L2 - Curvature pinching.

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L^2-Curvature pinching

MAUNG MIN-OO and ERNST A. RUH

§1. Introduction

For a Riemannian manifold (M^n, g), we will denote by K its sectional curvature, by $Rm = R^l_{ijk}$ the Riemannian curvature tensor, by $Rc = R_{jk} = R^l_{ljk}$ the Ricci curvature, and by $R = R^k_k = g^{jk}R_{jk}$ its scalar curvature. We normalize our curvature tensors so that the sphere S^n of radius 1 has $K = 1$. If M is compact, we denote by d its diameter, by V its volume and we define: $r = ⨍ R = 1/V \int R$ to be the average scalar curvature. In general we will use the notation $⨍ = 1/V \int$ to denote the average integral.

To measure the deviation from constant sectional curvature we introduce the tensor:

$$\widetilde{Rm} = \tilde{R}^l_{ijk} = R^l_{ijk} - \frac{r}{n(n-1)} g^l_{ijk}, \tag{1.01}$$

where $g^l_{ijk} = g_{jk}g^l_i - g_{ik}g^l_j$ is the curvature tensor of the standard sphere S^n. We will call $\widetilde{Rm}$ the reduced curvature tensor.

The first result of this paper is the following pinching theorem for the average L^2-norm of the reduced curvature tensor. If we replace the L^2-norm by the stronger pointwise C^0-norm, in the following, then the corresponding theorem would be simply the classical differentiable pinching theorem for the case of the sphere and hyperbolic space. (See [8] for the case $r > 0$ and [5] for $r < 0$). In fact our proof consists of showing that the weaker pinching assumptions do in fact imply the stronger one after a small smoothing perturbation obtained by following Hamilton's Ricci flow [9] for a short time.

THEOREM 1. *For any $n \geq 3$ and $\Lambda > 0$, there exists an $\epsilon(n, \Lambda) > 0$, depending only on n and Λ, such that if a compact Riemannian manifold (M^n, g)*

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satisfies:

(i) $r \neq 0$

(ii) $d^2 \max |K| \leq \Lambda^2$

(iii) $\frac{1}{r^2} \fint |\widetilde{Rm}|^2 < \epsilon(n, \Lambda)$,

then M admits a metric $\bar{g}$ *of constant sectional curvature* $K(\bar{g}) \equiv \dfrac{r}{n(n-1)}$.

The next result deals with the case where $r = 0$, i.e. with Riemannian manifolds which are almost flat in the L^2-sense, and generalizes the well known theorems of Gromov [6] and Ruh [14]. In fact, we show that our result can be reduced to the known theorems.

THEOREM 2. *For any* $n \geq 3$ *and* $\Lambda > 0$, *there exists an* $\epsilon(n, \Lambda) > 0$, *depending only on n and* Λ, *such that if a compact Riemannian manifold* (M^n, g) *satisfies*:

(i) $d^2 \max |K| \leq \Lambda^2$.

(ii) $d^4 \fint |Rm|^2 < \epsilon(n, \Lambda)$,

then M is diffeomorphic to a compact quotient of a nilpotent Lie group by a discrete group of isometries.

Our third result deals with almost Einstein manifolds and generalizes a previous result of the first author [12] on C^0-almost Einstein metrics. To formulate the theorem we introduce the reduced Ricci curvature to be the tensor:

$$\tilde{R}_{ij} = \tilde{R}^k_{kij} = R_{ij} - \frac{r}{n} g_{ij} = \widetilde{Rc}. \tag{1.02}$$

THEOREM 3. *For any* $n \geq 3$ *and* $\Lambda > 0$, *there exists an* $\epsilon(n, \Lambda) > 0$, *depending only on n and* Λ, *such that if a compact Riemannian manifold* (M^n, g) *satisfies*:

(i) $r < 0$

(ii) $d^2 \max |K| \leq \Lambda^2$

(iii) $\frac{1}{r^2} \fint |\widetilde{Rc}|^2 < \epsilon(n, \Lambda)$,

then M admits an Einstein metric $\bar{g}$ *of constant negative Ricci curvature.*

The basic method used to prove the above theorems is to deform the metric in the direction of its Ricci curvature as was first successfully done by R. S. Hamilton in [9]. This flow of metrics, which by [9] exists, at least for a short positive time, determines a non-linear parabolic evolution equation for the curvature tensor and its various components. The main idea in this work is to show that a weak L^2-pinching assumption on some appropriate component of the curvature would lead after a short time along the flow, to a C^0-pinching condition for the same curvature component, thus reducing our results to known theorems.

The basic technique used to achieve this is the classical Moser iteration method together with some recent estimates for Sobolev constants and isoperimetric inequalities as obtained by S. Gallot [2]. This work is a natural continuation of our papers [11] [12] and relies on some of the computations and methods therein.

