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# Complete minimal surfaces with index one and stable constant mean curvature surfaces

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## Introduction

Minimal surfaces and surfaces with constant mean curvature in the Euclidean space $R^3$ are critical points of variational problems. In the first case the surfaces have critical area for all variations with compact support. In the second one their area is critical for compactly supported variations that leave constant the signed volume "enclosed" by the surface.

So problems on stability and index of their Jacobi operator arises naturally. Do Carmo, Peng [2], Fischer-Colbrie, Schoen [4], and Pogorelov [11], have proved independently that the plane is the only stable complete minimal surface in $R^3$. Quite recently Fischer-Colbrie [3], and Gulliver, Lawson [6], [5] showed that a complete orientable minimal surface has finite index (in the sense that the index of the Jacobi operator on every bounded domain has a uniform upper bound) if and only if it has finite total curvature. The study of the stability for surfaces with constant mean curvature was made, in the compact case, by Barbosa and do Carmo [1]. They proved that the sphere is the only compact hypersurface immersed in a Euclidean space with constant mean curvature, which is stable.

In this paper we first prove that:

"*The Catenoid and the Enneper's surface are the only orientable complete minimal surfaces in $R^3$ with index equal to one*".

This gives a new characterization of these famous examples.

Our second result gives a complete solution to the global stability problem in the constant mean curvature case. Concretely we will prove the following fact:

"*The only orientable complete surfaces with constant mean curvature in $R^3$ which are stable, are the plane and the sphere*".

We also observe that a complete surface with constant mean curvature has finite index if and only if it is compact or a minimal surface with finite total curvature (examples of compact surfaces other than spheres are constructed by Wente [12]).

Finally we prove two properties of the Gauss map for complete surfaces in $R^3$ with non-zero constant mean curvature, which generalize a result of Hoffman, Osserman and Schoen [9].

The methods that we use in this paper are based, for the most part, on Fischer-Colbrie's work [3].

## Preliminaries

Let $(M, ds^2)$ be a complete Riemannian surface. We denote by $K$, $\Delta$ and $dA$ the Gaussian curvature, the Laplacian and the canonical measure corresponding to the metric $ds^2$. Given a smooth function $q : M \to R$ we consider on $M$ the operator $L = \Delta + q$. We denote by $Q$ the quadratic form associated to $L$, that is

$$Q(u, u) = -\int_M (Lu)u\, dA = \int_M [|\nabla u|^2 - qu^2]\, dA,$$

where $u$ has compact support on $M$. For any bounded domain $\Omega$ in $M$, we denote by $\operatorname{Ind}(L, \Omega)$ the index of $Q$ restricted to $H^0_{1,2}(\Omega)$. By definition, the index of $L$, $\operatorname{Ind}(L)$, is the supremum of the numbers $\operatorname{Ind}(L, \Omega)$. Given a minimal isometric immersion of $M$ in $R^3$, $\phi : M \to R^3$, we consider on $M$ the Jacobi operator $L = \Delta + |\sigma|^2$, $\sigma$ being the second fundamental form of the immersion. We define the index of $M$ as $\operatorname{Ind}(M) = \operatorname{Ind}(L)$. If $\operatorname{Ind}(M) = 0$ we say that $M$ is stable.

If $\phi : M \to R^3$ is a isometric immersion with constant mean curvature $H$, we consider also the Jacobi operator on $M$, $L = \Delta + |\sigma|^2$. Given a bounded domain $\Omega$ in $M$ we define the signed volume enclosed by $\Omega$ as

$$V(\Omega) = \tfrac{1}{3}\int_\Omega \langle \phi, N\rangle\, dA$$

where $N : M \to S^2(1)$ is the Gauss map of $M$.

