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On group homomorphisms inducing mod-p cohomology isomorphisms

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Let $\rho : F_1 \to F_2$ be a homomorphism of finite groups F_1 and F_2 inducing an isomorphism $H^*(F_2; Z/p) \to H^*(F_1; Z/p)$, p a fixed prime. By a result of S. Jackowski [5] it is known that then

(i) $\ker(\rho)$ is of order prime to p,
(ii) $\operatorname{im}(\rho)$ has index prime to p.

Simple examples show that in general (i) and (ii) alone do not suffice for ρ to induce a Z/p-cohomology isomorphism. The purpose of this note is to describe necessary and sufficient conditions on ρ in group theoretic terms for ρ to induce an H^*Z/p-isomorphism. It turns out to be natural to work in the more general setting of compact Lie groups. The following notations and terminology will be used throughout this note.

For $\rho : G \to H$ a morphism of compact Lie groups we write

$$C(\rho) = \{h \in H \mid h\rho(g) = \rho(g)h \quad \text{for all } g \in G\}$$

for the centralizer of ρ,

$$N(\rho) = \{h \in H \mid h\rho(G) = \rho(G)h\}$$

for the normalizer of ρ, and

$$W(\rho) = N(\rho)/C(\rho)$$

for the Weyl group of ρ. Note that $W(\rho)$ is a compact Lie group. It is a finite group, if for instance $\rho(G)$ is a finite subgroup of H. In case $\rho : T \to G$ stands for the inclusion of a maximal torus into a compact connected Lie group, $W(\rho) = W(G)$, the classical Weyl group of G. As usual

$$\operatorname{Rep}(G, H)$$

stands for the representations of G in H, that is, the set of H-conjugacy classes of continuous homomorphisms $G \to H$. For p a prime we write

$$Q_p(G)$$

for the Quillen-category of finite p-subgroups of G; its objects are the finite p-subgroups of G, and morphisms $P_1 \to P_2$ are homomorphisms of the form $c^g : x \to g^{-1}xg$ for some $g \in G$.

Our theorem then takes the following form.

THEOREM. *Let $\rho : G \to H$ be a morphism of compact Lie groups and let p be a prime. Then the following are equivalent*:

(A) $H^*B\rho : H^*(BH; Z/p) \to H^*(BG; Z/p)$ *is an isomorphism.*

(B) $\operatorname{Rep}(\rho) : \operatorname{Rep}(\pi, G) \to \operatorname{Rep}(\pi, H)$ *is a bijection for every finite p-group π.*

(C) $Q_p(\rho) : Q_p(G) \to Q_p(H)$ *is an equivalence of categories.*

REMARK. The reader verifies easily that (B) implies

(Bi): $\ker(\rho)$ contains no element of order p.

(Bii): every finite p-subgroup in H is conjugate to a subgroup in $\rho(G) \subset H$.

These statements generalize (i) and (ii) above to the case of compact Lie groups.

Before proving the Theorem, we want to recall some basic facts on homotopy fixed-points. All spaces considered are supposed to be of the homotopy type of CW-complexes. If X denotes a π-space, π a group, one writes $X^{h\pi}$ for the homotopy fixed-point space of the π action on X. It is by definition equal to $\operatorname{map}_\pi(E\pi, X)$, the space of π-maps from the universal π-space $E\pi$ to X. The space of fixed-points X^π maps naturally to $X^{h\pi}$ and the induced map

$$(Z/p)_\infty(X^\pi) \to ((Z/p)_\infty X)^{h\pi}, \tag{1}$$

is known to be an equivalence, if π is a finite p-group and X a finite dimensional π-space (Theorem of Carlsson, Miller and Lannes, cf. [2]). The functor $(Z/p)_\infty(-)$ denotes the Bousfield–Kan Z/p-completion functor [1]. It has the basic property that it turns $H_*(\ ; Z/p)$-isomorphisms into homotopy equivalences. The following lemma is implicit in Carlsson's paper [2].

LEMMA 1. *Let X be a finite dimensional π-space, π a finite p-group. Then the natural map*

$$X^{h\pi} \to ((Z/p)_\infty X)^{h\pi}$$

induces a bijection of connected components.

