component maps with respect to the decomposition

$$M_{\Gamma_i(\Delta)} = \bigoplus_{\sigma \in G \backslash \Gamma_i(\Delta)} M_{G/G_\sigma}$$

obtained by choosing representatives of the orbits of G on Δ. The component mapping $M_{G/G_\sigma} \to M_{G/G_\tau}$ is zero unless σ is a face of $g\tau$ for some $g \in G$, in which case its evaluation at a G-set Ω is

$$(\sigma \mid g\tau) M^*(1 \times \pi^{G_\sigma}_{G_{g\tau}} \cdot c_g) : M(\Omega \times G/G_\sigma) \to M(\Omega \times G/G_\tau)$$

in the case of the first sequence, or

$$(\sigma \mid g\tau)M_*(1 \times \pi^{G_\sigma}_{G_{g\tau}} \cdot c_g) : M(\Omega \times G/G_\tau) \to M(\Omega \times G/G_\sigma)$$

in the case of the second sequence, where $c_g : G/G_\tau \to G/G_{g\tau}$ is conjugation and $\pi^{G_\sigma}_{G_{g\tau}} : G/G_{g\tau} \to G/G_\sigma$ is the canonical quotient map. On evaluation at G (or, equivalently, at the G-set consisting of a single point) these sequences become the sequences in Theorem A. The sequences in Theorem B are resolutions of M by induced functors, and they take a particularly nice form. They invite comparison with the resolutions by functors M_Ω constructed by Dress [7]. The sequences here have the advantage that they are smaller and easier to compute with in specific examples.

In Section 2 we describe many situations in which Theorem A can be applied, to do with computing the values of specific Mackey functors, and also deducing a structure theorem for the chain complex of K. S. Brown's simplicial complex of p-subgroups of G. I have found Theorem A most useful for computing cohomology groups, and indeed this was the origin of this paper. Theorem A implies the conclusion of the Theorem A in [20] which expresses the cohomology as an alternating sum. I noticed that this alternating sum could be given a more satisfactory conceptual basis by arranging the terms in an exact sequence. I then found that the same thing could be done with Mackey functors. More recently Theorem A has found application in other areas, and we can refer the reader to [19] for an application in connection with Alperin's conjecture.

In Section 4 we give the proofs of Theorems A and B, and for this Section 3 on Mackey functors in a prerequisite. Finally in Section 5 we make deductions about the chain complex of Δ, and in particular about Brown's complex. The main result here is stated as Theorem 2.7.1, which describes the chain complex of Brown's complex. There is also a result, stated as 2.6.1, which says that the quotient by G of Brown's simplicial complex is mod p acyclic.

2. Applications

2.1. *Suitable simplicial complexes*

We start by describing some specific examples of simplicial complexes which can be used in Theorems A and B. In our applications Δ will always be obtained as the simplicial complex of chains in a partially ordered set. If $\mathscr{P}$ is a poset the corresponding simplicial complex $\Delta = \Delta(\mathscr{P})$ has as its n-simplices the set

$$\Gamma_n(\Delta) = \{x_0 < \cdots < x_n \mid x_i \in P\}.$$

The subsimplices of an n-simplex are the shorter subchains. If G acts on $\mathscr{P}$ as a group of poset automorphisms there is induced a simplicial action of G on $\Delta(P)$. A G-simplicial complex constructed in this way will always satisfy condition (i) of the theorem, provided the poset in finite, because if x, y are two vertices of a simplex σ then they are comparable as elements of the poset, and they must therefore lie in distinct G-orbits.

There are three posets which we use all the time, consisting of p-subgroups of G. We fix a prime p and put

$$\mathscr{S}_p(G) = \{\text{all non-identity } p\text{-subgroups of } G\}$$

$$\mathscr{A}_p(G) = \{\text{all non-identity elementary abelian } p\text{-subgroups of } G\}$$

$$\mathscr{B}_p(G) = \{H \leq G \mid H = O_p N_G(H), H \neq 1\}.$$

G acts on each of these posets by conjugating the subgroups. It turns out that from a certain point of view we get effectively the same answer in our applications no matter which of these three posets we use. This is a consequence of the following result.

2.1.1. THEOREM (Quillen [14], Bouc [1], Thévenaz, Webb [18]). *The three simplicial complexes $\Delta(\mathscr{S}_p(G))$, $\Delta(\mathscr{A}_p(G))$ and $\Delta(\mathscr{B}_p(G))$ are all G-homotopy equivalent.*

For a discussion of what this means see [22]. A consequence is that if any one of these simplicial complexes satisfies (ii) then so do the others. The significance of $\mathscr{B}_p(G)$ is that when G is a finite Chevalley group in defining characteristic p, $\Delta(\mathscr{B}_p(G))$ is the barycentric subdivision of the building of G. In particular $\Delta(\mathscr{B}_p(G))$ and the building are G-homotopic.

As well as the simplicial complexes associated to the above posets we will be using posets of the form $\mathscr{P} = \mathscr{S}_p(G) - \mathscr{Y}$, where $\mathscr{Y}$ is some set of subgroups of G closed under conjugation and taking subgroups. For these simplicial complexes the fixed point condition (ii) in the theorem will be verified using;

2.1.2. LEMMA. (i) Let Δ *be the simplicial complex of one of $\mathscr{S}_p(G)$, $\mathscr{A}_p(G)$ or $\mathscr{B}_p(G)$, and let H be a subgroup of G which has a non-trivial normal p-subgroup. Then Δ^H is contractible.*

(ii) *Let $\mathscr{Y}$ be a set of subgroups of G closed under taking subgroups and conjugation, and let $\Delta = \Delta(\mathscr{S}_p(G) - \mathscr{Y})$. Let H be a subgroup of G which has a non-trivial normal p-subgroup H_p with $H_p \in S_p(G) - \mathscr{Y}$. Then Δ^H is contractible.*

Proof. Part (i) in the case of $\Delta(\mathscr{S}_p(G))$ follows from (ii) by taking $\mathscr{Y}$ to be empty. Now the result for $\Delta(\mathscr{A}_p(G))$ and $\Delta(\mathscr{B}_p(G))$ follows in view of 2.1.1.

We prove (ii) by contracting Δ^H onto $\{H_p\}$. Now

$$(\Delta(\mathscr{S}_p(G) - \mathscr{Y}))^H = \Delta((\mathscr{S}_p(G) - \mathscr{Y})^H).$$

Take any $K \in (\mathscr{S}_p(G) - \mathscr{Y})^H$. The chain of inequalities $K \leq K \cdot H_p \geq H_p$ in $(\mathscr{S}_p(G) - \mathscr{Y})^H$ contracts $(\mathscr{S}_p(G) - \mathscr{Y})^H$ onto $\{H_p\}$, as in [4, p. 268]. □

Of course, one may try to apply the theorem with whatever simplicial complex and Mackey functor one may have in mind. Other candidate simplicial complexes come from the sporadic geometries which people construct for certain simple groups [16].

2.2. Overview of the Mackey functors to which the theorem applies

The notion of a Mackey functor will be described in Section 3. We first describe in general terms a proceedure we can follow which produces a Mackey functor M and simplicial complex Δ satisfying the conditions of Theorem A. In order to apply Theorem A to some candidate Mackey functor M we may have in mind, the first step is to find a set $\mathscr{X}$ of subgroups of G with respect to which M is projective, and preferably a minimal such set. Now choose a simplicial complex Δ satisfying (i) of Theorem A, and such that Δ^H is contractible for as many of the subgroups $H \in \mathscr{X}$ as possible. To avoid triviality, G should not stabilize any simplex of Δ. Take $\mathscr{Y}$ to be a set of subgroups of G closed under conjugation and taking subgroups so that $\mathscr{Y} \supseteq \{H \in \mathscr{X} \mid \Delta^H \text{ is not contractible}\}$. We define two more Mackey functors

$$M(H, \mathscr{Y}) = \sum_{Y \in \mathscr{Y}, Y \leq H} \operatorname{ind}_Y^H M(Y), \qquad M_{/\mathscr{Y}}(H) = M(H)/M(H, \mathscr{Y}).$$

One sees that if the original Mackey functor M were to satisfy $M(Y) = 0$ for all $Y \in \mathscr{Y}$, then we would have $M = M_{/\mathscr{Y}}$.

2.2.1. PROPOSITION. *Conditions* (i)–(iv) *of Theorem A are satisfied for the quadruple* $M_{/\mathscr{Y}}$, $\mathscr{X}$, $\mathscr{Y}$ *and* Δ.

This result is proved in Section 3.6, where it is also shown that $M_{/\mathscr{Y}}$ is characterised as the largest quotient Mackey functor of M which vanishes on the subgroups in $\mathscr{Y}$.

We now give a list of examples of how the above construction turns out in

particular cases. These will be examined in greater detail in the ensuing sections. We fix a prime p. We use the following symbols to denote some familiar Mackey functors.

$r(G)$—the ring of virtual complex characters
$Br(G)$—the ring of Brauer characters in characteristic p
$a(G)$—the Green ring of FG-modules, where F is a field of characteristic p
$A(G)$—the Green ring $\mathbb{Q} \otimes_{\mathbb{Z}} a(G)$ with rational coefficients
$\hat{H}^n(G, V)$—Tate cohomology with coefficient module V
$\hat{H}^n(G, V)_p$—the p-torsion subgroup of Tate cohomology
$H^n_{\mathscr{Y}}(G, V)$—cohomology relative to $\mathscr{Y}$

In the case of $a(G)$ we will need to suppose F is a splitting field for G, but not in the case of $A(G)$.

In the next statement we list $\mathscr{X}$ and $\mathscr{Y}$ so that the reader can see what is going on, but as far as practical applications are concerned we might as well forget them because they do not appear in the conclusions of Theorems A or B. For the sake of generality we have put $\Delta(\mathscr{S}_p(G) - \mathscr{Y})$ all the time, but the simplest situation is when $\mathscr{Y}$ consists just of the p'-subgroups of G, and in this case $\mathscr{S}_p(G) - \mathscr{Y} = \mathscr{S}_p(G)$.

We will use the following terminology:

DEFINITION. Let $\mathscr{Z}$ be a class of groups. We say a group H is $\mathscr{Z}$ mod p if H has a normal p-subgroup $H_p \lhd H$ with $H/H_p \in \mathscr{Z}$. Recall that a subgroup is said to be *Brauer elementary* if it is the direct product of a cyclic group with a q-group, for some prime q.

2.2.2. THEOREM. *Conditions* (i)–(iv) *of Theorem A are satisfied with the following choices of* M, $\mathscr{X}$, $\mathscr{Y}$ and Δ.

$M(G)$	*type of subgroups in* $\mathscr{X}$	$\mathscr{Y}$	Δ
$r_{/\mathscr{Y}}(G)$	*Brauer elementary*	$\supseteq p'$*-subgroups*	$\Delta(\mathscr{S}_p(G) - \mathscr{Y})$
$Br_{/\mathscr{Y}}(G)$	*Brauer elementary*	$\supseteq p'$*-subgroups*	$\Delta(\mathscr{S}_p(G) - \mathscr{Y})$
$a_{/\mathscr{Y}}(G)$	*Brauer elementary mod p*	$\supseteq p'$*-subgroups*	$\Delta(\mathscr{S}_p(G) - \mathscr{Y})$
$A_{/\mathscr{Y}}(G)$	*cyclic mod p*	$\supseteq p'$*-subgroups*	$\Delta(\mathscr{S}_p(G) - \mathscr{Y})$
$\hat{H}^n(G, V)_p$	*p-subgroups*	$\{1\}$	$\Delta(\mathscr{S}_p(G))$
$H^n_{/\mathscr{Y}}(G, V)_p$	*p-subgroups*	$\supseteq \{1\}$	$\Delta(\mathscr{S}_p(G) - \mathscr{Y})$
$H^n_{\mathscr{Y}}(G, V)_p, n \geq 1$	*p-subgroups*	$\supseteq \{1\}$	$\Delta(\mathscr{S}_p(G) - \mathscr{Y})$

The meaning of $\supseteq$ in the third column is that we may take $\mathcal{Y}$ to be any set of subgroups closed under taking subgroups and conjugation which contains the named set of subgroups.