Theorems similar in spirit to the above results have also been obtained by Gao [4]. The main results of [4] deal with purely $L^{n/2}$-pinching assumptions on the curvature and hence are weaker than our a priori assumption on $d^2 \max|K|$. On the other hand, Gao requires a somewhat restrictive assumption on the lower bound of the volume or the injectivity radius, which is needed in order to appeal to an abstract compactness theorem due to Gromov. This kind of assumption would rule out any version of Theorem 2 above.

§2. The evolution equations

We follow R. S. Hamilton's basic paper [9] and consider the Ricci flow:

$$\dot{g} = \frac{d}{dt} g = -2Rc + \frac{2}{n} r(0) g, \tag{2.01}$$

where $r(0)$ is a constant which we choose to be the average scalar curvature of the initial metric $g(0)$ at time $t = 0$. This differs in normalization from the equation used by Hamilton [9] since $r(0)$ is constant in time. It is proved in [9] that, on a compact manifold, this flow of metrics can be integrated for a maximal time interval $[0, T)$ such that if $T < \infty$, then $\lim_{t \to T} \max|K(t)| = \infty$.

We will freely use here the notation and also some formulas of [12]. For example, the Laplacians used here will be non-negative operators, which is opposite the sign convention of [9].

In terms of the reduced Ricci curvature $\widetilde{Rc} = \tilde{R}_{ij}$ introduced in (1.02) the basic evolution equation (2.01) becomes $\dot{g} = -2\widetilde{Rc}$.

If μ denotes the volume form of g, then we have

$$\dot{\mu} = -\tilde{R}\mu, \tag{2.02}$$

where the dot denotes time derivative, and $\tilde{R} = R - r(0)$, is the trace of $\widetilde{Rc}$. The rate of change of the total volume V is given by:

$$\frac{d}{dt}\log V(t) = -\fint \tilde{R}. \tag{2.03}$$

The standard decomposition of the curvature tensor into its irreducible components is:

$$R^l_{ijk} = \frac{R}{n(n-1)} g^l_{ijk} + Z^l_{ijk} + W^l_{ijk}, \tag{2.04}$$

Here W denotes the Weyl conformal curvature tensor, and Z, the traceless Ricci curvature tensor of type $(1,3)$ is given by:

$$Z^l_{ijk} = \frac{1}{n-2}(z_{jk}g^l_i + g_{jk}z^l_i - z_{ik}g^l_j - g_{ik}z^l_j), \tag{2.05}$$

where $z_{ij} = R_{ij} - (R/n)g_{ij}$ is the trace free Ricci tensor of type $(0,2)$.

The basic evolution equation for the whole Riemannian curvature tensor $Rm = R^l_{ijk}$, regarded as a 2-form with values in $gl(TM) = T^*M \otimes TM$, as derived in Thm. 7.1 of [9] or in Lemma 4 and formulas (2.14) and (2.15) of [11] is:

$$\frac{\partial}{\partial t}Rm + \bar{\Delta}Rm + Q = 0, \tag{2.07}$$

where $\bar{\Delta} = \nabla^*\nabla = -\mathrm{tr}\,\nabla^2$ is the rough Laplacian and the quadratic term Q is given by:

$$\begin{aligned} Q^l_{ijk} &= R^{pq}_{ij}R^l_{pqk} + 2R^{pl}_{iq}R^q_{pjk} - 2R^{pl}_{jq}R^q_{pik} \\ &\quad + R^l_{pjk}R^p_i + R^l_{ipk}R^p_j + R^l_{ijp}R^p_k - R^p_{ijk}R^l_p. \end{aligned} \tag{2.08}$$

Expanding Q in terms of the decomposition:

$$R^l_{ijk} = \tilde{R}^l_{ijk} + \frac{r(0)}{n(n-1)} g^l_{ijk} \quad \text{and} \quad R^l_i = \tilde{R}^l_i + \frac{r(0)}{n} g^l_i$$

we obtain:

$$
\begin{aligned}
Q^l_{ijk} = {} & \tilde{R}^{pq}_{ij}\tilde{R}^l_{pqk} + 2\tilde{R}^{pl}_{iq}\tilde{R}^q_{pjk} - 2\tilde{R}^{pl}_{jq}\tilde{R}^q_{pik} \\
& + \tilde{R}^l_{pjk}\tilde{R}^p_i + \tilde{R}^l_{ipk}\tilde{R}^p_j + \tilde{R}^l_{ijp}\tilde{R}^p_k - \tilde{R}^p_{ijk}\tilde{R}^l_p \\
& + \frac{r(0)}{n(n-1)}(\tilde{R}^{pq}_{ij}g^l_{pqk} + 2\tilde{R}^p_{iq}g^q_{pjk} - 2\tilde{R}^{pl}_{jq}g^q_{pik}) \\
& + \frac{r(0)}{n}(\tilde{R}^l_{pjk}g^p_i + \tilde{R}^l_{ipk}g^p_j + \tilde{R}^l_{ijp}g^p_k - \tilde{R}^p_{ijk}g^1_p) \\
& + \frac{r(0)}{n(n-1)}(g^{pq}_{ij}\tilde{R}^l_{pqk} + 2g^{pl}_{iq}\tilde{R}^q_{pjk} - 2g^{pl}_{jq}\tilde{R}^q_{pik} \\
& + g^l_{pjk}\tilde{R}^p_i + g^l_{ipk}\tilde{R}^p_j + g^l_{ijp}\tilde{R}^p_k - g^p_{ijk}\tilde{R}^l_p) \\
& + \left(\frac{r(0)}{n(n-1)}\right)^2(g^{pq}_{ij}g^l_{pqk} + 2g^{pl}_{iq}g^q_{pjk} - 2g^{pl}_{jq}g^q_{pik}) \\
& + \frac{r(0)^2}{n^2(n-1)}(g^l_{pjk}g^p_i + g^l_{ipk}g^p_j + g^l_{ijp}g^p_k - g^p_{ijk}g^l_p) \\
= {} & \tilde{Q}^1_{ijk} + \frac{r(0)}{n(n-1)}(-2\tilde{R}^l_{ijk} - 2\tilde{R}^l_i g_{jk} - 2\tilde{R}^l_{kij} + 2\tilde{R}^l_j g_{ik} + 2\tilde{R}^l_{kji}) \\
& + \frac{r(0)}{n}(\tilde{R}^l_{ijk} + \tilde{R}^l_{ijk} + \tilde{R}^l_{ijk} - \tilde{R}^l_{ijk}) \\
& + \frac{r(0)}{n(n-1)}(-2\tilde{R}^l_{ijk} + 2\tilde{R}^l_{jki} - 2g^l_i\tilde{R}_{jk} - 2\tilde{R}^l_{ikj} + 2g^l_j\tilde{R}_{ik} + g_{jk}\tilde{R}^l_i \\
& - g^l_j\tilde{R}_{ki} - g_{ik}\tilde{R}^l_j + g^l_i\tilde{R}_{kj} + g^l_i\tilde{R}_{jk} - g^l_j\tilde{R}_{ik} - g_{jk}\tilde{R}^l_i + g_{ik}\tilde{R}^l_j) \\
& + \left(\frac{r(0)}{n(n-1)}\right)^2(-2g^l_{ijk} - 2(n-1)g_{jk}g^l_i - 2g^l_{kij} + 2(n-1)g_{ik}g^l_j + 2g^l_{kji}) \\
& + \frac{r(0)^2}{n^2(n-1)}(2g^l_{ijk}).
\end{aligned}
$$

where we have substituted the definition $g^l_{ijk} = g_{jk}g^l_i - g_{ik}g^l_j$ and where

$$
\begin{aligned}
\tilde{Q}^l_{ijk} = {} & \tilde{R}^{pq}_{ij}\tilde{R}^l_{pqk} + 2\tilde{R}^{pl}_{iq}\tilde{R}^q_{pjk} - 2\tilde{R}^{pl}_{jq}\tilde{R}^q_{pik} \\
& + \tilde{R}^l_{pjk}\tilde{R}^p_i + \tilde{R}^l_{ipk}\tilde{R}^p_j + \tilde{R}^l_{ijp}\tilde{R}^p_k - \tilde{R}^p_{ijk}\tilde{R}^l_p
\end{aligned}
\tag{2.09}
$$

is quadratic in the reduced curvature $\widetilde{Rm} = \tilde{R}^l_{ijk}$.

Using now the first Bianchi identity and collecting terms we obtain:

$$Q^l_{ijk} = \tilde{Q}^l_{ijk} + \frac{2r(0)}{n}\tilde{R}^l_{ijk} - \frac{2r(0)}{n(n-1)}(g_{jk}\tilde{R}^l_i - g_{ik}\tilde{R}^l_j). \tag{2.10}$$

We note that the terms which are quadratic in the scalar curvature and g cancel away nicely. Since

$$\begin{aligned}\frac{\partial}{\partial t}g^l_{ijk} &= -2\tilde{R}_{jk}g^l_i + 2\tilde{R}_{ik}g^l_j \\ &= -2z_{jk}g^l_i + 2z_{ik}g^l_j - 2\frac{\tilde{R}}{n}g^l_{ijk},\end{aligned} \tag{2.11}$$

we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \bar{\Delta}\right)\tilde{R}^l_{ijk} &= \left(\frac{\partial}{\partial t} + \bar{\Delta}\right)R^l_{ijk} - \frac{r(0)}{n(n-1)}\frac{\partial}{\partial t}g^l_{ijk} \\ &= -\tilde{Q}^l_{ijk} - \frac{2r(0)}{n}\tilde{R}^l_{ijk} \\ &\quad + \frac{2r(0)}{n(n-1)}(g_{jk}\tilde{R}^l_i - g_{ik}\tilde{R}^l_j + \tilde{R}_{jk}g^l_i - \tilde{R}_{ik}g^l_j) \\ &= -\tilde{Q}^l_{ijk} + \frac{2r(0)}{n(n-1)}\left(\frac{\tilde{R}}{n}g^l_{ijk} - Z^l_{ijk} - (n-1)W^l_{ijk}\right) \\ &= -\tilde{Q}^l_{ijk} + \frac{2r(0)}{n}\left(\tilde{R}^l_{ijk} - \frac{n}{n-1}Z^l_{ijk} - 2W^l_{ijk}\right) \quad \text{using (2.04)}\end{aligned}$$