Proof. From the equivalence (1) we see that

$$\pi_0((Z/p)_\infty X)^{h\pi} \cong \pi_0(X^\pi).$$

By [2, VI.12],

$$\pi_0(X^{h\pi}) \cong \pi_0(((Z/p)_\infty^{\mathrm{tot}} X)^\pi),$$

where the space $((Z/p)_\infty^{\mathrm{tot}} X)^\pi$ consists of a disjoint union of certain partial completions of the components of X^π (cf. [2, IV.3]). Therefore,

$$\pi_0(((Z/p)_\infty^{\mathrm{tot}} X)^\pi) \cong \pi_0(X^\pi)$$

and the lemma follows.

We will also need the following result which applies to arbitrary (not necessarily finite dimensional) spaces X.

LEMMA 2. *Suppose X is an i-connected π-space, π a finite p-group and $i \geq 2$. Then the canonical map*

$$\Theta : X^{h\pi} \to ((Z/p)_\infty X)^{h\pi}$$

induces a π_0-bijection, and isomorphisms

$$\pi_j(X^{h\pi}, x) \to \pi_j(((Z/p)_\infty X)^{h\pi}, \Theta x)$$

for $j < i$ and all $x \in X^{h\pi}$.

Proof. Since X is i-connected, $(Z/p)_\infty X$ is i-connected too and it follows that the fibre F of $X \to (Z/p)_\infty X$ is $(i-1)$-connected, with uniquely p-divisible homotopy groups. The (homotopy) fibre F_y of Θ over a point $y \in ((Z/p)_\infty X)^{h\pi}$ may be identified with $F^{h\pi}$ for some action of π on F. Since F is 1-connected mod-p acyclic, $F^{h\pi}$ is p-acyclic too [3, 2.3]. Thus F_y is non-empty and connected, which implies that Θ is a π_0-bijection. The obstruction theory spectral sequence

$$H^*(\pi, \underline{\pi_* F}) \Rightarrow \pi_*(F^{h\pi})$$

then collapses, because the groups $\pi_k F$ are all uniquely p-divisible, and it follows that

$$\pi_k(F^{h\pi}) \cong (\pi_k F)^\pi$$

for all k. In particular, $\pi_k(F^{h\pi}) = \pi_k(F_y) = 0$ for $k < i$ since F is $(i-1)$-connected. It follows then that Θ is a π_j-isomorphism for $j < i$.

Proof of the Theorem. (A) $\Rightarrow$ (B). By Dwyer–Zabrodsky [3] one has a natural bijection

$$\operatorname{Rep}(\pi, G) \to \pi_0 \operatorname{map}(B\pi, BG), \tag{2}$$

associating with a homomorphism $\varphi : \pi \to G$ the component of $\operatorname{map}(B\pi, BG)$ containing $B\varphi$; we denote that component by $\operatorname{map}(B\pi, BG)_\varphi$. As $B\rho : BG \to BH$ is an $H_*(\ ; Z/p)$-isomorphism, the induced map

$$\operatorname{map}(B\pi, (Z/p)_\infty BG) \to \operatorname{map}(B\pi, (Z/p)_\infty BH), \tag{3}$$

is an equivalence. Thus, to prove (B) it suffices to show that for a general compact Lie group G

$$\pi_0(\operatorname{map}(B\pi, BG)) \to \pi_0(\operatorname{map}(B\pi, (Z/p)_\infty BG)), \tag{4}$$

is a bijection. This is certainly so for $G = SU(n)$ as we see from Lemma 2 (trivial π-action on $BSU(n)$). In the general case we choose an embedding $\epsilon : G \to SU(n)$ for some n, and we look at the fibration

$$SU(n)/G \to BG \to BSU(n), \tag{5}$$

If we fix a map $\sigma : \pi \to SU(n)$ which factors through $G \subset SU(n)$, then we obtain a fibration sequence

$$Z \to \coprod_\alpha \operatorname{map}(B\pi, BG)_{\sigma_\alpha} \to \operatorname{map}(B\pi, BSU(n))_\sigma, \tag{6}$$