Proof. We deduce this from 2.2.1. We must show that Δ satisfies (i) and (ii), which it does from 2.1.2 and the remarks in Section 2.1. Note in this connection that if a Brauer elementary subgroup is not a p'-group then it has a non-trivial normal p-subgroup, so 2.1.2 may be applied. We must also show that M is $\mathcal{X}$-projective in the various cases. This is discussed in Section 3.4. □

2.2.3. REMARK. It will be apparent from Proposition 3.4.2 that if we start with an $\mathcal{X}$-projective Mackey functor N, let $\mathcal{Z}$ be some class of subgroups closed under taking subgroups and conjugation, and put $M(H) = N(H, \mathcal{Z})$ then $M_{/\mathcal{Y}}$ is also $\mathcal{X}$-projective. By this means quadruples such as

$$\begin{aligned} M(G) &= a(G, \mathcal{Z})/a(G, \mathcal{Y}), \\ \mathcal{X} &= \text{subgroups 'Brauer elementary mod } p\text{'}, \\ \mathcal{Y} &= \text{any subconjugacy closed set such that } \{p'\text{-subgroups}\} \subseteq \mathcal{Y} \subseteq \mathcal{Z}, \\ \Delta &= \Delta(S_p(G)) \end{aligned}$$

may be included in the list in Theorem 2.2.2. There is also some freedom in the choice of Δ in that we may take $\Delta = \Delta(\mathcal{P})$ where $\mathcal{P}$ is any G-invariant poset satisfying $\mathcal{S}_p(G) - \mathcal{Y} \subseteq \mathcal{P} \subseteq \mathcal{S}_p(G)$ (see Lemma 2.1.2(ii)).

2.3. Character rings

In using Theorem A to compute either a character ring or a Green ring in the manner of Theorem 2.2.2 it is necessary to factor out the span of the projective modules (or the characters of the projective modules, in the case of a character ring). To be explicit about this, let R be a complete discrete valuation ring in characteristic 0 containing a primitive $|G|$th root of unity with residue field of characteristic p. Then

$$\begin{aligned} r(G, p'\text{-subgroups}) &= \text{the } \mathbb{Z}\text{-span of characters of projective } RG\text{-modules} \\ Br(G, p'\text{-subgroups}) &= \text{the } \mathbb{Z}\text{-span of Brauer characters of projective } FG\text{-modules} \\ a(G, p'\text{-subgroups}) &= \text{the } \mathbb{Z}\text{-span of projective } FG\text{-modules} \\ A(G, p'\text{-subgroups}) &= \text{the } \mathbb{Q}\text{-span of projective } FG\text{-modules.} \end{aligned}$$

For Br and a we require F to be a splitting field. These statements follow from [8] and [5]. In view of this, $\mathrm{Br}_{/p'\text{-subgroups}}(G) = \mathbb{Z}^{l(G)}/C(G)\mathbb{Z}^{l(G)}$ where $C(G)$ is the Cartan

matrix of G and $l(G)$ is the number of p-regular conjugacy classes of G, and this is a finite abelian p-group whose torsion coefficients are the elementary divisors of $C(G)$. Theorem A now states that this group is computable as the end term in an exact sequence where all the remaining information is p-locally determined.

2.3.1. COROLLARY. *With the above notation, there is a short exact sequence*

$$0 \to \mathbb{Z}^{l(G)}/C(G)\mathbb{Z}^{l(G)} \to \bigoplus_{\sigma \in G\backslash \Gamma_0(\Delta)} \mathbb{Z}^{l(G_\sigma)}/C(G_\sigma)\mathbb{Z}^{l(G_\sigma)} \to \cdots \to \bigoplus_{\sigma \in G\backslash \Gamma_d(\Delta)} \mathbb{Z}^{l(G_\sigma)}/C(G_\sigma)\mathbb{Z}^{l(G_\sigma)} \to 0$$

where Δ is any one of $\Delta(\mathscr{S}_p(G))$, $\Delta(\mathscr{A}_p(G))$ or $\Delta(\mathscr{B}_p(G))$, and where the maps correspond to restrictions of Brauer characters, some with minus signs. There is a similar short exact sequence with maps in the opposite direction corresponding to induction of characters.

By equating the alternating sum of the terms in either of the above sequences to zero one obtains a formula for the number of elementary divisors of $C(G)$ of a given size as an alternating sum of these numbers for the groups G_σ. This formula has been observed by Robinson [15].

In a similar way to the above, $r_{/p'\text{-subgroups}}(G)$ is the quotient of $r(G)$ by the span of the columns of the characteristic p decomposition matrix D. Since the rank of D equals the number of p-regular conjugacy classes in G, $r_{/p'\text{-subgroups}}(G)$ has torsion free rank equal to the number of classes of elements of order divisible by p. We obtain a result which was first observed in a different form by Knörr.

2.3.2. COROLLARY. *Let p be a prime and for each finite group G let $m(G)$ be the number of conjugacy classes of elements of G of order divisible by p. Then*

$$m(G) = \sum_{\sigma \in G\backslash \Delta} (-1)^{\dim \sigma} m(G_\sigma)$$

where Δ is any one of $\Delta(\mathscr{S}_p(G))$, $\Delta(\mathscr{P}_p(G))$ or $\Delta(\mathscr{B}_p(G))$.

2.4. Cohomology

In [20] and [21] I obtained the cohomology $\hat{H}^n(G, V)_p$ additively as an alternating sum $\Sigma_{\sigma \in G\backslash \Delta}(-1)^{\dim \sigma}\hat{H}^n(G_\sigma, V)_p$ valid in $K_0(\mathrm{Ab}, 0)$, the Grothendieck group of finite abelian groups with relations given only by direct sum decompositions. Such an expression as an alternating sum is evidently a consequence of the result here,

but I demonstrated the existence of the alternating sum in [21] under a weaker hypothesis on Δ, which in fact is a necessary and sufficient condition for its existence. The split exact sequence formulation has certain advantages. In [21] I required the coefficient module V to be finitely generated in order that the cohomology groups be finite, and hence that the alternating sum would make sense. With the exact sequence here we may perfectly well have non-finitely generated modules. With this sequence it is also possible to determine the cup product structure of cohomology. To do this, note that in the start of the sequence

$$0 \to \hat{H}^*(G, \mathbb{Z})_p \xrightarrow{(\mathrm{res}^G_{G_\sigma})} \bigoplus_{\sigma \in G\backslash\Gamma_0(\Delta)} \hat{H}^*(G_\sigma, \mathbb{Z})_p \to \cdots$$

the first morphism embeds $\hat{H}^*(G, \mathbb{Z})_p$ by a sum of restriction maps, and this is a ring homomorphism. We know the cohomology ring of G once we know the cohomology rings of the G_σ for $\sigma \in \Gamma_0(\Delta)$, and once we have computed the kernel of the map $\oplus_{\sigma \in G\backslash\Gamma_0(\Delta)} \hat{H}^*(G_\sigma, \mathbb{Z})_p \to \oplus_{\sigma \in G\backslash\Gamma_1(\Delta)} \hat{H}^*(G_\sigma, \mathbb{Z})_p$.

2.5. Equivariant cohomology

In the situation where G acts on a simplicial complex Δ there is defined the equivariant cohomology $\hat{H}^n_G(\Delta, V)$ and in case $\Delta = \Delta(\mathscr{S}_p(G))$ it is a result of Brown [4, X.7.2] that this has the same p-torsion subgroup as $\hat{H}^n(G, V)_p$. The restriction sequences of Theorem A applied to cohomology groups $M(G) = \hat{H}^n(G, V)$, $n \in \mathbb{Z}$, but with the left hand terms $M(G)$ removed, make up the E_1 page of the equivariant cohomology spectral sequence associated to this situation, the restriction maps giving the differential [4, VII.8.1]. Because of exactness of these sequences (except where $M(G)$ has been removed), the p-torsion part of the homology of the E_1 page consists just of groups $\hat{H}^n(G, V)_p$ concentrated on the fibre. This is then the E_2 page, and its differential is the zero map. We thus obtain the following result.

2.5.1. COROLLARY. *Let G be a finite group and Δ a G-simplicial complex such that for every simplex $\sigma \in \Delta$ the vertices of σ lie in distinct G-orbits. Suppose that for every non-identity p-subgroup $H \leq G$, Δ^H is contractible. Then the equivariant cohomology spectral sequence*

$$E_1^{r,s} = \bigoplus_{\sigma \in G\backslash\Gamma_r(\Delta)} \hat{H}^s(G_\sigma, V)_p \Rightarrow \hat{H}^{r+s}_G(\Delta, V)_p$$

stops at the E_2 page, and the only non-zero terms $E_2^{r,s}$ lie on the fibre.

2.6. Relative cohomology and the groups $H^n_{/\mathcal{Y}}(G, V)$

Let $\mathcal{Y}$ be a class of subgroups of G closed under taking subgroups and conjugation. The cohomology of G relative to $\mathcal{Y}$ is defined as follows. Let

$$P. = \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

be a $\mathcal{Y}$-split $\mathcal{Y}$-projective resolution of $\mathbb{Z}$. This means that the sequence splits on restriction to every $Y \in \mathcal{Y}$, and every indecomposable summand of every module which appears is projective relative to some $Y \in \mathcal{Y}$. The relative cohomology is defined as

$$H^n_{\mathcal{Y}}(G, V) = H_n(\mathrm{Hom}_{\mathbb{Z}G}(P., V)).$$

These are not the same as the groups $H^n_{/\mathcal{Y}}(G, V)$ defined previously as $H^n_{/\mathcal{Y}}(G, V) = H^n(G, V)/\Sigma_{Y \in \mathcal{Y}} \operatorname{cores}^G_Y H^n(Y, V)$. As an example of the difference, take $G = A \times B$ where A and B are both cyclic of order 2 and let $\mathcal{Y} = \{A, \{1\}\}$. One readily sees that a minimal $\mathcal{Y}$-split $\mathcal{Y}$-projective resolution of $\mathbb{Z}$ is periodic, being the splice of short exact sequences $0 \to \mathbb{Z} \to \mathbb{Z}\uparrow^G_A \to \mathbb{Z} \to 0$, and hence $H^n_{\mathcal{Y}}(G, V)$ is periodic. On the other hand putting $k = \mathbb{Z}/2\mathbb{Z}$ we have $\dim_k H^n(G, k) = n + 1$ and $\dim_k H^n(A, k) = 1$ so $\dim_k H^n_{/\mathcal{Y}}(G, k)$ is unbounded as a function of n. This demonstrates that $H^n_{\mathcal{Y}}(G, V)$ and $H^n_{/\mathcal{Y}}(G, V)$ behave quite differently. They do, however, share the important property that they vanish if $G \in \mathcal{Y}$.

The following is a corollary of Theorem A applied to $H^0_{/\mathcal{Y}}(G, \mathbb{Z})_p$. Recall that a space is said to be mod p *acyclic* if it has the same homology with $\mathbb{Z}/p\mathbb{Z}$ coefficients as a point.