The evolution equation satisfied by the reduced curvature $\widetilde{Rm}$ is therefore:

$$\left(\frac{\partial}{\partial t} + \bar{\Delta}\right)\tilde{R}^l_{ijk} = -\tilde{Q}^l_{ijk} + \frac{2r(0)}{n}\left(\tilde{R}^l_{ijk} - \frac{n}{n-1}Z^l_{ijk} - 2W^l_{ijk}\right) \tag{2.12}$$

Taking now the trace with respect to i and l in (2.12) we find that $\widetilde{Rc}$ satisfies:

$$\left(\frac{\partial}{\partial t} + \bar{\Delta}\right)\tilde{R}_{ij} = -\tilde{q}_{ij} + \frac{2r(0)}{n}\left(\tilde{R}_{ij} - \frac{n}{n-1}z_{ij}\right) \tag{2.13}$$

where

$$\tilde{q}_{ij}\tilde{Q}^k_{kij} = 2\tilde{R}^q_{ipj}\tilde{R}^p_q + \tilde{R}_{pi}\tilde{R}^p_j + \tilde{R}_{pj}\tilde{R}^p_i$$

is a trace free symmetric tensor. From this it follows that the scalar curvature satisfies:

$$\begin{aligned}\frac{\partial}{\partial t}R + \Delta R = \frac{\partial}{\partial t}\tilde{R} + \Delta\tilde{R} &= 2\,|\widetilde{Rc}|^2 + \frac{2}{n}r(0)\tilde{R} \\ &= 2\,|Rc|^2 - \frac{2}{n}r(0)R \\ &= 2\,|z|^2 + \frac{2}{n}R\tilde{R}. \qquad (2.14)\end{aligned}$$

Taking the scalar product of (2.12) with $\widetilde{Rm}$, we get:

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right)|\widetilde{Rm}|^2 + |\nabla\widetilde{Rm}|^2 + \langle\tilde{Q}, \widetilde{Rm}\rangle \\ &= \frac{2r(0)}{n}\left(|\widetilde{Rm}|^2 - \frac{n}{n-1}|Z|^2 - 2\,|W|^2\right) + \langle\widetilde{Rc} * \widetilde{Rm}, \widetilde{Rm}\rangle, \qquad (2.15)\end{aligned}$$

where the last term $(\widetilde{Rc} * \widetilde{Rm})^l_{ijk} = (g^p_i\tilde{R}^q_j + \tilde{R}^p_i g^q_j)\tilde{R}^l_{pqk}$ arises from the fact that we also have to differentiate the norm we use to measure the curvature.

Since the terms $\langle\tilde{Q}, \widetilde{Rm}\rangle$ and $\langle\widetilde{Rc} * \widetilde{Rm}, \widetilde{Rm}\rangle$ are cubic in the reduced curvature $\widetilde{Rm}$, we have therefore the parabolic inequalities:

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right)|\widetilde{Rm}|^2 + |\nabla\widetilde{Rm}|^2 \le \frac{2r(0)}{n}|\widetilde{Rm}|^2 + c(n)\,|\widetilde{Rm}|^3 \quad \text{if } r(0) > 0 \qquad (2.16)$$

and

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right)|\widetilde{Rm}|^2 + |\nabla\widetilde{Rm}|^2 &\le -\frac{2r(0)}{n}\left(\frac{n}{n-1}|Z|^2 + 2\,|W|^2\right) + c(n)\,|\widetilde{Rm}|^3 \\ &\le -2r(0)\,|\widetilde{Rm}|^2 + c(n)\,|\widetilde{Rm}|^3 \quad \text{in case } r(0) < 0. \qquad (2.17)\end{aligned}$$

For the case $r(0) = 0$, we can refer directly to the evolution equation (2.07) of the total curvature and deduce that:

$$\frac{1}{2}\left(\frac{\partial}{\partial t}+\Delta\right)|Rm|^2+|\nabla Rm|^2 \leq c(n)\,|Rm|^3 \tag{2.18}$$

Integration of the above inequalties (2.16) and (2.17) gives us:

$$\frac{1}{2}\frac{\partial}{\partial t}\int|\widetilde{Rm}|^2+\int|\nabla\widetilde{Rm}|^2 \leq \frac{2\,|r(0)|}{n}\int|\widetilde{Rm}|^2+c(n)\int|\widetilde{Rm}|^3-\frac{1}{2}\int\tilde{R}\,|\widetilde{Rm}|^2, \tag{2.19}$$

where the last term arises from differentiating the volume form μ.