where $\sigma_\alpha : \pi \to G$ runs over all G-conjugacy classes for which $\epsilon\sigma_\alpha$ is $SU(n)$-conjugate to σ. We can identify Z with the space of sections of the fibration

$$SU(n)/G \to E\pi \underset{\pi}{\times} (SU(n)/G) \to B\pi$$

which is obtained by pulling back (5) along $B\sigma : B\pi \to BSU(n)$. As a result

$$Z \cong \operatorname{map}_\pi(E\pi, SU(n)/G) = (SU(n)/G)^{h\pi}$$

where π acts on $SU(n)/G$ via σ. Since $BSU(n)$ is simply connected, $(Z/p)_\infty(-)$ turns

(5) into a fibration sequence

$$(Z/p)_\infty(SU(n)/G) \to (Z/p)_\infty BG \to (Z/p)_\infty BSU(n), \tag{7}$$

which will give rise, as before, to a fibration

$$(((Z/p)_\infty(SU(n)/G))^{h\pi} \to \operatorname{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)} \to \operatorname{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma, \tag{8}$$

where map $(B\pi, (Z/p)_\infty BG)_{R(\sigma)}$ denotes the disjoint union of those connected components of map $(B\pi, (Z/p)_\infty BG)$ which map to map $(B\pi, (Z/p)_\infty BSU(n))_\sigma$, the component of $(Z/p)_\infty B\sigma$ of map $(B\pi, (Z/p)_\infty BSU(n))$. To ensure that the map in (4) is bijective it obviously suffices to check that

$$\pi_0\left(\coprod_\alpha \operatorname{map}(B\pi, (Z/p)_\infty BG)_{\sigma_\alpha}\right) \to \pi_0(\operatorname{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)}), \tag{9}$$

is bijective for every $\sigma : \pi \to SU(n)$ which factors through $G \subset SU(n)$. For this, consider the natural map of the fibration (6) to that of (8). Because

$$\pi_1 \operatorname{map}(B\pi, BSU(n))_\sigma \to \pi_1 \operatorname{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma$$

is an isomorphism (Lemma 2) we see that (9) is a bijection, if the map on fibres

$$\pi_0(SU(n)/G)^{h\pi} \to \pi_0((Z/p)_\infty(SU(n)/G))^{h\pi}$$

is a bijection. But this is the case by Lemma 1.

(B) $\Rightarrow$ (C). We first check that $Q_p(\rho)$ induces a bijection on isomorphism classes of objects. Let A, B be finite p-subgroups of G with $\rho(A)$ and $\rho(B)$ isomorphic as objects of $Q_p(H)$ so that there exists an $h \in H$ with $c^h : \rho(A) \to \rho(B)$ a group isomorphism. Note that $A \to \rho(A)$ is injective in view of (B). Thus, there is a group isomorphism $\Theta : A \to B$ rendering the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \rho(A) \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle c^h} \\ B & \xrightarrow{\rho} & \rho(B) \end{array}$$

commutative; we will show that $\Theta = c^g$ for some $g \in G$, proving that A is isomorphic to B in $Q_p(G)$. Namely, because the bijection

$$\mathrm{Rep}\,(\rho) : \mathrm{Rep}\,(A, G) \to \mathrm{Rep}\,(A, H)$$

maps the class of $\tilde{\Theta} : A \to G$, $(x \to \Theta x)$, to $\rho\tilde{\Theta} = c^h\rho : A \to H$, which is the same as the image under Rep (ρ) of the inclusion $A \subset G$, we infer that $\tilde{\Theta}$ is G-conjugate to this inclusion; thus $\Theta = c^g : A \to B$ for some $g \in G$. This shows that $Q_p(G) \to Q_p(H)$ is one–one on isomorphism classes of objects. Actually, the same argument shows that $Q_p(\rho)$ is full: for any objects $A, B \in Q_p(G)$, the induced map of Q_p-morphisms

$$\mathrm{Mor}\,(A, B) \to \mathrm{Mor}\,(\rho(A), \rho(B))$$

is surjective.