2.6.1. COROLLARY. *Let $\mathcal{Y}$ be a set of p-subgroups of G closed under conjugation and taking subgroups. If $\mathcal{Y}$ does not contain a Sylow p-subgroup of G then $\Delta(\mathcal{S}_p(G) - \mathcal{Y})/G$ is* mod p *acyclic.*

Proof. We apply Theorem A with $M(G) = H^0_{/\mathcal{Y}}(G, \mathbb{Z})_p$. In this context the mapping 'ind' of the Mackey functor is cores, and $H^0_{/\mathcal{Y}}(G, \mathbb{Z})$ is the quotient of $H^0(G, \mathbb{Z}) = \mathbb{Z}$ by the sum of the images $\mathrm{Im}(\operatorname{cores}^G_Y) = |G : Y|\mathbb{Z}$, $Y \in \mathcal{Y}$. Thus $H^0_{/\mathcal{Y}}(G, \mathbb{Z})_p = \mathbb{Z}/p^\alpha\mathbb{Z}$ where p^α is the highest power of p dividing all $|G : Y|$, $Y \in \mathcal{Y}$. I claim that $p \mid |G_\sigma : Y|$ for every simplex $\sigma \in \Delta(\mathcal{S}_p(G) - \mathcal{Y})$ and every subgroup $Y \leq G_\sigma$, $Y \in \mathcal{Y}$. This is equivalent to saying that no Sylow p-subgroup of G_σ is in $\mathcal{Y}$. To see this, suppose that σ is the chain $P_0 < P_1 < \cdots < P_r$ where the P_i are all p-subgroups not in $\mathcal{Y}$. Then $P_0 \leq G_\sigma$ and it follows that no Sylow p-subgroup of G_σ is in $\mathcal{Y}$, otherwise one of them would contain P_0, and P_0 would be in $\mathcal{Y}$, a contradiction.

Write Δ for $\Delta(\mathscr{S}_p(G) - \mathscr{Y})$. Theorem A now states that the sequence

$$0 \to H^0_{/\mathscr{Y}}(G, \mathbb{Z})_p \to \bigoplus_{\sigma \in G\backslash\Gamma_0(\Delta)} H^0_{/\mathscr{Y}}(G_\sigma, \mathbb{Z})_p \to \cdots \to \bigoplus_{\sigma \in G\backslash\Gamma_d(\Delta)} H^0_{/\mathscr{Y}}(G_\sigma, \mathbb{Z})_p \to 0$$

is split exact. For each orbit of simplices there corresponds in this sequence a non-trivial cyclic p-group. We now reduce the sequence modulo p (to reduce all cyclic groups to $\mathbb{Z}/p\mathbb{Z}$). Because of the splitting the sequence remains exact:

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \bigoplus_{\sigma \in G\backslash\Gamma_0(\Delta)} \mathbb{Z}/p\mathbb{Z} \to \cdots \to \bigoplus_{\sigma \in G\backslash\Gamma_d(\Delta)} \mathbb{Z}/p\mathbb{Z} \to 0.$$

The component morphisms are the reductions mod p of the $(\sigma \mid g\tau) \cdot c_{g^{-1}} \cdot \mathrm{res}^{G_\sigma}_{G_{g\tau}}$, which acting on $\mathbb{Z}$ are just $(\sigma \mid g\tau)$, so we have the augmented cellular cochain complex of $G\backslash\Delta$ with $\mathbb{Z}/p\mathbb{Z}$ coefficients. Its dual is the chain complex, and is also acyclic. □

By Theorem 2.1.1, $G\backslash\Delta(\mathscr{S}_p(G))$, $G\backslash\Delta(\mathscr{A}_p(G))$ and $G\backslash\Delta(\mathscr{B}_p(G))$ are homotopy equivalent, and hence they are all mod p acyclic, provided $p \mid |G|$. Corollary 2.6.1 thus improves on 8.2(i) in [20] where it was shown that these spaces had Euler characteristic one. It is conceivable that they are always contractible.

2.7. A structure theorem for the chain complex of $\mathscr{S}_p(G)$

If $C. = \cdots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$ is a chain complex of modules over some ring R, we say $C.$ is *acyclic split* if it has zero homology and for every r the sequence $0 \to \ker d_r \to C_r \to \mathrm{Im}\, d_r \to 0$ is split. If $C.$ is augmented by an epimorphism $C_0 \to R$, by saying $C.$ is *acyclic split augmented* we mean the same condition applied to the augmented complex (i.e. zero reduced homology, all the sequences split, and the augmentation splits). Let $\mathbb{Z}_p$ be the ring of p-adic integers. The following theorem applies when Δ is the simplicial complex of $\mathscr{S}_p(G)$, $\mathscr{A}_p(G)$ or $\mathscr{B}_p(G)$.

2.7.1. THEOREM. *Let Δ be a G-simplicial complex satisfying*

(i) *For every simplex $\sigma \in \Delta$ the vertices of σ lie in distinct G-orbits.*

(ii) *For every non-identity p-subgroup $H \leq G$, Δ^H is contractible.*

Let $C.(\Delta)$ be the chain complex of Δ over $\mathbb{Z}_p$, as a complex of $\mathbb{Z}_pG$-modules. Then $C.(\Delta) = D. \oplus P.$ where $D.$ is an acyclic split augmented subcomplex, and $P.$ is a subcomplex of projective $\mathbb{Z}_pG$-modules.

We may immediately deduce that $C.(\Delta)$ and $P.$ have the same reduced homology. Another implication of the theorem is obtained by considering the Lefschetz module $\Sigma_{i=0}^d (-1)^i C_i(\Delta)$.

2.7.2. COROLLARY. *In the circumstances of the last theorem,*

$$\mathbb{Z}_p \equiv \sum_{\sigma \in G\backslash\Delta} (-1)^{\dim \sigma} \mathbb{Z}_p \uparrow_{G_\sigma}^{G} \text{ (mod projectives).}$$

This congruence holds in the Green ring of $\mathbb{Z}_p G$-lattices.

This was the conclusion of Theorem A in [20] and one implication of the theorem in [21].

We derive Theorem 2.7.1 from our split exact sequences of cohomology groups. The proof is immediate from the next two results, proved in Section 5.

2.7.3. PROPOSITION. *Let $M(G) = H^n(G, V)_p$ for some integer $n \geq 1$ and $\mathbb{Z}G$-module V. In this situation the sequences in Theorem A are* $\mathrm{Ext}^n_{\mathbb{Z}_p G}(\tilde{C}.(\Delta), V \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ *and* $\mathrm{Ext}^n_{\mathbb{Z}_p G}(\tilde{C}.(\Delta)^*, V \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ *where $*$ denotes the contragredient and $\tilde{C}.(\Delta)$ is the augmented chain complex of Δ.*

2.7.4. THEOREM. *Let R denote either $\mathbb{Z}_p$* or *$\mathbb{Z}/p\mathbb{Z}$ and let $C.$ be a finite dimensional chain complex of RG-modules which are permutation modules. The following conditions are equivalent:*

(i) *For some integer $n \geq 1$,* $\mathrm{Ext}^n_{RG}(C., V)$ *is acyclic for all RG-modules V.*

(ii) *For some integer $n \geq 1$,* $\mathrm{Ext}^n_{RG}(C., V)$ *is acyclic split for all RG-*modules *V.*

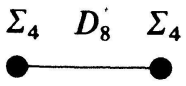
(iii) *$C.$ has a subcomplex $D.$ which is acyclic split over RG so that the quotient $C./D.$ is a complex of projective modules.*

(iv) *$C. = D. \oplus P.$ as complexes of RG-modules where $D.$ is acyclic split over RG, and $P.$ is a complex of projective modules.*

2.8. An example

We take an uncomplicated example to illustrate the principles involved in the previous sections, where the statements we make were more fully explained. Let $G = GL(3, 2)$ and let Δ be the building of G. This is the graph whose vertices are the subgroups Σ_4, the edges are the subgroups D_8, and the two end vertices of an edge are the two Σ_4 subgroups containing the D_8. These are the proper parabolic subgroups of G.

The quotient $G\backslash\Delta$ is the graph



with representative stabilizers as shown. We choose the representative vertices and edge so that the two vertices are actually the end vertices in Δ of the edge, and equivalently so that the particular copies of the subgroups Σ_4 contain the D_8 subgroup. These Σ_4 subgroups are representatives of the two conjugacy classes in G of such subgroups. For a Mackey functor M as prescribed in Section 2.2 we obtain split exact sequences as follows:

$$0 \to M(G) \xrightarrow{\binom{\text{res}}{\text{res}}} M(\Sigma_4) \oplus M(\Sigma_4) \xrightarrow{(-\text{res},\text{res})} M(D_8) \to 0$$

$$0 \leftarrow M(G) \xleftarrow{(\text{ind},\text{ind})} M(\Sigma_4) \oplus M(\Sigma_4) \xleftarrow{\binom{-\text{ind}}{\text{ind}}} M(D_8) \leftarrow 0.$$

Taking $M(G)$ to be Tate cohomology $\hat{H}^n(G, V)_2$ and specializing to the case $\hat{H}^{-2}(G, \mathbb{Z})_2 = H_1(G, \mathbb{Z})_2 = (G/G')_2$ we obtain the sequence

$$0 \to (G/G')_2 \to \Sigma_4/\Sigma_4' \oplus \Sigma_4/\Sigma_4' \to D_8/D_8' \to 0$$

with $\pm$ the classical transfer homomorphism as the component morphisms, and also the sequence

$$0 \leftarrow (G/G')_2 \leftarrow \Sigma_4/\Sigma_4' \oplus \Sigma_4/\Sigma_4' \leftarrow D_8/D_8' \leftarrow 0$$

with the component homomorphisms given by $\pm$ inclusion of subgroups. We obtain similar sequences with the 2-torsion subgroup of the Schur multiplier $\hat{H}^{-3}(G, \mathbb{Z})_2 = H_2(G, \mathbb{Z})_2$, one of which identifies as

$$0 \to C_2 \to C_2 \oplus C_2 \to C_2 \to 0.$$

The Mackey functor $M(G) = \mathrm{Br}_{/p'\text{-subgroups}}(G)$ takes as its value the abelian group which is a direct sum of cyclic groups whose orders are the elementary divisors of the Cartan matrix. For this Mackey functor the first sequence is

$$0 \to C_8 \to C_8 \oplus C_8 \to C_8 \to 0.$$

The group $r_{/p'\text{-subgroups}}(G)$ has torsion-free rank the number of conjugacy classes of elements in G of order divisible by p. With $p = 2$ the first sequence for this Mackey functor is

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \oplus \mathbb{Z}^3 \to \mathbb{Z}^4 \to 0.$$

Turning to the chain complex $C.(\Delta)$ of Δ over $\mathbb{Z}_2$, one knows from the Solomon–Tits theorem that the reduced homology is zero except in dimension 2, where it is the Steinberg module $St(G)$, a projective $\mathbb{Z}_2G$-module. In the notation of Theorem 2.7.1 this is the homology of the subcomplex P. so we have an exact sequence

$$0 \to St(G) \to P_2 \to P_1 \to P_0 \to 0$$

and since all modules are projective, it splits. It follows from 2.7.1 that the augmented chain complex $\tilde{C}.(\Delta)$ is the direct sum of an acyclic split complex and the homology group $St(G)$. This splitting has also been observed for Chevalley groups by Kuhn and Mitchell [11].

3. Mackey functors

3.1. *First notions*

The notion of a Mackey functor is designed to capture the common features of such constructions as character rings, cohomology groups, certain K-groups, various representation rings, etc., and in the form in which we will take them they were first defined by Dress [7]. Dress's work followed soon after work of Green [9], who made a similar definition. The Mackey functors here may be subsumed into Green's theory of G-functors by considering them as G-functors with zero multiplication. We choose not to do this because Green's definition of relative projectivity is not the appropriate one to use here. In the present context the definition of relative projectivity given by Dress is more fundamental and convenient, and it is for this reason that we follow Dress. Our concepts agree with those of Yoshida [23, 24] and Sasaki [17] but we differ in terminology. They use Green's term 'G-functor' to mean a Mackey functor regarded as a being defined on the subgroups of G, but this differs from Green's usage. They reserve the term 'Mackey functor' for a Mackey functor regarded as being defined on G-sets, which is the same as the terminology of Dress. Here we prefer to regard these two ways of viewing Dress's Mackey functors as being little more than a change of notation, and so we will call them both Mackey functors.

We will summarise the work of Dress [7], often taking a different point of view.