The above estimate still holds for the case $r(0)=0$, i.e.,

$$\frac{1}{2}\frac{\partial}{\partial t}\int|Rm|^2+\int|\nabla Rm|^2 \leq c(n)\int|Rm|^3-\frac{1}{2}\int\tilde{R}\,|Rm|^2. \tag{2.20}$$

Finally, the following evolution equations for the reduced Ricci curvature were derived in [12]. ($\widetilde{Rc}=h$ in the notation of that paper).

$$\left(\frac{\partial}{\partial t}+\tilde{\Delta}\right)\tilde{R}_{ij}=\frac{2}{n}r(0)\tilde{R}_{ij}-q_{ij}, \tag{2.21}$$

where

$$\begin{aligned} q_{ij} &= Q^k_{kij} \\ &= 2R^q_{ipj}R^p_q+R_{pi}R^p_j+R_{pi}R^p_i \\ &= 2R^q_{ipj}z^p_q+R_{pi}z^p_j+R_{pj}z^p_i, \end{aligned}$$

and hence

$$\frac{1}{2}\left(\frac{\partial}{\partial t}+\Delta\right)|\widetilde{Rc}|^2+|\nabla\widetilde{Rc}|^2 \leq c(n)\,|Rm|\,|\widetilde{Rc}|^2, \tag{2.22}$$

$$\frac{1}{2}\frac{d}{dt}\int|\widetilde{Rc}|^2 \leq \frac{2}{n}r(0)\int|\widetilde{Rc}|^2+c(n)\int|Rm|\,|\widetilde{Rc}|^2. \tag{2.23}$$

§3. Moser iteration

We begin by normalizing the initial metric $g(0)$ at $t=0$. We assume:

$$\max|K(0)|=1, \qquad d(0)\leq\Lambda \tag{3.01}$$

This implies that $|r(0)| \leq n(n-1)$ and $|Rm(0)| \leq c(n)$, where from now on, by abuse of notation, $c(n)$ will denote any constant depending only on the dimension.

Since $|Rm|^2$ satisfies a parabolic inequality (2.18) with a cubic non-linearity in the zero$^{\text{th}}$ order terms, the usual maximum principle shows that there exists a universal time $T = T(n) > 0$, depending only on the dimension n, such that:

$$\max |K(t)| \leq 2 \quad \text{for all } t \in [0, 3T] \tag{3.02}$$

This implies $\max |Rm(t)| \leq c(n)$ and hence also $\max |\widetilde{Rc}(t)| \leq c(n)$ for $0 \leq t \leq 3T$. It follows that all the metrics $g(t)$, $t \in [0, 3T]$ are uniformly bounded. This is because the change in the metric satisfies:

$$\max_{|v|=1} \left| \frac{d}{dt} \log g_t(v, v) \right| \leq 2 \max_{|v|=1} |\widetilde{Rc}(t)| \leq c(n)$$

and hence for $v \neq 0$

$$\exp(-c(n)T) \leq \frac{g_t(v, v)}{g_0(v, v)} \leq \exp(c(n)T),$$

which also gives a volume estimate:

$$c(n, T)^{-1} \leq \frac{V(t)}{V(0)} \leq c(n, T) \quad \text{for all } t \in [0, 3T].$$

Our assumptions on the initial curvature in Theorems 1, 2 and 3 are:

$$\begin{aligned} &\text{(i)} \quad ⨍ |\widetilde{Rm}(0)|^2 \leq r(0)^2 \epsilon \\ &\text{(ii)} \quad ⨍ |Rm(0)|^2 \leq d^{-4} \epsilon \\ &\text{(iii)} \quad ⨍ |\widetilde{Rc}(0)|^2 \leq r(0)^2 \epsilon, \qquad r(0) < 0 \end{aligned} \tag{3.03}$$

where $⨍$ denotes the average value.

Using the notation $\| \ \|_2$ for the L^2-norm, we have from (2.19), (2.20), (2.23), and the uniform bound on $|Rm|$ in $[0, 3\mathrm{T}]$ the following inequality:

$$\frac{d}{dt} \|T(t)\|_2^2 \leq c(n) \, \|T(t)\|_2^2 \quad \text{for } t \in [0, 3T] \tag{3.04}$$

for any of the tensors $T = \widetilde{Rm}$, Rm, or $\widetilde{Rc}$. This implies the following estimate for their L^2-norms:

$$\|T(t)\|_2^2 \le \|T(0)\|_2^2 \exp(c(n)t) \quad \text{for } t \in [0, 3T] \tag{3.05}$$

In order to proceed to a C^0-estimate we use the pointwise inequality:

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right)|T|^2 + |\nabla T|^2 \le c(n)\,|Rm|\,|T|^2,$$

which by (2.16), (2.17), (2.18), and (2.22) holds for all the curvature tensors we are interested in.

