Null k-cobordant links in S3.

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Null k-cobordant links in S^3

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Recently, our understanding of the set $C(m)$ of equivalence classes of links $(\coprod_{i=1}^{m} S^1 \hookrightarrow S^3)$ under the equivalence relation of link concordance has been expanded tremendously. The main advances were brought to us through a series of works by Cochran, le Dimet, Levine, Orr and Habegger-Lin. See [C1], [CO], [Di], [Le1,2], [O] and [HL1]. Here one should notice that although in [HL1], only the equivalence relation of link-homotopy, which is much weaker than link concordance, was concerned, the results in [HL1] are prototypical for the forthcoming work [HL2] in which we will give $C(m)$ an orbit space picture which will lead to a refinement of Milnor's $\bar{\mu}$-invariants and a corresponding computable obstruction theory. Moreover, we will show in this paper that the notion of link-homotopy can be explored to help us to draw some strong conclusions about link concordance invariants.

The so-called k-cobordance is another equivalence relation among links in S^3 rose naturally in the work of Sato, Cochran and Orr (see [Sa], [C1], and [O]). Here is the definition.

DEFINITION. Let $L = \{L_1, L_2, \ldots, L_m\}$ and $L' = \{L'_1, L'_2, \ldots, L'_m\}$ be ordered, oriented tame links in S^3. We say that L is k-cobordant to L' where k is a positive integer if there are disjointly imbedded, compact, connected, oriented surfaces $V_1, V_2, \ldots, V_m$ in $S^3 \times [0, 1]$ with $\partial V_i = \partial_0 V_i \cup \partial_1 V_i$ such that for each $i = 1, 2, \ldots, m$, we have

(1) $V_i \cap S^3 \times 0 = \partial_0 V_i = L_i$, $V_i \cap S^3 \times 1 = \partial_1 V_i = L'_i$; and

(2) there is a tubular neighborhood $V_i \times D^2$ of V_i in $S^3 \times [0, 1]$ which extends the "longitudinal" ones of $\partial V_i = L_i \cup L'_i$ in $S^3 \times 0$ and $S^3 \times 1$ respectively such that the image of the homomorphism

$$\frac{\pi_1(V_i)}{\langle \partial_0 V_i \rangle}\left(\text{or } \frac{\pi_1(V_i)}{\langle \partial_1 V_i \rangle}\right) \subset \pi_1(V_i) \to \pi_1(V_i \times \partial D^2) \to \pi_1\left(S^3 \times [0, 1] \setminus \coprod_{i=1}^{m} V_i\right)$$

lies in the k-th term of the lower central series of the last group.

Here $\langle \partial_0 V_i \rangle$ and $\langle \partial_1 V_i \rangle$ are the subgroups of $\pi_1(V_i)$ normally generated by the boundary loops $\partial_0 V_i$ and $\partial_1 V_i$ respectively.

We will call $V = \amalg_{i=1}^{m} V_i$ a *k-cobordance* between L and L'. If L' is an unlink, then we say that L is *null k-cobordant.*

With this definition, we have the following conjecture of T. Cochran and K. Orr.

CONJECTURE. A link L is null k-cobordant if and only if Milnor's $\bar{\mu}$-invariants of L with length less than or equal to $2k$ vanish.

Suppose a link L in S^3 is null k-cobordant. Then, by a theorem of Dwyer in [D] which generalized the main theorem of [St], the inclusion

$$S^3 \backslash L \hookrightarrow S^3 \times [0, 1] \backslash V$$

induces isomorphisms between the fundamental groups modulo the n-th terms of their lower central series for $n \leq k+1$. Since the longitudes of the unlink L' are trivial, it is easy to see that the longitudes of L lie in the $2k$-th term of the lower central series of $\pi_1(S^3 \times [0, 1] \backslash V)$. So we can only conclude that the $\bar{\mu}$-invariants of L with length less than or equal to $k+1$ vanish. From this observation, we see that the Cochran–Orr conjecture is quite strong and it seems to be impossible to attack this conjecture with the standard machineries in homology theory.

As announced by K. Igusa and K. Orr, one direction of the Cochran–Orr conjecture, that the vanishing of the $\bar{\mu}$-invariants of weight less than or equal to $2k$ implies null k-cobordism, is true. They also announced that null k-cobordism implies the vanishing of the $\bar{\mu}$-invariants of weight less than or equal to $2k-1$. The purpose of this paper is to give a proof of the other direction of the full conjecture. Here are our theorem and its corollary.