DEFINITION. A *Mackey functor* for G over a ring R is a function

$$M : \{\text{subgroups of } G\} \to R\text{-mod}$$

with morphisms

$$\mathrm{ind}_K^H : M(K) \to M(H)$$

$$\mathrm{res}_K^H : M(H) \to M(K)$$

$$c_g : M(H) \to M({}^gH)$$

whenever $K \leq H$ are subgroups of G and $g \in G$, such that

(0) $\mathrm{ind}_H^H, \mathrm{res}_H^H, c_h : M(H) \to M(H)$ are the identity morphisms for all subgroups H and $h \in H$

$$\left.\begin{array}{l}(1)\ \mathrm{res}_J^K \mathrm{res}_K^H = \mathrm{res}_J^H \\ (2)\ \mathrm{ind}_K^H \mathrm{ind}_J^K = \mathrm{ind}_J^H\end{array}\right\} \text{ for all subgroups } J \leq K \leq H$$

(3) $c_g c_h = c_{gh}$ for all $g, h \in G$

$$\left.\begin{array}{l}(4)\ \mathrm{res}_{{}^gK}^{{}^gH} c_g = c_g \mathrm{res}_K^H \\ (5)\ \mathrm{ind}_{{}^gK}^{{}^gH} c_g = c_g \mathrm{ind}_K^H\end{array}\right\} \text{ for all subgroups } K \leq H \text{ and } g \in G$$

(6) $\mathrm{res}_J^H \mathrm{ind}_K^H = \Sigma_{x \in J \backslash H/K} \mathrm{ind}_{J \cap {}^xK}^J c_x \mathrm{res}_{{}^xJ \cap K}^K$ for all subgroups $J, K \leq H$.

The most important of these axioms is (6), which is the *Mackey decomposition formula*. As is well-known, this definition is equivalent to the definition given by Dress, except that Dress works with a more general domain of definition. We now remind the reader of Dress's version.

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *bifunctor* $M = (M_*, M^*) : \mathscr{C} \to \mathscr{D}$ is a pair consisting of a covariant functor $M_* : \mathscr{C} \to \mathscr{D}$ and a contravariant functor $M^* : \mathscr{C} \to \mathscr{D}$ such that on objects, $M_*(C) = M^*(C)$ for all $C \in Ob(\mathscr{C})$. We write $M(C)$ for the common value of these functors.

Let G-set denote the category of finite left G-sets. The morphisms in this category are the G-equivariant mappings. A *Mackey functor* is a bifunctor $M : G\text{-set} \to R\text{-mod}$ such that the following two conditions hold:

(1) for every pullback diagram

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{\alpha} & \Omega_2 \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \Omega_3 & \xrightarrow[\delta]{} & \Omega_4 \end{array}$$

in G-set we have $M^*(\delta)M_*(\gamma) = M_*(\beta)M^*(\alpha)$.

(2) The two mappings $\Omega \to \Omega \cup \Psi \leftarrow \Psi$ into the disjoint union define an isomorphism $M(\Omega \cup \Psi) \cong M(\Omega) \oplus M(\Psi)$ via M_*.

The purpose of giving these definitions is that we will sometimes switch notation and write either $M(H)$ or $M(G/H)$ for the value of the Mackey functor M, using either the subgroup H as the argument as in the first definition, or the G-set G/H as in the second. In technical arguments it is usually easier to work with the second definition. On the other hand it is generally the case that whenever one has a naturally occuring example of a Mackey functor, one most easily recognizes that it is a Mackey functor by verifying the first definition. Thus both definitions have their uses.

3.2. Relative projectivity

For the sake of establishing notation we recall the definition of relative projectivity from [7]. Using the G-set notation, let S be a fixed G-set and M a Mackey functor. Then

$$M_S : T \to M(T \times S)$$

$$M_S^*(f) = M^*(f \times id_S), \qquad M_{S*}(f) = M_*(f \times id_S)$$

defines a Mackey functor M_S, as one easily checks. Projection $\mathrm{pr} : T \times S \to T$ onto the first coordinate defines a natural transformation of bifunctors

$$\theta_S : M_S \to M, \qquad \theta_S(T) = M_*(\mathrm{pr}).$$

DEFINITION. The Mackey functor M is called *S-projective* if θ_S is split-surjective as a natural transformation of Mackey functors. If $\mathscr{X}$ is a set of subgroups of G we put $X = \bigcup_{H \in \mathscr{X}} G/H$ and say M is *$\mathscr{X}$-projective* if it is X-projective.

The most powerful general method we have of showing that a particular Mackey functor M is projective relative to a set of subgroups $\mathscr{X}$ depends on the theorem of Dress [7], which states that in case M happens to be a *Green functor* (in the sense of Dress) then it is sufficient that

$$\sum_{H \in \mathscr{X}} \mathrm{ind}_H^G : \bigoplus_{H \in \mathscr{X}} M(H) \to M(G)$$

be surjective. Furthermore, a *Green module* over a Green functor which is $\mathscr{X}$-projective is itself $\mathscr{X}$-projective. Using this result we are able to determine the relative projectivity of all the Mackey functors we consider here, and we summarize this information in the following table.

Mackey functor	projective relative to
$r(G)$	Brauer elementary subgroups
$\mathrm{Br}(G)$	Brauer elementary subgroups
$a(G)$	subgroups Brauer elementary mod p
$A(G)$	subgroups cyclic mod p
$\hat{H}^n(G, V)_p$	p-subgroups
$H^n_{\mathscr{Y}}(G, V)_p, n \geq 1$	p-subgroups

These assertions of relative projectivity follow, respectively, from Brauer's induction theorem, Brauer's induction theorem together with surjectivity of the decomposition map, Dress's induction theorem [8], Conlon's induction theorem [5], and for the cohomology groups the fact that corestriction from a Sylow p-subgroup is surjective on the p-torsion subgroup.

We will combine this with the work of Section 3.4 to deduce the $\mathscr{X}$-projectivity of the Mackey functors in the list in Theorem 2.2.2.

3.3. Induction and restriction of Mackey functors

It will be important for us later on to interpret the notion of relative projectivity in terms of induction and restriction of Mackey functors. These are introduced in [17], where they are attributed to Yoshida. We present them in a different way using G-set notation. First we go back to G-sets, where we have a notion of induction and restriction

$$\uparrow_H^G : H\text{-set} \to G\text{-set}$$

and

$$\downarrow_H^G : G\text{-set} \to H\text{-set}$$

whenever H is a subgroup of G. Induction is defined by the formula

$$\Omega\uparrow_H^G = G \times_H \Omega$$

which, by definition, is the set of equivalence classes of $G \times \Omega$ under the equivalence relation $(gh, \omega) \sim (g, h\omega)$, $g \in G$, $h \in H$, $\omega \in \Omega$. It has the property that $(H/K)\uparrow_H^G = G/K$. The restriction $\Omega\downarrow_H^G$ is defined simply to be Ω regarded as an H-set by restriction of the action.

We define induction and restriction of Mackey functors. We denote the category of Mackey functors on G by Mack (G). Now induction and restriction are functors

$$\uparrow_H^G : \text{Mack}\,(H) \to \text{Mack}\,(G)$$

and

$$\downarrow_H^G : \text{Mack}\,(G) \to \text{Mack}\,(H).$$

They are defined in a formal fashion by

$$M\uparrow_H^G(\Omega) = M(\Omega\downarrow_H^G)$$
$$M\downarrow_H^G(\Omega) = M(\Omega\uparrow_H^G).$$

To make the connection with the notion of relative projectivity we will need the following lemma.

3.3.1. LEMMA. (i) $\Omega\downarrow_H^G\uparrow_H^G \cong \Omega \times G/H$
(ii) $M_{G/H} \cong M\downarrow_H^G\uparrow_H^G$

Proof. (i) The isomorphism is given by

$$G \times_H \Omega \to \Omega \times G/H$$
$$(g, \omega) \mapsto (\omega, gH).$$

(ii) Evaluating at a G-set Ω we have

$$\begin{aligned} M_{G/H}(\Omega) &= M(\Omega \times G/H) \\ &\cong N(\Omega\downarrow_H^G\uparrow_H^G) \\ &= M\downarrow_H^G\uparrow_H^G(\Omega). \end{aligned}$$

Because the isomorphism in part (i) is natural with respect to Ω we deduce that we have a natural isomorphism between $M_{G/H}$ and $M\downarrow_H^G\uparrow_H^G$. □

It is because of the identification of $M_{G/H}$ just given that we refer to the sequences in Theorem B as resolutions of M by induced functors. We point out a further property of induction and restriction, but we will not use it in this paper. It turns out that $\uparrow_H^G$ is both the left and the right adjoint of $\downarrow_H^G$. With the identification of $M_{G/H}$ just given the natural transformation $\theta_{G/H} : M_{G/H} \to M$, which arises in the

definition of projectivity relative to G/H, corresponds to the counit $M{\downarrow}_H^G{\uparrow}_H^G \to M$ of one of these adjunctions.

In Section 4.2 we will need to know the following result, which has an easy direct proof.

3.3.2. LEMMA. *Induction of Mackey functors is exact.*

Proof. Given an exact sequence of Mackey functors for H

$$0 \to L \to M \to N \to 0$$

the induced sequence

$$0 \to L{\uparrow}_H^G \to M{\uparrow}_H^G \to N{\uparrow}_H^G \to 0$$

is exact if and only if it is exact on evaluation at each G-set Ω. But this is

$$0 \to L(\Omega_H^G) \to M(\Omega_H^G) \to N(\Omega_H^G) \to 0$$

which is exact because it is an evaluation of the original sequence. □

3.4. The construction of $M_{/\mathscr{Y}}$ and the proof of 2.2.1

In Section 2.2, starting with a Mackey functor M and a set $\mathscr{Y}$ of subgroups of G closed under taking subgroups and conjugation, we defined Mackey functors

$$M(H, \mathscr{Y}) = \sum_{Y \in \mathscr{Y}, Y \leq H} \operatorname{ind}_Y^H M(Y), \qquad M_{/\mathscr{Y}}(H) = M(H)/M(H, \mathscr{Y}).$$

We should observe first that these are Mackey functors, and for this it suffices to note that $M(H, \mathscr{Y})$ is stable under ind and res and c_g. This is straightforward from the axioms. We now prove 2.2.1. Conditions (i) and (ii) of Theorem A hold immediately by construction. It remains to verify conditions (iii) and (iv). These will follow from 3.4.1 and 3.4.3.