Applying the Cauchy-Schwarz inequality, dividing through by $|T|$, and using the uniform bound on $|Rm|$, we obtain the *linear* parabolic inequality:

$$\left(\frac{\partial}{\partial t} + \Delta\right)|T| \le c(n)\,|T| \quad \text{for } t \in [0, 3T] \tag{3.06}$$

where we interpret the inequality in the weak sense at the points where $T = 0$.

We will now apply the Moser iteration technique to the above inequality (3.06) to obtain C^0-estimates for all the quantities $|\widetilde{Rm}|$, $|Rm|$, and $|\widetilde{Rc}|$. We will show that after some short time the C^0-norms are controlled by the average L^2-norms up to a constant depending only on the dimension n and the constant Λ. Since this estimate is a basic ingredient of this paper, we will prove a general Lemma about the Moser iteration technique on a compact manifold.

LEMMA. *Let M be a compact manifold and let $g(t)$ be a smooth one parameter family of Riemannian metrics for $t \in [0, 3T]$ with $T < \infty$ and suppose that for some constant $B \ge 0$, we have a uniform estimate*:

$$\max_{|v|=1}\left|\frac{d}{dt} g_t(v, v)\right| \le B \quad \text{for } t \in [0, 3T], \tag{3.07}$$

where the norm used is with respect to the metric at time t.

Assume further that there exists $t' \in [0, 3T]$ such that the diameter d and the Ricci curvature of the metric $g(t')$ at time t' satisfies the estimate:

$$d^2 \min_{|v|=1} Ric(v, v) \ge -(n-1)H^2 \quad \textit{for some } H \ge 0. \tag{3.08}$$

Let $u: [0, 3T] \times M \to [0, \infty]$ *be a non-negative function with square integrable first derivatives satisfying the parabolic inequality*:

$$\frac{\partial}{\partial t} u + \Delta u \leq A u \tag{*}$$

in the weak sense, where Δ *is the time dependent Laplacian with respect to the metric* $g(t)$ *at time* t, *and* A *is a constant.*

Then, there exists a constant $c(n, T, A, B, H)$ *depending only on the arguments indicated such that the following estimate holds*:

$$\underset{(t,x)\in[2T,3T]\times M}{\text{maximum}} |u(t, x)|^2 \leq C(n, T, A, B, H) d^n \fint_T^{3T} \fint_M |u(t, x)|^2 \, dx \, dt \tag{**}$$

where d *is the minimum diameter of all the metrics* $\{g(t) \mid t \in [0, 3T]\}$, $\fint$ *denotes the average integral, and maximum stands for the essential maximum.*

We give a proof of the above Lemma following closely Moser's original paper [13]. The only technical point we have to take care of is the fact that the metric and hence the volume form we are using is changing with time. First the assumption (3.07) implies:

$$\max_{|v|=1} \left| \frac{d}{dt} \log g_t(v, v) \right| \leq B$$

and hence

$$c(T, B)^{-1} \leq \frac{g_{t_1}(v, v)}{g_{t_2}(v, v)} \leq c(T, B) \quad \text{for all } t_1, t_2 \in [0, 3T] \text{ and } v \neq 0. \tag{3.09}$$

This implies in particular that the diameters and the volumes of all the metrics are equivalent:

$$\text{(i)} \quad c(T, B)^{-1} \leq \frac{\text{diam}\,(g(t_1))}{\text{diam}\,(g(t_2))} \leq c(T, B) \tag{3.10}$$

and

$$\text{(ii)} \quad c(n, T, B)^{-1} \leq \frac{V(t_1)}{V(t_2)} \leq c(n, T, B) \quad \text{for all } t_1, t_2 \in [0, 3T].$$

For the rate of change of the volume form μ we have

$$\left|\left(\frac{\partial}{\partial t}\mu\right)\mu^{-1}\right| \le c(n, B). \tag{3.11}$$

We also need to establish a uniform bound for the *Sobolev constant* C_{Sob} for the Riemannian manifold M, which appears in the Sobolev inequality:

$$\|u\|_{2m}^2 \le C_{\mathrm{Sob}}(\|du\|_2^2 + \|u\|_2^2) \quad \text{where } m = \frac{n}{n-2}. \tag{3.12}$$

It is well known that the best constants in Sobolev inequalities are determined by the *isoperimetric constant* defined by:

$$C_{\mathrm{iso}} = \inf\{(\mathrm{vol}\,\partial D)^n/(\mathrm{vol}\,D)^{n-1}\}, \tag{3.13}$$

where the infinum is taken over all (not necessarily connected) open submanifolds $D^n \subset M^n$ with smooth boundary ∂D^{n-1} and with $2\,\mathrm{vol}\,(D) \le \mathrm{vol}\,(M)$.