If P is any finite p-subgroup of H, we apply (B) with $\pi = P$ to infer a commutative diagram

$$\begin{tikzcd} & G \arrow[d, "\rho"] \\ P \arrow[ur, "f"] \arrow[r, "c^h"'] & H \end{tikzcd}$$

Thus $P \in Q_p(H)$ is isomorphic, as object of $Q_p(H)$, to $\rho(fP)$, showing that $Q_p(G) \to Q_p(H)$ is onto on isomorphism classes of objects.

It remains to check that $Q_p(\rho)$ is faithful, i.e., that for any $A, B \in Q_p(G)$

$$\mathrm{Mor}\,(A, B) \to \mathrm{Mor}\,(\rho A, \rho B)$$

is injective. But this is obvious because Mor $(A, B) \subset$ Hom (A, B), Mor $(\rho A, \rho B) \subset$ Hom$(\rho A, \rho B)$ and $\rho : B \to \rho(B)$ is a group isomorphism as observed earlier.

(C) $\Rightarrow$ (A). Define a cofunctor $F : Q_p(G) \to Ab$ by mapping P to $H^*(BP; Z/p)$. The natural map

$$\mathrm{Res} : H^*(BG; Z/p) \to \varprojlim F$$

is then an isomorphism. In the case of a finite group G this follows from the classical result describing $H^*(BG; Z/p)$ in terms of the stable elements in the cohomology of a p-Sylow subgroup of G; the general case was dealt with in [4, Theorem 2.3]. The implication (C) $\Rightarrow$ (A) is then plain.

The next result is an immediate consequence of the Theorem. It relates Weyl-groups of maps with group cohomology.

COROLLARY 1. *Let $\rho : G \to H$ be a map of compact Lie groups inducing an isomorphism $H^*(BH; Z/p) \to H^*(BG, Z/p)$. Then for every homomorphism $\varphi : \pi \to G$ with π a finite p-group, the induced map of Weyl-groups*

$$\rho_* : W(\varphi) \to W(\rho\varphi)$$

is a group isomorphism.

Proof. Note that $W(\varphi)$ is the automorphism group of the object $\varphi(\pi) \in Q_p(G)$; similarly for $W(\pi\varphi)$. Thus part (C) of the theorem shows that the natural map $W(\varphi) \to W(\rho\varphi)$ is an isomorphism.

It seems surprising that Z/p-cohomology information can contain such precise information on Weyl-groups, which are in general not p-groups. The following application shall illustrate this; as a variation of the theme we use rational cohomology information as input.

COROLLARY 2. *Let $\rho : G \to H$ be a map of connected compact Lie groups inducing an isomorphism*

$$H^*(BH; Q) \to H^*(BG; Q).$$

Then ρ induces an isomorphism of Weyl-groups $W(G) \to W(H)$.

Proof. Choose a prime p large enough such that $H^*B\rho : H^*(BH; Z/p) \to H^*(BG; Z/p)$ is an isomorphism (any prime which does not divide the order of the kernel and cokernel of the map $H_*(G; Z) \to H_*(H; Z)$ will do). Clearly, G and H have the same rank and, because in addition there is no element of order p in the kernel of ρ, ρ maps a maximal torus $T(G) \subset G$ onto a maximal torus $\rho T(G) = T(H) \subset H$. The union of the finite p-subgroups is dense in $T(G)$ and $T(H)$. As a result, we can find a finite p-subgroup $\pi \subset T(G)$ with centralizer $C(\pi) = C(T(G)) = T(G)$, and $C(\rho\pi) = C(T(H)) = T(H)$; here we used the fact that in a compact Lie group closed subgroups satisfy the descending chain condition and that in a connected compact Lie group, a maximal torus is its own centralizer. Similarly, we may assume that the normalizer of π satisfies $N(\pi) = N(T(G))$, and $N(\rho\pi) = N(T(H))$. Then it follows that the induced map of Weyl-groups $W(G) \to W(H)$ is an isomorphism as one sees by applying the previous Corollary to the given map $\rho : G \to H$ and the inclusion map $\varphi : \pi \to G$.

Of course, this corollary could also be proved in a more conventional way by observing that the hypothesis implies that $\rho : G \to H$ induces an isomorphism of associated Lie algebras.

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