3.4.1. PROPOSITION. *$M_{/\mathscr{Y}}$ is the largest quotient of M which vanishes on the subgroups in $\mathscr{Y}$.*

Proof. First, $M_{/\mathscr{Y}}$ vanishes on $\mathscr{Y}$ because if $H \in \mathscr{Y}$ then $M(H, \mathscr{Y})$ contains $M(H) = \operatorname{ind}_H^H M(H)$ so $M_{/\mathscr{Y}}(H) = M(H)/M(H) = 0$. To say it is the largest means

that whenever we have an epimorphism of Mackey functors $v : M \to N$ (i.e. a natural transformation) and $N(Y) = 0$ for all $Y \in \mathscr{Y}$ then there is a factorisation

$$\begin{tikzcd} M \arrow[rr, "v"] \arrow[dr] & & N \\ & M_{/\mathscr{Y}} \arrow[ur] & \end{tikzcd}$$

Consider the short exact sequence of Mackey functors

$$0 \to M(\ ,\mathscr{Y}) \to M \to M_{/\mathscr{Y}} \to 0.$$

Now for any $Y \in \mathscr{Y}$, $v(\mathrm{ind}_Y^H(M(Y))) = \mathrm{ind}_Y^H v(M(Y)) = 0$ since v is a natural transformation, and $vM(Y) \subseteq N(Y) = 0$. Thus v vanishes on $M(\ ,\mathscr{Y})$ and hence v lifts to a mapping $M_{/\mathscr{Y}} \to N$ are required. □

3.4.2. PROPOSITION. *Suppose M is $\mathscr{X}$-projective. Then $M(\ ,\mathscr{Y})$ and $M_{/\mathscr{Y}}$ are $\mathscr{X}$-projective.*

We first need to prove:

3.4.3. LEMMA. *Let $T = \bigcup_{Y \in \mathscr{Y}} G/Y$. Then $M(\ ,\mathscr{Y})$ is the image of the natural transformation $\theta_T : M_T \to M$.*

Proof. Recall from 3.2 that $M_T(\Omega) = M(\Omega \times T)$ for any G-set Ω, and $\theta_T = M_*(pr) : M(\Omega \times T) \to M(\Omega)$. To say that $M(\ ,\mathscr{Y})$ is the image of θ_T means that $M(\Omega, \mathscr{Y})$ is the image of θ_T for every G-set Ω. Using a mix of subgroup and G-set notation,

$$\begin{aligned} \theta_T M_T(H) &= \theta_T M((G/H) \times T) \\ &= \sum_{Y \in \mathscr{Y}} \sum_{x \in H \backslash G/Y} \mathrm{ind}_{H \cap {}^xY}^H M(H \cap {}^xY) \end{aligned}$$

and this is $M(H, \mathscr{Y})$ since all terms in the above sum are contained in $M(H, \mathscr{Y})$; and conversely taking the terms with $x = 1$ we obtain $\mathrm{ind}_{H \cap Y}^H(M(H \cap Y))$ for all $Y \in \mathscr{Y}$ in the above sum, and this includes every term $\mathrm{ind}_Y^H M(Y)$ with $Y \in \mathscr{Y}$, $Y \leq H$. □

Proof of 3.4.2. Let $S = \bigcup_{X \in \mathscr{X}} G/X$ and $T = \bigcup_{Y \in \mathscr{Y}} G/Y$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & M(\ ,\mathscr{Y})_S \arrow[r] \arrow[d, "\theta_s"] & M_S \arrow[r] \arrow[d, "\theta_s"] & M_{/\mathscr{Y}S} \arrow[r] \arrow[d, "\theta_s"] & 0 \\ 0 \arrow[r] & M(\ ,\mathscr{Y}) \arrow[r] & M \arrow[r] & M_{/\mathscr{Y}} \arrow[r] & 0. \end{tikzcd}$$

The bottom row is exact, by definition of $M_{/\mathscr{Y}}$. The top row is obtained by evaluating the bottom on $\Omega \times S$, and it follows that the top row is exact, because the bottom row is exact on every evaluation. The diagram commutes, because the horizontal arrows are natural transformations and the vertical arrows are obtained from evaluation on $\Omega \times S \to \Omega$. We are given that M is $\mathscr{X}$-projective, so there is a natural transformation $\alpha : M \to M_S$ which splits θ_S, i.e. $\theta_S\alpha = 1_M$. We show that α sends $M(\ ,\mathscr{Y})$ to $M(\ ,\mathscr{Y})_S$. This is because for any G-set Ω,

$$\begin{aligned}
\alpha(M(\Omega, \mathscr{Y})) &= \alpha \operatorname{Im}(M(\Omega \times T) \to M(\Omega)) && \text{by 3.4.3}\\
&\subseteq \operatorname{Im}(M_S(\Omega \times T) \to M_S(\Omega)) && \text{from the diagram below}\\
&= \operatorname{Im}(M(\Omega \times T \times S) \to M(\Omega \times S))\\
&= \operatorname{Im}(M(\Omega \times S \times T) \to M(\Omega \times S)) && \text{by swapping } S \text{ and } T\\
&= M(\Omega, \mathscr{Y})_S.
\end{aligned}$$

The containment above follows by considering the commutative diagram

$$\begin{array}{ccc}
M_S(\Omega \times T) & \longrightarrow & M_S(\Omega)\\
\uparrow{\scriptstyle \alpha_{\Omega \times T}} & & \uparrow{\scriptstyle \alpha_{\Omega}}\\
M(\Omega \times T) & \longrightarrow & M(\Omega).
\end{array}$$

It follows that α restricts to a splitting of θ_S on $M(\ ,\mathscr{Y})_S$ and induces a splitting of θ_S on $M_{/\mathscr{Y}S}$. Thus $M(\ ,\mathscr{Y})$ and $M_{/\mathscr{Y}}$ are both $\mathscr{X}$-projective. □

4. Proof of the main theorems

4.1. *Semisimplicial constructions*

Throughout the constructions we consider we will be working with a G-simplicial complex Δ which satisfies

(*) for every simplex $\sigma \in \Delta$, the vertices of σ lie in distinct G-orbits.

If Δ does not have this property then its barycentric subdivision does.

We will be interested in two consequences of (*), the first of which is the fact that in the presence of (*) every fixed point set Δ^H is also a simplicial complex. The other consequence is the following lemma.

4.1.1. LEMMA. *Let Δ be a G-simplicial complex satisfying* (*). *Then we may put a G-invariant partial order on the vertices of Δ in such a way that in every simplex the vertices are totally ordered.*

Proof. Totally order the set of orbits of vertices. The partial order we require has $x < y$ if and only if the orbit of x is less than the orbit of y. □

A G-simplicial complex satisfying the conclusion of this lemma is sometimes called an *ordered* G-simplicial complex, and we might as well be working in the category of ordered G-simplicial complexes.

We will need to use the language of semisimplicial objects, and for this we refer the reader to [13]. Starting with an ordered G-simplicial complex Δ we may form the associated simplicial set

$$S(\Delta) = \cdots \Gamma_2(\Delta) \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}\; \Gamma_1(\Delta) \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow}\; \Gamma_0(\Delta)$$

where the face maps $\partial_j : \Gamma_n(\Delta) \to \Gamma_{n-1}(\Delta)$ are

$$\partial_j(v_0, \ldots, v_n) = (v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$$

and the degeneracy maps are

$$s_j(v_0, \ldots, v_n) = (v_0, \ldots, v_j, v_j, \ldots, v_n).$$

It is immediate from the fact that G preserves the order of vertices in every simplex that all the ∂_j and s_j are G-equivariant mappings. We thus see that in fact $S(\Delta)$ is a simplicial G-set. We really wish to consider the augmented simplicial object associated to Δ, which is

$$\tilde{S}(\Delta) = \cdots \Gamma_2(\Delta) \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}\; \Gamma_1(\Delta) \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow}\; \Gamma_0(\Delta) \to \Gamma_{-1}(\Delta)$$

where $\Gamma_{-1}(\Delta)$ is a single point, and this is also a simplicial G-set.

We now perform some further operations. These rely on the observation that whenever we are given a functor $\Theta : G\text{-set} \to \mathscr{C}$ where $\mathscr{C}$ is some category then the diagram $\Theta\tilde{S}(\Delta)$ is a simplicial object in $\mathscr{C}$. Suppose we are given a Mackey functor M for G (over a commutative ring R). Let Mack(G) denote the category of Mackey functors for G over the ring R. We define two functors Θ_M, $\Theta^M : G\text{-set} \to \mathrm{Mack}(G)$ in the following fashion, such that Θ_M is covariant and Θ^M is contravariant. For a G-set Ω we put $\Theta_M(\Omega) = \Theta^M(\Omega) = M_\Omega$. Given a morphism $\alpha : \Omega \to \Psi$ of G-sets we define $\Theta_M(\alpha) : M_\Omega \to M_\Psi$ and $\Theta^M(\alpha) : M_\Psi \to M_\Omega$ as follows. The effect of $\Theta_M(\alpha)$

on the Mackey functor M_Ω evaluated at a G-set X is $M_*(1 \times \alpha) : M(X \times \Omega) \to M(X \times \Psi)$. The effect of $\Theta^M(\alpha)$ is similarly $M^*(1 \times \alpha) : M(X \times \Psi) \to M(X \times \Omega)$. In Mack($G$) we may now form the chain complexes of Mackey functors associated to $\Theta_M \tilde{S}(\Delta)$. These will be denoted $CC(\Theta_M \tilde{S}(\Delta))$ and $CC(\Theta^M \tilde{S}(\Delta))$. Explicitly, we mean the objects in the first chain complex to be the $\Theta_M(\Gamma_r(\Delta))$ with differential $\Sigma_{j=0}^r (-1)^j \Theta_M(\partial_j)$, and similarly for the second one. We summarise the operations we have just performed, writing Θ instead of Θ_M or Θ^M:

$$\begin{aligned} G\text{-simplicial complex} &\xrightarrow{\tilde{S}} \text{simplicial } G\text{-set} \\ &\xrightarrow{\Theta} \text{simplicial Mackey functor} \\ &\xrightarrow{CC} \text{chain complex of Mackey functors.} \end{aligned}$$

We will show that the chain complexes we finish with are the ones specified in Theorem B.

The chain complexes in Theorem A are the evaluations at the trivial G-set (or, equivalently, at G) of the chain complexes in Theorem B. We can therefore construct them by repeating what we have just done, and evaluating always at the trivial G-set. A more direct proceedure is indicated by the following scheme:

$$\begin{aligned} G\text{-simplicial complex} &\xrightarrow{\tilde{S}} \text{simplicial } G\text{-set} \\ &\xrightarrow{M} \text{simplicial } R\text{-module} \\ &\xrightarrow{CC} \text{chain complex of } R\text{-modules.} \end{aligned}$$

Here M denotes either M_* or M^*. We apply M_* and M^* to all the mappings and objects which make up $\tilde{S}(\Delta)$. We thus obtain two simplicial R-modules denoted $M_*\tilde{S}(\Delta)$ and $M^*\tilde{S}(\Delta)$. In the category of R-modules, we may now form the chain complexes of R-modules associated to $M_*\tilde{S}(\Delta)$ and $M^*\tilde{S}(\Delta)$ and these are denoted $CC(M_*\tilde{S}(\Delta))$ and $CC(M^*\tilde{S}(\Delta))$.

4.1.2. LEMMA. *The sequences in Theorem A are* $CC(M^*\tilde{S}(\Delta))$ *and* $CC(M_*\tilde{S}(\Delta))$. *The sequences in Theorem B are* $CC(\Theta^M\tilde{S}(\Delta))$ *and* $CC(\Theta_M\tilde{S}(\Delta))$.

Proof. We give the proof for the sequences in Theorem B and then indicate the modifications we must make for the sequences in Theorem A.