The precise relation of C_{iso} with the optimal C_{Sob} appearing in (3.12) is then: (see for example [2], [10]):

$$C_{\mathrm{Sob}} = c(n)C_{\mathrm{iso}}^{-2/n}. \tag{3.14}$$

By its very definition, C_{iso} is a C^0-invariant of the metric and by our information Lipschitz estimate (3.09) for the C^0-norms of the metrics we have

$$C(n, B, T)C_{\mathrm{iso}}(t'') \ge C_{\mathrm{iso}}(t') \ge C(n, B, T)^{-1}C_{\mathrm{iso}}(t'') \quad \text{for } t', t'' \in [0, 3T]. \tag{3.15}$$

Now by results due to S. Gallot [2] (see also [1] and [3]), which are based on an isoperimetric inequality of M. Gromov [7], we know that $C_{\mathrm{iso}}V^{-1}$ can be bounded from below by a constant depending only on an upper bound for the diameter and a lower bound for the Ricci curvature. More explicitly, under the assumption (3.08) we have, according to [2(I), Theorem 1.1], the following estimate:

$$C_{\mathrm{iso}} \ge V d^{-n} I(n, H) \tag{3.16}$$

where

$$I(n, H)^{-1} = \frac{1}{H}\int_0^H \left(\mathscr{C}\mathscr{H}(H)\cosh t + \frac{1}{nH}\sinh t\right)^{n-1} dt$$

with

$$\mathscr{C}\mathscr{H}(H) = H^{-1}\int_0^{H/2} (\cosh s)^{n-1}\, ds \quad \text{in case } H > 0$$

and if $H = 0$, we set $I(n, 0) = 2^{n-1}$.

Therefore, by (3.14):

$$\underset{t\in[0,3T]}{\text{maximum}}\ C_{\text{Sob}}(t)V(t)^{2/n}\, d(t)^{-2} \leq c(n, B, T, H). \tag{3.17}$$

If $u \geq 0$ is a sub solution of $(*)$ then for any $q > 0$, we compute:

$$\left(\frac{\partial}{\partial t} + \Delta\right)u^q = qu^{q-1}\left(\frac{\partial}{\partial t} + \Delta\right)u - q(q-1)u^{q-2}\,|du|^2 \leq Aqu^q.$$

By setting: $v_0 = u$ and $v_{k+1} = v_k^p$, for $k = 0, 1, \ldots$ with $p = \dfrac{n+2}{n}$ we obtain:

$$\left(\frac{\partial}{\partial t} + \Delta\right)v_k \leq Ap^k \cdot v_k \quad \text{for } k = 0, 1, \ldots, \infty, \tag{3.18}$$

which shows that the powers u^{p^k} also satisfy $(*)$ except that the constant A has to be replaced by $A_k = Ap^k$.

If $\chi(t)$ is a function of t alone, and if v satisfies $(*)$ with a constant A', then we have:

$$\begin{aligned}
&\frac{d}{dt}(\chi^2\|v\|_2^2) + 2\chi^2\,\|dv\|_2^2 - 2\chi\dot{\chi}\,\|v\|_2^2 \\
&\quad = 2\chi^2\int(v\dot{v} + v\,\Delta v) + \chi^2\int v^2\dot{\mu} \\
&\quad \leq (2A' + c(n, B))\chi^2\,\|v\|_2^2 \quad \text{by (3.11).} \\
&\quad \leq 4A'\chi^2\,\|v\|_2^2
\end{aligned} \tag{3.19}$$

where we assume, without loss of generality that $2A' \geq c(n, B)$.

For any $0 < T \leq t_k < t_k + \tau_k = t_{k+1} \leq 2T$, we choose a cut-off function $\chi(t)$ satisfying: $\chi \equiv 0$ on $[0, t_k]$, $\chi \equiv 1$ on $[t_{k+1}, \infty]$ and $0 \leq \dot{\chi} < 2\tau_k^{-1}$.

By integrating inequality (3.19) over the interval $[t_k, 3T]$, and neglecting the first term, which is nonnegative, we obtain the following energy estimate:

$$\int_{t_{k+1}}^{3T} \|dv(t)\|_2^2 \leq c(n, B)(A' + \tau_k^{-1}) \int_{t_k}^{3T} \|v(t)\|_2^2 . \tag{3.20}$$

On the other hand, by integrating on $[t_k, \bar{t}]$ where $\bar{t} \in [t_{k+1}, 3T]$ is chosen to be such that:

$$\underset{t_{k+1} \leq t \leq 3T}{\text{maximum}} \|v(t)\|_2^2 \leq 2 \|v(\bar{t})\|_2^2$$

and neglecting the non-negative energy term, we have

$$\underset{t_{k+1} \leq t \leq 2T}{\text{maximum}} \|v(t)\|_2^2 \leq 4(A' + \tau_k^{-1}) \int_{t_k}^{3T} \|v(t)\|_2^2 . \tag{3.21}$$