THEOREM 2.1. *Let L and L' be m-component links which are k-cobordant. If $m \leq 2k$, then L and L' are link-homotopic.*

COROLLARY 2.2. *If L and L' are k-cobordant, then Milnor's $\bar{\mu}$-invariants of L and L' with length less than or equal to $2k$ are the same. In particular, if L is null k-cobordant, then $\bar{\mu}$-invariants of L with length less than or equal to $2k$ vanish.*

Thus, there is a nice characterization of null k-cobordant links in S^3. See [C2] for various consequences of the Cochran–Orr conjecture.

Here is the organization of this paper. In §1, we will reduce a k-cobordance to a sequence of concordances and elementary k-cobordances. Our theorem and its corollary are proved in §2. We work in the PL category.

§1. Elementary k-cobordances

In this section, we analyse the definition of k-cobordances between two links. Suppose V is a k-cobordance between two links $L=\{L_1, L_2, \ldots, L_m\}$ and $L'=\{L'_1, L'_2, \ldots, L'_m\}$. It has a normal direction in $S^3 \times [0, 1]$ which extends the zero framing normal directions of L and L' in $S^3 \times 0$ and $S^3 \times 1$ respectively. For a loop c on V, let c^+ be the push-off of c away from V in $S^3 \times [0, 1]$ along that normal direction. Then the homomorphism

$$\pi_1(V_i) \to \pi_1(S^3 \times [0, 1] \backslash V)$$

in the definition of k-cobordance is given by sending $\langle c \rangle$ to $\langle c^+ \rangle$. We first have the following lemma.

LEMMA 1.1. *One can isotopy V in $S^3 \times [0, 1]$ relative to the boundary such that for some $t \in (0, 1)$, $V \cap (S^3 \times [0, t])$ is a concordance from L to an m-component link K in $S^3 \times t$; $V \cap (S^3 \times [t, 1])$ is a concordance of an m-component link K' in $S^3 \times t$ to L'; and $V \cap (S^3 \times t)$ is a surface with m components such that the boundary of each component of this surface consists of one component of K and one component of K'. Moreover, the normal direction of the surface $V \cap (S^3 \times t)$ in S^3 restricted on K (and K') is the zero framing normal direction of K (and K') in $S^3 \times t$.*

Proof. This is proved by the usual general position argument. Consider the projection $p : S^3 \times [0, 1] \to [0, 1]$. We assume $p|V$ is a Morse function on V. We can arrange the critical points of V so that index 0 critical points lie in $S^3 \times (0, 1/3)$, index 1 critical points in $S^3 \times (1/3, 2/3)$ and index 2 critical points in $S^3 \times (2/3, 1)$. Moreover, we can arrange index 1 critical points within $S^3 \times (1/3, 2/3)$ in any order when they are ordered by their heights (the values of the projection). Thus, we can isotopy the surface V in $S^3 \times I$ relative to the boundary so that for some t_1, t_2 with $0 < t_1 < t_2 < 1$, $V \cap (S^3 \times [0, t_1])$ is a concordance from L to an m-component link $V \cap (S^3 \times t_1)$ in $S^3 \times t_1$; $V \cap (S^3 \times [t_2, 1])$ is a concordance of an m-component link $V \cap (S^3 \times t_2)$ in $S^3 \times t_2$ to L' and $V \cap (S^3 \times [t_2, t_1])$ is a surface with m components such that the projection restricted on this surface has no critical points of index 0 and 2. We can make $t_2 - t_1$ arbitrarily small so that finally, the surface $V \cap (S^3 \times [t_1, t_2])$ can be isotopied into some 3-sphere level $S^3 \times t$. Each component of this surface in $S^3 \times t$ may be considered as obtained from band summing an annulus with a connected surface whose boundary is also connected. If the annulus is twisted in $S^3 \times t$, we can push it into $S^3 \times [t, 1]$ or $S^3 \times [0, t]$ as a part of the concordance. In this way, we can achieve our assertion about the normal direction of the surface in $S^3 \times t$. □

Now suppose that we have arranged a k-cobordance V between m-component links L and L' as described in Lemma 1.1. Denote $V' = V \cap (S^3 \times t)$, a surface with m connected components such that $\partial V' = K \cup K'$ and the boundary of each component of V' consists of two components belonging to K and K' respectively. The normal direction of V in $S^3 \times [0, 1]$ restricted on V' is consistant with the normal direction of V' in $S^3 \times t$.