We work first with $CC(\Theta^M\tilde{S}(\Delta))$. The typical Mackey functor in $CC(\Theta^M\tilde{S}(\Delta))$ is $\Theta^M(\Gamma_r(\Delta)) = M_{\Gamma_r}(\Delta)$ and since

$$\Gamma_r(\Delta) \cong \bigcup_{\sigma \in G\backslash\Gamma_r(\Delta)} G/G_\sigma$$

broken up according to the given set of representatives σ for the G-orbits on $\Gamma_r(\Delta)$, we have

$$M_{\Gamma_r(\Delta)} \cong \bigoplus_{\sigma \in G\backslash\Gamma_r(\Delta)} M_{G_\sigma}.$$

The differentials in the chain complexes are $\Sigma_{j=0}^{r}(-1)^j\Theta^M(\partial_j)$. If $\tau = (v_0 \ldots v_r)$ is an r-simplex with the vertices taken in order then $\partial_j(\tau) = (v_0 \ldots v_{j-1}, v_{j+1} \ldots v_r)$. Putting $\partial_j(\tau) = \rho$, say, we use the notation $(\rho \mid \tau) = (-1)^j$, and this is what we mean by the 'orientation of the embedding of ρ in τ'. Thus the component of the differential between the orbit containing τ in $\Gamma_r(\Delta)$ and the orbit containing ρ in $\Gamma_{r-1}(\Delta)$ is $(\rho \mid \tau)\Theta^M(\partial_j)$ where on the orbit containing τ, ∂_j is the map

$$\begin{aligned} \partial_j : G/G_\tau &\to G/G_\rho \\ gG_\tau &\mapsto gG_\rho. \end{aligned}$$

Suppose that in fact ρ was not one of our originally chosen orbit representatives, but σ was, and that $\sigma = g\rho$ for some $g \in G$. Then σ is a face of $g\tau$. Identifying the orbit containing ρ and σ with G/G_σ we now write ∂_j as

$$\begin{aligned} \partial_h : G/G_\tau &\to G/G_\sigma \\ xG_\tau &\mapsto xg^{-1}G_\sigma. \end{aligned}$$

This factorizes as

$$G/G_\tau \xrightarrow{c_g} G/{}^gG_\tau = G/G_{g\tau} \xrightarrow{\pi_{G_{g\tau}}^{G_\sigma}} G/G_\sigma,$$

where c_g is the map $xG_\tau \mapsto xg^{-1}\,{}^gG_\tau$ and $\pi_{G_{g\tau}}^{G_\sigma}$ is $xG_{g\tau} \mapsto xG_\sigma$. Applying Θ^M to this we conclude that the component morphism $M_{G_\sigma} \to M_{G_\tau}$ is

$$(\sigma \mid g\tau)\Theta^M(\pi_{G_{g\tau}}^{G_\sigma} c_g) = (\sigma \mid g\tau)M^*(1 \times \pi_{G_{g\tau}}^{G_\sigma} \cdot c_g) : M(? \times G/G_\sigma) \to M(? \times G/G_\tau).$$

The proof for the second chain complex $CC(\Theta_M\tilde{S}(\Delta))$ is similar to this, but we apply Θ_M instead of Θ^M. Thus the component morphism $M_{G_\tau} \to M_{G_\sigma}$ is

$$(\sigma \mid g\tau)\Theta_M(\pi_{G_{g\tau}}^{G_\sigma} c_g) = (\sigma \mid g\tau)M_*(1 \times \pi_{G_{g\tau}}^{G_\sigma} \cdot c_g) : M(? \times G/G_\tau) \to M(? \times G/G_\sigma).$$

These are the mappings described in Theorem B.

For the sequence in Theorem A we proceed in the same way but we only have to apply the functors M^* and M_*. Thus the component morphisms are

$$\phi_{\tau\sigma} = (\sigma \mid g\tau)M^*(\pi^{G_\sigma}_{G_{g\tau}} c_g) = (\sigma \mid g\tau) \cdot c_{g^{-1}} \cdot \mathrm{res}^{G_\sigma}_{G_{g\tau}}$$

and

$$\psi_{\sigma\tau} = (\sigma \mid g\tau)M_*(\pi^{G_\sigma}_{G_{g\tau}} c_g) = (\sigma \mid g\tau) \cdot \mathrm{ind}^{G_\sigma}_{G_{g\tau}} \cdot c_g. \qquad \square$$

Remark. It is clear from the above discussion that the sequences in Theorem A are obtained by evaluating the sequences in Theorem B at the trivial G-set. Because of this, Theorem A is a consequence of Theorem B, since a sequence of Mackey functors which is exact and split must be exact and split on each evaluation. In view of this we will only prove Theorem B in the next sections. A direct proof of Theorem A may be obtained by following the arguments we give and replacing Θ_M and M_* and Θ^M with M^*.

4.2. The reduction step

We suppose we are given a G-simplicial complex Δ, a Mackey functor M and sets of subgroups $\mathscr{X}$ and $\mathscr{Y}$ satisfying the conditions of Theorem A. We will show

4.2.1. PROPOSITION. *To prove Theorems A and B it suffices to assume the structure of G is such that $G \in \mathscr{X}$.*

The proof of this is obtained by combining the next two results. If H is a subgroup of G we let $M{\downarrow}^G_H$ denote M with the domain of definition restricted to the subgroups of H. Similarly $\Delta{\downarrow}^G_H$ denotes Δ regarded as an H-simplicial complex by restriction of the action. We put

$$\mathscr{X}_H = \{K \in \mathscr{X} \mid K \leq H\} \quad \text{and} \quad \mathscr{Y}_H = \{K \in \mathscr{Y} \mid K \leq H\}.$$

It is an elementary observation that the hypotheses of Theorem A are inherited by the quadruple $M{\downarrow}^G_H$, $\Delta{\downarrow}^G_H$, $\mathscr{X}_H$ and $\mathscr{Y}_H$. The immediate implication of the next results is that it suffices to show that $CC(\Theta_{M{\downarrow}^G_H}\tilde{S}(\Delta{\downarrow}^G_H)){\uparrow}^G_H$ and $CC(\Theta^{M{\downarrow}^G_H}\tilde{S}(\Delta{\downarrow}^G_H)){\uparrow}^G_H$ are split acyclic for all $H \in \mathscr{X}$. Since induction is exact (and hence preserves direct sums), it suffices to show that $CC(\Theta_{M{\downarrow}^G_H}\tilde{S}(\Delta{\downarrow}^G_H))$ and $CC(\Theta^{M{\downarrow}^G_H}\tilde{S}(\Delta{\downarrow}^G_H))$ are split acyclic for all $H \in \mathscr{X}$, and in view of the observation made just previously this is the same as assuming that $G \in \mathscr{X}$.

4.2.2. LEMMA. *$CC(\Theta_M \tilde{S}(\Delta))$ is a direct summand of*

$$\bigoplus_{H \in \mathscr{X}} CC(\Theta_{M\downarrow_H^G} \tilde{S}(\Delta\downarrow_H^G))\uparrow_H^G.$$

Similarly $CC(\Theta^M \tilde{S}(\Delta))$ is a direct summand of

$$\bigoplus_{H \in \mathscr{X}} CC(\Theta^{M\downarrow_H^G} \tilde{S}(\Delta\downarrow_H^G))\uparrow_H^G.$$

4.2.3. LEMMA. *A direct sum of chain complexes is split acyclic if and only if its summands are split acyclic.*

Proof of 4.2.2. This is an exercise in using the relative projectivity of M. We work first in an abstract setting where we have a functor $\Theta : G\text{-set} \to \mathscr{C}$ for some abelian category $\mathscr{C}$, thus obtaining a simplicial object $\Theta(\tilde{S}(\Delta))$ in $\mathscr{C}$. In our application we will take $\Theta = \Theta_M$ or Θ^M. Suppose we have another functor $\Xi : G\text{-set} \to \mathscr{C}$ and natural transformations $\alpha : \Xi \to \Theta$, $\beta : \Theta \to \Xi$ such that $\alpha\beta = 1$. Then α, β give rise to simplicial maps

$$\Theta(\tilde{S}(\Delta)) \xrightarrow{\beta} \Xi(\tilde{S}(\Delta)) \xrightarrow{\alpha} \Theta(\tilde{S}(\Delta))$$

and hence to maps of chain complexes

$$CC(\Theta(\tilde{S}(\Delta))) \xrightarrow{\beta} CC(\Xi(\tilde{S}(\Delta))) \xrightarrow{\alpha} CC(\Theta(\tilde{S}(\Delta)))$$

such that the composite of the two maps is the identity. By this means we deduce that $CC(\Theta(\tilde{S}(\Delta)))$ is a direct summand of $CC(\Xi(\tilde{S}(\Delta)))$ as chain complexes.

In our situation we put $X = \bigcup_{H \in \mathscr{X}} G/H$, and take $\Theta = \Theta_M$ and $\Xi = \Theta_{M_X}$. The projectivity of M relative to $\mathscr{X}$ means that the natural transformation $\theta_X : M_X \to M$ is a split epimorphism. We will show that θ_X gives rise to a natural transformation $\alpha : \Theta_{M_X} \to \Theta_M$. At a G-set Ω we define the effect of α to be another natural transformation $\alpha_\Omega : \Theta_{M_X}(\Omega) = (M_X)_\Omega \to M_\Omega = \Theta_M(\Omega)$, whose effect at a G-set Ψ is

$$(M_X)_\Omega(\Psi) = M_X(\Psi \times \Omega) \xrightarrow{\theta_X} M(\Psi \times \Omega) = M_\Omega(\Psi).$$

It is apparent that α is natural with respect to Ω because when we expand the G-sets as direct products the G-set morphisms which take place in the Ω factor commute with morphisms which take place in the other factors. In a similar fashion the natural transformation $M \to M_X$ which splits θ_X gives rise to a natural transformation $\beta : \Theta_M \to \Theta_{M_X}$, and we have $\alpha\beta = 1$. We have now shown that the abstract

situation just described in the previous paragraph is in force here, and we deduce that $CC(\Theta_M \tilde{S}(\Delta))$ is a direct summand of $CC(\Theta_{M_X} \tilde{S}(\Delta))$.

Evidently $CC(\Theta_{M_X} \tilde{S}(\Delta)) \cong \oplus_{H \in \mathscr{X}} CC(\Theta_{M_{G/H}} \tilde{S}(\Delta))$ and it remains to show that $\Theta_{M_{G/H}} \tilde{S}(\Delta) \cong (\Theta_{M\downarrow_H^G} \tilde{S}(\Delta \downarrow_H^G))\uparrow_H^G$. This is immediate from the fact that $\Theta_{M_{G/H}}$ factorizes as a composite

$$G\text{-set} \xrightarrow{\downarrow_H^G} H\text{-set} \xrightarrow{\Theta_{M\downarrow_H^G}} \mathrm{Mack}(H) \xrightarrow{\uparrow_H^G} \mathrm{Mack}(G).$$

We see this by considering the chain of isomorphisms

$$\begin{aligned}
\Theta_{M_{G/H}}(\Omega)(\Psi) &= (M_{G/H})_\Omega(\Psi) \\
&= M_{G/H}(\Psi \times \Omega) \\
&= M(\Psi \times \Omega \times G/H) \\
&\cong M((\Psi \times \Omega)\downarrow_H^G \uparrow_H^G) \\
&= M\downarrow_H^G(\Psi\downarrow_H^G \times \Omega\downarrow_H^G) \\
&= (M\downarrow_H^G)_{\Omega\downarrow_H^G}(\Psi\downarrow_H^G) \\
&= ((M\downarrow_H^G)_{\Omega\downarrow_H^G})\uparrow_H^G(\Psi).
\end{aligned}$$

The argument which shows that $CC(\Theta^M \tilde{S}(\Delta))$ is a direct summand of

$$\bigoplus_{H \in \mathscr{X}} CC(\Theta^{M\downarrow_H^G}(\Delta\downarrow_H^G)\uparrow_H^G$$

is similar. □

Proof of 4.2.3. Suppose $C_1 \oplus C_2$ is a direct sum of chain complexes. Then $H_*(C_1 \oplus C_2) \cong H_*(C_1) \oplus H_*(C_2)$ so if $C_1 \oplus C_2$ is acyclic, so are its summands.

As for the splitting, we recall from [22, 7.1] (and it is easy to prove) that a chain complex with differential d is split if and only if there is a chain map α of degree $+1$ with $d\alpha d = d$. The differential on $C_1 \oplus C_2$ is $d = (d_1, d_2)$, where d_1 and d_2 are the differentials on C_1 and C_2. Suppose that $C_1 \oplus C_2$ is split by a map α of degree $+1$ with $d\alpha d = d$. Then $\pi_1 \alpha i_1$ splits C_1, where

$$i_1 : C_1 \to C_1 \oplus C_2 \quad \text{and} \quad \pi_1 : C_1 \oplus C_2 \to C_1$$

are inclusion and projection, since $d_1 \pi_1 \alpha i_1 d_1 = \pi_1 d\alpha d i_1 = \pi_1 d i_1 = d_1$. □

4.3. The direct argument with subgroups in $\mathscr{X}$

Our argument uses notions from topology, and in particular the notion of G-homotopy. Given topological spaces X and Y on which G acts as a group of homeomorphisms, we say that G-equivariant maps $f, g : X \to Y$ are *G-homotopic* if they are homotopic by a homotopy $H : X \times I \to Y$ which is itself a G-equivariant map, where the unit interval I has the trivial G-action. The spaces X and Y are *G-homotopy equivalent* if there are G-equivariant maps between them in each direction so that the composites in both directions are G-homotopic to the identity.