The Sobolev inequality (3.12) implies

$$\left(\fint v^{2m}\right)^{1/m} \leq C_{\text{Sob}} V^{2/n} \fint (|dv|^2 + v^2)$$

and by the Hölder inequality:

$$\fint v^{2p} \leq \left(\fint v^{2m}\right)^{1/m} \left(\fint v^2\right)^{2/n}, \quad \text{since} \quad \frac{1}{m} + \frac{2}{n} = 1 \quad \text{and} \quad p = \frac{2}{n} + 1.$$

Combining them we have

$$\fint v^{2p} \leq C_{\text{Sob}} V^{2/n} \left(\fint v^2\right)^{2/n} \fint (|dv|^2 + v^2)$$

and hence by integration with respect to t, we get

$$\int_{t_{k+1}}^{3T} \fint_M v^{2p} \leq \underset{t \in [t_{k+1}, 3T]}{\text{maximum}} \left\{ C_{\text{Sob}}(t) V(t)^{2/n} \left(\fint_M v^2\right)^{2/n} \right\} \int_{t_k}^{3T} \fint_M (|dv|^2 + v^2). \tag{3.22}$$

Substituting now the estimates (3.20) and (3.21) and using the fact that we have uniform estimates (3.10), and (3.15) for the volume $V(t)$, diameters $d(t)$

and the Sobolev constants of all the metrics $g(t)$ we get

$$\int_{t_{k+1}}^{3T} \int_M v^{2p} \leq c(n, B, T, H)\, d^2\left((A' + \tau_k^{-1}) \int_{t_k}^{3T} \int_M v^2\right)^p, \tag{3.23}$$

where d is the diameter of any of the metrics $g(t)$ for $t \in [0, 3T]$.

Choosing now a partition: $T = t_0 < \cdots < t_k = T(2 - p^{-k}) < \cdots < t_\infty = 2T$ of $[T, 2T]$ such that

$$\tau_k = t_{k+1} - t_k = \frac{2}{n+2} T p^{-k}$$

and applying (3.23) to the inequality (3.18) satisfied by the powers v_k of u on the intervals $[t_k, 3T]$ we get:

$$\int_{t_{k+1}}^{3T} \int v_k^{2p} \leq c(n, T, B, H)\, d^2\left((A_k + \tau_k^{-1}) \int_{t_k}^{3T} \int v_k^2\right)^p,$$

i.e.,

$$\int_{t_{k+1}}^{3T} \int v_{k+1}^2 \leq c(n, T, B, H)\, d^2\left(A_k' \int_{t_k}^{3T} \int v_k^2\right)^p \tag{3.24}$$

with

$$A_k' = A_k + \tau_k^{-1} = Ap^k + \frac{n+2}{2T} p^k = \left(A + \frac{n+2}{2T}\right) p^k.$$

If we set $L(k) = (\int_{t_k}^{3T} \int v_k^2)^{p^{-k}}$, then (3.24) can be expressed as

$$\begin{aligned} L(k+1) &\leq L(0) \prod_{j=0}^{k} (c(n, T, B, H)\, d^2)^{p^{-j-1}} \left(A + \frac{n+2}{2T}\right)^{p^{-j}} p^{jp^{-j}} \\ &\leq c(n, T, A, B, H)\, d^n L(0) \end{aligned}$$

This proves Lemma 1, because

$$\lim_{k \to \infty} L(k) = \underset{(t,x) \in [2T, 3T] \times M}{\text{maximum}} |u(t, x)|^2.$$

Applying Lemma 1 to the linear parabolic inequality (3.06) satisfied by the curvature tensors $\widetilde{Rm}$, Rm, and $\widetilde{Rc}$ in the time interval $[0, 3T]$ chosen above, and using the L^2-estimate (3.05) now reduces the proofs of Theorems 1, 2 and 3 respectively to their known C^0-versions [8], [5], [6], [14] and [12].

Finally, we check that al the constants appearing in this case depend only on the dimension n and the a-priori bound Λ:

(i) after the initial scaling (3.01) the time $T > 0$ was chosen to depend only on n;

(ii) the constant B of the assumption (3.17) is given by $|\widetilde{Rc}|$ and hence can be estimated by $c(n)\,|Rm| \le c(n)$ in the time interval $[0, 3T]$

(iii) the constant H of (3.08) can be estimated by Λ

(iv) the constant A is estimated by $c(n)\,|Rm| \le c(n)$ in the given time interval.

(v) the diameter $d(0)$ at time 0 is normalized to be $\le \Lambda$.

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Department of Mathematics
McMaster University
Hamilton, Ontario
Canada L8S 4K1

Department of Mathematics
The Ohio State University
Columbus, Ohio 43210
USA
and
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel
Switzerland

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