LEMMA 1.2. *Let c be a simple closed curve on V' not parallel to $\partial V'$, then $\langle c \rangle$ lies in the k-th terms of the lower central series of $\pi_1(S^3 \times t \backslash K)$ and $\pi_1(S^3 \times t \backslash K')$ respectively.*

Proof. We first notice that the push-off of c away from V' in $S^3 \times t$ along the normal direction of V' in $S^3 \times t$ is c^+.

For $\varepsilon > 0$ small enough, we may assume that

$$(S^3 \times [t - \varepsilon, t], V \cap (S^3 \times [t - \varepsilon, t])) = (S^3, K) \times [t - \varepsilon, t]$$

and

$$(S^3 \times [t, t + \varepsilon], V \cap (S^3 \times [t, t + \varepsilon])) = (S^3, K') \times [t, t + \varepsilon].$$

We can assume $c \times (t - \varepsilon) = c^+$.

As proved in [Cl] as well as in [O] by using a theorem of Dwyer in [D] which generalized the main theorem of [St], the inclusions

$$S^3 \times 0 \backslash L \hookrightarrow S^3 \times [0, 1] \backslash V \hookleftarrow S^3 \times 1 \backslash L'$$

induce isomorphisms between the fundamental groups modulo the n-th terms of their lower central series for $n \le k + 1$. Moreover, since $V \cap S^3 \times [0, t - \varepsilon]$ is a concordances from L to $K \times (t - \varepsilon)$ so that the inclusions induce isomorphisms between $\pi_1(S^3 \times 0 \backslash L)$ and $\pi_1((S^3 \backslash K) \times (t - \varepsilon))$ modulo any terms of the corresponding lower central series, we conclude that that $\langle c \times (t - \varepsilon) \rangle$ lies in the k-th term of the lower central series of $\pi_1((S^3 \backslash K) \times (t - \varepsilon))$. This proves the lemma for K and similarly for K'. □

Notice that $S^3 \times [t - \varepsilon, t + \varepsilon]$ is actually a k-cobordance between $K \times (t - \varepsilon)$ and $K' \times (t + \varepsilon)$. We are led to the following definition by this observation.

Let $L = \{L_1, L_2, \ldots, L_n\}$ be a link in S^3. Let a and b be two simple loops in $S^3 \backslash L$ such that $\langle a \rangle$ and $\langle b \rangle$ belong to the k-th term of the lower central series of $\pi_1(S^3 \backslash L)$. Suppose $a \cap b$ is a single point. Then $a \cup b$ is the spine of a punctured torus T in $S^3\backslash L$. Let A be an annulus such that $\partial A = \partial_0 \cup \partial_1$ where ∂_0 and ∂_1 are components of ∂A, ∂_0 is a component of L, say L_1, and the linking number of ∂_0 and ∂_1 is zero. We also assume A is disjoint with T as well as the components of L other than $L_1 = \partial_0$. Let δ be an arc in S^3 which is disjoint with L and meets A and T on ∂_1 and ∂T with its two end points respectively. Form a band sum $A \#_\delta T$. Let $L_1' = \partial(A \#_\delta T)$. Then, the links L and $L' = \{L_1', L_2, \ldots, L_m\}$ are k cobordant. We say L' is obtained from L by an *elementary k-cobordance.* Lemma 1.1 and 1.2 obviously imply the following proposition.

PROPOSITION 1.3. *Suppose L and L' are k-cobordant links. Then there are links K and K' concordant to L and L' respectively such that K' is obtained from K by a sequence of elementary k-cobordances.*

§2. Milnor's $\bar{\mu}$-invariants

Let L be a link in S^3. The *link group* (or *reduced fundamental group*) of L, $\mathfrak{g}(L)$, is defined to be the quotient group of $\pi_1(S^3\backslash L)$ obtained by adding relations which say that two meridians of a component of L commute. The group $\mathfrak{g}(L)$ is invariant under link-homotopy. Here are two properties of this group $\mathfrak{g}(L)$ which are useful for us:

(1) If the link L has m components, then the $(m+1)$-th term of the lower central series of $\mathfrak{g}(L)$ is trivial; and
(2) If l and l' are two simple loops in $S^3\backslash L$ such that $\langle l\rangle = \langle l'\rangle$ in $\mathfrak{g}(L)$, then the links $\{L, l\}$ and $\{L, l'\}$ are link-homotopic.