The final lines in our argument will rely on the following observation.

4.3.1. LEMMA. *Let Δ_1 and Δ_2 be G-simplicial complexes satisfying* (*) *which are G-homotopy equivalent, and let Θ denote one of the functors Θ_M, Θ^M, M_* or M^* as in Section* 4.1. *Then $CC(\Theta\tilde{S}(\Delta_1))$ and $CC(\Theta\tilde{S}(\Delta_2))$ are chain homotopy equivalent.*

Proof. For each G-simplicial complex Δ we will need to consider the simplicial G-set of *singular* simplices of $|\Delta|$, which we will denote $\Sigma(\Delta)$, and also the corresponding augmented simplicial G-set $\tilde{\Sigma}(\Delta)$. We will show two things, firstly that $\Delta_1 \simeq_G \Delta_2$ implies $CC(\Theta\tilde{\Sigma}(\Delta_1)) \simeq CC(\Theta\tilde{\Sigma}(\Delta_2))$; and secondly that $CC(\Theta\tilde{S}(\Delta)) \simeq CC(\Theta\tilde{\Sigma}(\Delta))$ for any G-simplicial complex Δ. Putting these pieces together we obtain the conclusion of Lemma 4.3.1, since $CC(\Theta\tilde{S}(\Delta_1)) \simeq CC(\Theta\tilde{\Sigma}(\Delta_1)) \simeq CC(\Theta\tilde{\Sigma}(\Delta_2)) \simeq CC(\Theta\tilde{S}(\Delta_2))$.

To start, it is immediate that $\Delta_1 \simeq_G \Delta_2$ implies $\tilde{\Sigma}(\Delta_1) \simeq \tilde{\Sigma}(\Delta_1)$ as in [25, p. 12], and since all mappings are G-equivariant we have a homotopy equivalence of simplicial G-sets. We now wish to apply the functor Θ to both sides. It is first necessary to say what this would mean, since Θ has so far only been defined on finite G-sets and we need to extend the domain of definition to infinite ones. We define Θ on an infinite G-set Ω by first expressing it as a union of its orbits $\Omega = \bigcup \Omega_i$ and then putting $\Theta(\Omega) = \oplus\Theta(\Omega_i)$. On morphisms we define Θ in terms of its components with respect to this direct sum decomposition, each component being the effect of Θ on the restriction of the morphism to the corresponding orbit. Now to show that $CC(\Theta\tilde{\Sigma}(\Delta_1)) \simeq CC(\Theta\tilde{\Sigma}(\Delta_2))$ we proceed as in [25, p. 100]. The essential matter is to show that if we have two semisimplicial mappings

$$f_0, f_1 : \tilde{\Sigma}(\Delta_1) \to \tilde{\Sigma}(\Delta_2)$$

which are homotopic by a homotopy H then the chain mappings $CC(\Theta(f_0))$ and

$CC(\Theta(f_1))$ are chain homotopic. Using notation borrowed from [25], each of the mappings

$$\begin{aligned} \kappa_i : \Sigma_n(\Delta_1) &\to \Sigma_{n+1}(\Delta_2) \\ x &\mapsto H(s_i x, (0 \ldots \overset{i}{0}1 \ldots 1)) \end{aligned}$$

is G-equivariant, so we may form $\Theta(\kappa_i)$. Now we put

$$k = \sum_{i=0}^{n} (-1)^i \Theta(\kappa_i)$$

and calculate in the usual way that $dk + kd = CC(\Theta(f_1)) - CC(\Theta(f_0))$. This is the same as the usual calculation except that all the terms have Θ applied to them. To be explicit about this, we have to show that

$$\begin{aligned} &\sum_{j=0}^{n+1} (-1)^j \Theta(\partial_j) \sum_{i=0}^{n} (-1)^i \Theta(\kappa_i) + \sum_{i=0}^{n-1} (-1)^i \Theta(\kappa_i) \sum_{j=0}^{n} (-1)^j \Theta(\partial_j) \\ &\quad = \Theta(f_1) - \Theta(f_0) \end{aligned}$$

We rely on the identities

$$\partial_j \kappa_i = \begin{cases} \kappa_{i-1}\partial_j & \text{if } j < i \\ \kappa_i \partial_{j-1} & \text{if } j > i+1 \end{cases}$$

$$\partial_{i+1}\kappa_i = \partial_{i+1}\kappa_{i+1}$$

$$\partial_0 \kappa_0 = f_1$$

$$\partial_{n+1}\kappa_n = f_0$$

which immediately show that terms cancel in pairs except for two of them, and hence the above equation holds. Now the desired assertion about chain homotopy equivalence of the complexes follows in the usual way from what we have just shown about mappings.

Finally we need to show that $CC(\Theta\tilde{S}(\Delta)) \simeq CC(\Theta\tilde{\Sigma}(\Delta))$ for any G-simplicial complex Δ. To do this we follow a standard treatment which shows that the singular and simplicial chain complexes of Δ are chain homotopy equivalent, such as [26]. One has the natural inclusion $CC(\tilde{S}(\Delta)) \hookrightarrow CC(\tilde{\Sigma}(\Delta))$ and in [26, p. 115] there is

constructed a map in the reverse direction which is a homotopy inverse to the inclusion. We have to observe that in this construction the homotopy inverse may be obtained on each orbit of simplices as an integer linear combination of G-equivalent maps. This is because the constructions at every stage are natural, and we may always proceed by making a definition on one element in an orbit and extending to the rest of the orbit so as to be equivariant for the action of G. We may now take instead the same linear combinations of these mappings but with Θ applied to them. The result is a chain map $CC(\Theta\tilde{\Sigma}(\Delta)) \to CC(\Theta\tilde{S}(\Delta))$, and it is a homotopy inverse to the canonical map $CC(\Theta\tilde{S}(\Delta)) \to CC(\Theta\tilde{\Sigma}(\Delta))$. One shows these facts by exactly the same arguments as in [26] except that Θ is applied to all the terms in the equations. For the validity of these equations after Θ has been applied one relies on the fact that in every case the equations simplify because terms cancel in pairs. Thus it makes no difference if we apply Θ everywhere. We omit the precise technicalities of this argument because they are complicated, and well-known □

We now give the proof of Theorem B. By virtue of 4.2.1 we may assume that $G \in \mathscr{X}$, and so we are reduced to proving the following special case.

4.3.2. THEOREM. *Let G be a finite group, M a Mackey functor for G, $\mathscr{Y}$ a set of subgroups of G closed under taking subgroups and conjugation, and Δ a G-simplicial complex of dimension d. Suppose that*

(i) *For every simplex $\sigma \in \Delta$ the vertices of σ lie in distinct G-orbits.*

(ii) *For every subgroup $H \notin \mathscr{Y}$, Δ^H is contractible.*

(iv) *For every $Y \in \mathscr{Y}$, $M(Y) = 0$.*

Then the complexes of Mackey functors

$$0 \to M \to M_{\Gamma_0(\Delta)} \to M_{\Gamma_1(\Delta)} \to \cdots \to M_{\Gamma_d(\Delta)} \to 0$$

and

$$0 \leftarrow M \leftarrow M_{\Gamma_0(\Delta)} \leftarrow M_{\Gamma_1(\Delta)} \leftarrow \cdots \leftarrow M_{\Gamma_d(\Delta)} \leftarrow 0$$

are split acyclic.

The assumptions of this theorem will now remain in force throughout this section. The key to the proof of the theorem is the following result.

4.3.3. LEMMA *Suppose $G \notin \mathscr{Y}$. Then Δ has a G-subcomplex E which is G-contractible and such that every simplex σ not in E has its stabilizer G_σ in $\mathscr{Y}$.*

Proof. We define $E = \bigcup_{H \leq G, H \notin \mathcal{Y}} \Delta^H$. This is a subcomplex, since if σ is fixed by H then so is every face of σ; and it is preserved under the action of G. Evidently all simplices outside E have their stabilizers in $\mathcal{Y}$. We show that E is G-contractible.

Because Δ^G is contractible it is non-empty, so E is non-empty. Choose a vertex $x \in \Delta^G$ and consider the inclusion mapping $i : \{x\} \to E$. We will show that this is a G-homotopy equivalence. For each subgroup $H \leq G$ there is a mapping of fixed points $i^H : \{x\} \to E^H$. By a theorem of Bredon ([12, Section II] see also [22]) it suffices to show that each map i^H is a homotopy equivalence. For this we only need show that E^H is contractable for all subgroups $H \leq G$. There are two cases. When $H \notin \mathcal{Y}$ we have $E^H = \Delta^H$, and this is contractible by hypothesis. When $H \in \mathcal{Y}$ we have $E^H = \bigcup_{H \leq K, K \notin \mathcal{Y}} \Delta^K$ since the right hand side is clearly contained in the left, and if $\sigma \in E^H$ then $H \leq G_\sigma \notin \mathcal{Y}$ by construction of E, so $\sigma \in \Delta^{G_\sigma}$ which is in the right hand side. Thus E^H is a union of contractible spaces, every non-empty intersection of which is also contractible since $\Delta^K \cap \Delta^J = \Delta^{\langle K,J \rangle}$. Hence E^H is homotopic to the nerve of the covering [3, p. 50]. Since the intersection of all the sets in the covering is non-empty, the nerve consists of a single simplex, which is contractible. □

4.3.4. LEMMA. *Suppose $G \notin \mathcal{Y}$. The complexes of Mackey factors obtained from Δ and E are the same.*

Proof. In the complex

$$0 \to M \to M_{\Gamma_0(\Delta)} \to M_{\Gamma_1(\Delta)} \to \cdots \to M_{\Gamma_d(\Delta)} \to 0$$

the typical term has the form

$$M_{\Gamma_i(\Delta)} = \bigoplus_{\sigma \in G \backslash \Gamma_i(\Delta)} M_{G/G_\sigma}.$$

If $\sigma \notin E$ then $G_\sigma \in \mathcal{Y}$ and hence $M_{G/G_\sigma} = M{\downarrow}^G_{G_\sigma}{\uparrow}^G_{G_\sigma} = 0$ since M is zero on subgroups of G_σ. Thus $M_{\Gamma_i(\Delta)} = M_{\Gamma_i(E)}$. Furthermore the restriction to $M_{\Gamma_i(E)}$ of the differential coming from Δ is the same as the differential coming from E, so the two complexes are the same.