As surfaces with constant mean curvature have critical area for variations which preserve the signed volume, the operator $L$ and the quadratic form $Q$, have geometric meaning only for functions $u \in H^0_{1,2}(\Omega)$ satisfying $\int_M u\, dA = 0$, see details in [1]. So we define a new index for $\Omega$, $\operatorname{Ind}_0(L, \Omega)$, as the index of $Q$ restricted to the space $\{u \in H^0_{1,2}(\Omega) / \int_\Omega u\, dA = 0\}$. We define the index of $M$, as surface with $H = \text{const.}$, by

$$\mathrm{Ind}_0(M) = \sup\{\mathrm{Ind}_0(L, \Omega)/\Omega \text{ bounded domain in } M\}.$$

Finally we say that $M$ is stable, as surface with $H = \text{const.}$, if $\mathrm{Ind}_0(M) = 0$. Observe that for minimal surfaces we have defined two different indices, $\mathrm{Ind}(M)$ and $\mathrm{Ind}_0(M)$.

## 1. Minimal surfaces with index 1

Let $M$ be an orientable complete minimal surface in $R^3$, and $N: M \to S^2(1)$ the Gauss map. If $\mathrm{Ind}(M)$ is finite then, from the result of Fischer-Colbrie [3], Theor. 2, $M$ has finite total curvature, and so there exist a compact Riemann surface $\Sigma$, such that $M$ is conformally equivalent to $\Sigma - \{p_1, p_2, \ldots, p_n\}$, $p_k \in \Sigma$, $k = 1, \ldots, n$. Moreover the Gauss map extends to $\{p_1, \ldots, p_n\}$ as a meromorphic map $N_1: \Sigma \to S^2(1)$, see Ossermann [10], Chapter 9.

Let $ds_1^2$ be a metric on $\Sigma$ compatible with the Riemann structure. We define the operator $L_1$ on $\Sigma$ by

$$L_1 = \Delta_1 + |\nabla_1 N_1|^2, \tag{1.1}$$

where $\Delta_1$ and $\nabla_1$ are the Laplacian and the gradient of the metric $ds_1^2$. From [3], Corol. 2 we have

$$\mathrm{Ind}(L_1) = \mathrm{Ind}(M) \tag{1.2}$$

we denote by $Q_1$ the quadratic form associated to $L_1$, i.e.

$$Q_1(u, u) = -\int_\Sigma u L_1 u \, dA_1, \quad \text{for all} \quad u \in H_{1,2}(\Sigma)$$

Finally we will need the following slight extension of a result of Hersch [7].

LEMMA 1. *Let $\rho: \Sigma \to R$ be a positive smooth function and $\Psi: \Sigma \to S^2(1)$ a non-constant meromorphic map. Then there exists a conformal transformation $g: S^2(1) \to S^2(1))$ such that*

$$\int_\Sigma \rho(g \circ \Psi) \, dA_1 = 0,$$

*where $dA_1$ is the canonical measure associated to $ds_1^2$.*

THEOREM 2. *Let $M$ be a complete orientable minimal surface in $R^3$. Then* $\operatorname{Ind}(M)=1$ *if and only if $M$ is either the Catenoid or the Enneper's surface.*

*Proof.* If $M$ is the Catenoid or the Enneper's surface it is proved in [3], p. 132 that $\operatorname{Ind}(M)=1$.

Suppose now $\operatorname{Ind}(M)=1$. Then by (1.2) we have $\operatorname{Ind}(L_1)=1$. Let $\rho$ be the first eigenfunction of $L_1$. Recall that $\rho$ is positive everywhere. By hypothesis for all $u \in C^1(\Sigma)$ such that $\int_\Sigma \rho u\, dA_1 = 0$, we have

$$Q_1(u, u) \geqq 0 \tag{1.3}$$

the equality holding if and only if $L_1 u = 0$.

Let $\Psi: \Sigma \to S^2(1)$ be a non-constant meromorphic map. Using Lemma 1 we have a conformal transformation, $g$, of $S^2(1)$ such that $\bar{\Psi} = g \circ \Psi: \Sigma \to S^2(1)$ satisfies

$$\int_\Sigma \rho \bar{\Psi}\, dA_1 = 0 \tag{1.4}$$

On the other hand

$$\begin{aligned} Q_1(\bar{\Psi}, \bar{\Psi}) &= \int_\Sigma (|\nabla_1 \bar{\Psi}|^2 - |\nabla_1 N_1|^2 |\bar{\Psi}|^2)\, dA_1 \\ &= \int_\Sigma (|\nabla_1 \bar{\Psi}|^2 - |\nabla_1 N_1|^2)\, dA_1 \\ &= 8\Pi[\text{degree}(\bar{\Psi}) - \text{degree}(N_1)], \end{aligned}$$