See [M1] or [HL1].

THEOREM 2.1. *Let L and L' be m-component links which are k-cobordant. If $m \le 2k$, then L and L' are link-homotopic.*

Proof. Since concordance implies link-homotopy ([Gi], [Go]), by virtue of Proposition 1.3, we can assume that L' is obtained from L by an elementary k-cobordance on a component l of L. Let l' be the component of L' so that $L\backslash l = L'\backslash l' = K$, where K is a link of $m-1$ components. Then, by the definition of elementary k-cobordant, $\langle l\rangle = \langle l'\rangle$ in $\pi_1(S^3\backslash K)$ modulo $2k$-commutators. Now $m \le 2k$, so $\langle l\rangle = \langle l'\rangle$ in $\mathfrak{g}(K)$ by the above property (1). This, by the property (2), implies that L and L' are link-homotopy. □

Milnor's *$\bar{\mu}$-invariants*, $\bar{\mu}(i_1, i_2, \ldots, i_n)$, of a link $L = \{L_1, L_2, \ldots, L_m\}$, where $1 \le i_\alpha \le m$ for $\alpha = 1, 2, \ldots, n$, are link concordance invariants which measure how deep the longitudes of L lie in the lower central series of $\pi_1(S^3 \backslash L)$. If the indices i_α, $\alpha = 1, 2, \ldots, n$ are all distinct, then the corresponding $\bar{\mu}$-invariant is also an invariant of link-homotopy.

Here we would rather think of a $\bar{\mu}$-invariant as being indexed by the components of a link. Suppose there are two indices taking as their values the same component of L. We can split that component of L into two parallel (zero linking number) copies and assign each copy to one of those two indices separately. Thus, we get a $\bar{\mu}$-invariant of the derived link. This $\bar{\mu}$-invariant is the same as the original one. See [M2]. In this way, we need only to consider $\bar{\mu}$-invariants with distinct indices of links obtained from the original one by splitting components into parallel copies.

COROLLARY 2.2. *If L and L' are k-cobordant, then Milnor's $\bar{\mu}$-invariants of L and L' with length less than or equal to $2k$ are the same. In particular, if L is a null k-cobordant, then $\bar{\mu}$-invariants of L with length less than or equal to $2k$ vanish.*

Proof. Suppose L (as well as L') has m components. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be an ordered set of positive integers and

$$|\Lambda| = \sum_{\alpha=1}^{m} \lambda_\alpha.$$

Let $D_\Lambda(L)$ be the link obtained from L by splitting its α-th component into λ_α parallel copies, $\alpha = 1, 2, \ldots, m$. Then $D_\Lambda(L)$ and $D_\Lambda(L')$ are still k-cobordant. If $|\Lambda| \le 2k$, by Theorem 2.1, $D_\Lambda(L)$ and $D_\Lambda(L')$ are link-homotopic.

A $\bar{\mu}$-invariant of L with length less than or equal to $2k$ is the same as a $\bar{\mu}$-invariant with distinct indices of $D_\Lambda(L)$ for some Λ with $|\Lambda| \le 2k$. Since $D_\Lambda(L')$ is link-homotopic to $D_\Lambda(L)$, they have the same $\bar{\mu}$-invariants with distinct indices. This implies that L and L' have the same $\bar{\mu}$-invariants with length less than or equal to $2k$. □

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Corollary 2.2 was first proved in my 1988 thesis [L]. After a revised version of that thesis being submitted, the referee found an elegant simplification of my original tedious proof which also led to a generalization as formulated in Theorem 2.1. I am deeply indebted to the referee for her/his contribution to this paper. To take this opportunity, I would like to express my appreciation to my thesis advisor Professor M. Freedman for his mathematical insight, encouragement and kindness. I also would like to thank T. Cochran and N. Habegger for their generous help. The support of the Sloan Foundation in 1987–88 with its Doctoral Dissertation Fellowships is gratefully acknowledged.

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