Proof of Theorem 4.3.2. If $G \in \mathcal{Y}$ then M is the zero Mackey functor, so we may assume $G \notin \mathcal{Y}$. We work with Θ_M the argument for Θ^M being similar. Let E be the subcomplex of Δ in Lemma 4.3.3. We have that $CC(\theta_M \tilde{S}(\Delta)) = CC(\Theta_M \tilde{S}(E))$. But E is G-contractible, so by 4.3.1 $CC(\Theta_M \tilde{S}(E))$ is chain homotopic to the zero complex. Therefore it has zero homology, and is split since splitting is preserved under chain homotopy equivalence [22, 7.2]. □

5. Proof of theorems on the chain complex of Δ

5.1. *Proof of Proposition* 2.7.3

Let $\mathbb{Z}_p$ denote the p-adic integers. For any $\mathbb{Z}G$-module V we will write $V_p = V \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The statement we are due to prove is:

2.7.3. PROPOSITION. *Take $M(G) = H^n(G, V)_p$ for some integer $n \geq 1$ and $\mathbb{Z}G$-module V. The sequences in Theorem A are $\mathrm{Ext}^n_{\mathbb{Z}_pG}(\tilde{C}.(\Delta), V_p)$ and $\mathrm{Ext}^n_{\mathbb{Z}_pG}(\tilde{C}.(\Delta)^*, V_p)$ where $*$ denotes the contragredient and $\tilde{C}.(\Delta)$ is the augmented chain complex of Δ.*

Proof. We verify first that the groups in the complexes are what they should be, then that the maps are correct. A typical chain group $C_r(\Delta)$ can be written $\bigoplus_{\sigma \in G\backslash\Gamma_r(\Delta)} \mathbb{Z}_p\uparrow^G_{G_\sigma}$, because it is the free $\mathbb{Z}_p$-module on the simplices in dimension r and these divide up into orbits, each giving a submodule $\mathbb{Z}_p\uparrow^G_{G_\sigma}$. Applying Ext we obtain

$$\begin{aligned}
\mathrm{Ext}^n_{\mathbb{Z}_pG}(C_r(\Delta), V_p) &\cong \bigoplus_{\sigma \in G\backslash\Gamma_r(\Delta)} \mathrm{Ext}^n_{\mathbb{Z}_pG}(\mathbb{Z}_p\uparrow^G_{G_\sigma}, V_p) \\
&\cong \bigoplus_{\sigma \in G\backslash\Gamma_r(\Delta)} \mathrm{Ext}^n_{\mathbb{Z}_pG_\sigma}(\mathbb{Z}_p, V_p) \\
&\cong \bigoplus_{\sigma \in G\backslash\Gamma_r(\Delta)} \mathrm{Ext}^n_{\mathbb{Z}G_\sigma}(\mathbb{Z}, V)_p \\
&= \bigoplus_{\sigma \in G\backslash\Gamma_r(\Delta)} H^n(G_\sigma, V)_p,
\end{aligned}$$

by means of standard isomorphisms. The fact that we can take completion at p outside the Ext term follows from [10, p. 233]. There is a similar chain of isomorphisms for the contragredient representations $C_r(\Delta)^*$, since they are permutation modules and $C_r(\Delta)^* \cong C_r(\Delta)$.

We have to show that $\phi_{\tau\sigma}$ and $\psi_{\sigma\tau}$ are the maps induced on the Ext groups by the homomorphism $\mathbb{Z}_p\uparrow^G_{G_\tau} \to \mathbb{Z}_p\uparrow^G_{G_\sigma}$ and its dual. We suppose here that σ is a face of $g\tau$. Evidently the maps have the right sign from the definition of the boundary operator. Regarding the modules now as free $\mathbb{Z}_p$-modules on the simplices, the map is $hg\tau \to h\sigma$, $h \in G$, and its dual is $h\hat{\sigma} \to hzg\hat{\tau}$, where z is the sum of a set of representatives for the cosets of $G_{g\tau}$ in G. A hat indicates the element of the dual basis corresponding to the bare-headed symbol. Because these maps take place in the first Ext variable and this commutes with what happens in the second variable, they give natural transformations of the cohomological functors $H^n(G_\sigma, V_p)$ and

$H^n(G_\tau, V_p)$, so it suffices to check that they coincide with the original definitions of $\phi_{\tau\sigma}$ and $\psi_{\sigma\tau}$ on fixed points (see [10]). On fixed points the maps are

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}_pG}(\mathbb{Z}_p\uparrow_{G_\sigma}^G, V_p) &\to \mathrm{Hom}_{\mathbb{Z}_pG}(\mathbb{Z}_p\uparrow_{G_\tau}^G, V_p)\\ (\sigma \mapsto v) &\mapsto (\tau \mapsto g^{-1}v)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}_pG}(\mathbb{Z}_p\uparrow_{G_\tau}^G, V_p) &\to \mathrm{Hom}_{\mathbb{Z}_pG}(\mathbb{Z}_p\uparrow_{G_\sigma}^G, V_p)\\ (\tau \mapsto v) &\mapsto (\sigma \mapsto zgv).\end{aligned}$$

These are respectively $c_{g^{-1}} \cdot \mathrm{res}_{G_{g\tau}}^{G_\sigma}$ and $\mathrm{ind}_{G_{g\tau}}^{G_\sigma} \cdot c_g$. □

5.2. Proof of Theorem 2.7.4

We first need a technical lemma. We work with modules over a finite dimensional algebra Λ, and if A and B are Λ-modules we use the notation $\underline{\mathrm{Hom}}_\Lambda(A, B)$ for the group of homomorphisms from A to B modulo those homomorphisms which factor through a projective module.

5.2.1. LEMMA. *Let Λ be a self-injective algebra finite dimensional over a field k, and let $\mu : A \to B$ be a monomorphism between finite dimensional Λ-modules. If the induced map $\underline{\mathrm{Hom}}_\Lambda(B, V) \to \underline{\mathrm{Hom}}_\Lambda(A, V)$ is surjective for all Λ-modules V then μ is split.*

With full homomorphism groups this result is immediate, and the point is that it works with homomorphisms modulo projectives.

Proof. Taking $V = A$, there exists $\phi : B \to A$ whose image in $\underline{\mathrm{Hom}}_\Lambda(A, V)$ is the same as that of 1_A, i.e. $\phi\mu \equiv 1$ (mod projective homomorphisms). For some n, $(\phi\mu)^n$ has the same image as $(\phi\mu)^{n+1}$, so $A = A_0 \oplus A_1$ where $(\phi\mu)^n$ is an automorphism on A_0 and is zero on A_1, and again $(\phi\mu)^n \equiv 1$ (mod projective homomorphisms). When we restrict the domain of μ to A_0 the monomorphism $\mu : A_0 \to B$ is split by $[(\phi\mu)^n]^{-1}(\phi\mu)^{n-1}\phi$, where $[(\phi\mu)^n]^{-1}$ denotes the inverse of $(\phi\mu)^n$ on A_0. Because $(\phi\mu)^n$ is zero on A_1, 1_{A_1} factors through an injective (=projective) module, and so A_1 is injective and $\mu : A_1 \to B$ is split. Hence μ is split as a morphism $A \to B$. □

We now prove Theorem 2.7.4. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are clear. We first prove (i) $\Rightarrow$ (iii). Because trivial source modules and all morphisms between them are liftable from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}_p$ [12, II, 12.4], it suffices to prove the decomposition of $C.$ in (iii) when $R = \mathbb{Z}/p\mathbb{Z} = k$, say.

We proceed by induction on the dimension d of the complex

$$C. = C_d \to \cdots \to C_0.$$

When $d = 0$ then $\mathrm{Ext}^n_{kG}(C_0, V) = 0$ for all V, so C_0 is projective. Now suppose $d > 0$ and the result is true for smaller dimensions. There is an isomorphism $\underline{\mathrm{Hom}}_{kG}(A. B) \cong \mathrm{Ext}^1_{kG}(A, \Omega^{-1}B)$ and we apply the condition in (i) to these groups. Let H_d be the top homology of $C.$, so

$$0 \to H_d \to C_d \to C_{d-1}$$

is exact. Then the composite

$$\underline{\mathrm{Hom}}_{kG}(C_{d-1}, V) \to \underline{\mathrm{Hom}}_{kG}(C_d, V) \to \underline{\mathrm{Hom}}_{kG}(H_d, V)$$

is zero and the left hand map is epi by the hypothesis (i), so the right hand map is zero and the image of 1_{C_d} in $\mathrm{Hom}_{kG}(H_d, C_d)$ (i.e. the inclusion map $H_d \to C_d$) factors through an injective module. Hence there is a factorisation $H_d \to I \to C_d$ where I is the injective hull of H_d, and $I \to C_d$ is injective since I and H_d have the same socle and this embeds in C_d. This means that $C_d \cong I \oplus Y$ for some submodule Y of C_d, and $H_d \subseteq I$. Also Y embeds in C^{d-1} by restriction of the map $C_d \to C_{d-1}$ and we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{kG}(C_{d-1}, V) \arrow[r] & \underline{\mathrm{Hom}}_{kG}(C_d, V) \arrow[d, "\cong"] \\
& \underline{\mathrm{Hom}}_{kG}(Y, V) \arrow[ul, phantom, "\searrow"]
\end{tikzcd}$$

since the summand I contributes zero to the homomorphisms modulo injectives. Hence the monomorphism $Y \to C_{d-1}$ splits by Lemma 5.2.1 and $C_{d-1} \cong Y \oplus W$ for some submodule W. Now

$$\underline{\mathrm{Hom}}_{kG}(C_{d-2}, V) \to \underline{\mathrm{Hom}}_{kG}(Y \oplus W, V) \to \underline{\mathrm{Hom}}_{kG}(Y, V) \to 0$$

is exact, and so

$$\underline{\mathrm{Hom}}_{kG}(C_{d-2}, V) \to \underline{\mathrm{Hom}}_{kG}(W, V)$$

is epi. Thus $W \to C_{d-2} \to \cdots \to C_0$ is a complex of dimension $d-1$ satisfying the induction hypothesis, so there exists an acyclic split subcomplex $X \to D_{d-2} \to \cdots \to D_0$ with $C_i = D_i \oplus P_i$ for $i \leq d-2$, $W = X \oplus P_{d-1}$. Put $D_{d-1} = X \oplus Y$, $D_d = Y$. Then $D_d \to \cdots \to D_0$ is our acyclic split complex D. and $C_d \cong Y \oplus I$, $C_{d-1} \cong Y \oplus W = D_{d-1} \oplus P_{d-1}$ as required. This completes the proof of the implication (i) $\Rightarrow$ (iii).

We now prove (iii) $\Rightarrow$ (iv), showing by induction on n that for all r with $0 \leq r \leq n$ we may choose a projective submodule P_r so that $C_r = D_r \oplus P_r$ and such that the differential d_r sends D_r into D_{r-1} and P_r into P_{r-1}. The induction starts at $n = 0$. Here we only need the additive decomposition $C_0 = D_0 \oplus C_0/D_0$ which results from the fact that C_0/D_0 is projective. Suppose now that $n > 0$ and the results holds for smaller values. Because C_n/D_n is projective we may write $C_n = D_n \oplus Q_n$ for some projeective module Q_n and represent $d_n|_{Q_n}$ in component form (β_n, α_n) corresponding to the decomposition $C_{n-1} = D_{n-1} \oplus P_{n-1}$. The picture of C is

$$\begin{tikzcd}
\cdots \arrow[r] & Q_n \arrow[r, "\alpha_n"] \arrow[d, phantom, "\oplus"] \arrow[dr, "\beta_n"] & P_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, phantom, "\oplus"] & P_{n-2} \arrow[r] \arrow[d, phantom, "\oplus"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "\gamma_n"] & D_{n-1} \arrow[r, "\gamma_{n-1}"] & D_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

where for $0 \leq r \leq n$, $d_r = \gamma_r \oplus \alpha_r$ with $d_r|_{D_r} = \gamma_r$. For any element $x \in Q_n$, $d_{n-1}\, d_n(x) = (\gamma_{n-1}\beta_n(x),\ \alpha_{n-1}\alpha_n(x)) = 0$, from which we see that $\gamma_{n-1}\beta_n = 0$ and $\operatorname{Im} \beta_n \subseteq \ker \gamma_{n-1} = \operatorname{Im} \gamma_n$. Hence by projectivity of Q_n there exists a map $\phi_n : Q_n \to D_n$ so that $\beta_n = \gamma_n \phi_n$. Define $P_n = \{(-\phi_n(x), x) \mid x \in Q_n\}$. This is an isomorphic image of Q_n under the map $x \mapsto (-\phi_n(x), x)$ so $P_n \cong Q_n$ is a projective submodule, and plainly $C_n = P_n \oplus D_n$. Furthermore for $y = (-\phi_n(x), x) \in P_n$,

$$\begin{aligned}
d_n(y) &= d_n((-\phi_n(x), x)) \\
&= (-\gamma_n \phi_n(x) + \beta_n(x), \alpha_n(x)) \\
&= (0, \alpha_n(x)) \in P_{n-1}
\end{aligned}$$

thus completing the induction step. □

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Received September 12, 1989