which combined with (1.3) and (1.4) say that

$$\text{degree}(\Psi) = \text{degree}(\bar{\Psi}) \geqq \text{degree}(N_1), \tag{1.5}$$

the equality holding if and only if $\Delta_1 \bar{\Psi} + |\nabla_1 N_1|^2 \bar{\Psi} = 0$. But as $\bar{\Psi}$ is a meromorphic map we have $\Delta_1 \bar{\Psi} + |\nabla_1 \bar{\Psi}|^2 \bar{\Psi} = 0$. So the equality (1.5) implies $|\nabla_1 \bar{\Psi}|^2 = |\nabla_1 N_1|^2$ everywhere.

Now we discuss the following cases:

a) $\Sigma$ is a sphere.

Taking $\Psi$ as the identity map, from (1.5) we conclude that degree $(N_1)=1$. So taking into account Osserman [10], Chapter 9, we conclude that $M$ is the Catenoid or the Enneper's surface.

b) genus $(\Sigma) \geqq 1$.

Recall that degree $(N_1) \geqq$ genus $(\Sigma)+1$, see Hoffman and Osserman [8], p. 75. If the inequality is strict, we construct via the Riemann–Roch Theorem a meromorphic map $\Psi$, with degree $(\Psi)=$ genus $(\Sigma)+1$, and (1.5) gives a contradiction.

If the equality holds, let $p \in \Sigma$ be a regular point of $N_1$ which is not a Weierstrass point of $\Sigma$. Again from the Riemann-Roch Theorem we obtain a meromorphic map, $\Psi$, of the same degree than $N_1$ which has $p$ as a point of ramification of order genus $(\Sigma)+1$. Then we ~~have~~ the equality in (1.5) and so $|\nabla_1 \bar{\Psi}|^2 = |\nabla_1 N_1|^2$ everywhere. But $|\nabla_1 \bar{\Psi}|^2(p)=0$ and $|\nabla_1 N_1|^2(p) \neq 0$. This is a contradiction and the theorem is proved.

## 2. Stable constant mean curvature surfaces

Let $(M, ds^2)$ be a complete Riemannian surface. Fischer-Colbrie has proved the following results.

THEOREM 3 [3]. *If the operator $L=\Delta+q$ has finite index on $M$, then there exist a compact set $C$ in $M$ and a positive function $u$ on $M$ with $Lu=0$ on $M-C$.*

THEOREM 4 [3]. *Let $C$ be a compact set in $M$ and $c$ a positive constant. If there exists a positive function $u$ on $M$ such that $\Delta u - Ku + cu \leqq 0$ on $M-C$, then $M$ is compact.*

Theorem 4 is not explicitly stated in [3]. A consequence of this result is mentioned at the introduction of her paper. The proof follows from a simplification of the arguments given in Theorem 1 of [3]. For completeness we give some hints how the proof of this Theorem should be modified in order to yield Theorem 4.

Suppose that $M$ is not compact. Following the proof of Theorem 1 in [3], and using the differential inequality $\Delta u - Ku + cu \leqq 0$, instead of inequality $\Delta u - Ku \leqq 0$ used in [3], we transform the least integral inequality of p. 127 in [3], into

$$\int_0^\infty [(\psi')^2 + 2\psi\psi'' + c\psi^2](u \circ \gamma)(s)\, ds \leqq 0, \tag{2.1}$$

for any smooth function $\psi:[0,\infty) \to R$ vanishing for large $s$, with $\psi(0)=0$ where $\gamma(s):[0,\infty) \to M-C$ is a certain curve parametrized by arc length.

Given $a>0$ we consider the function $\psi_a:[0,\infty)\to R$ defined by

$$\psi_a(s)=\sin\frac{\Pi s}{a} \quad \text{if} \quad 0\leqq s\leqq a,$$

$$\psi_a(s)=0 \qquad \text{if} \quad a\leqq s.$$

It is clear that (2.1) remains true if we take $\psi=\psi_a$. Then we have

$$\int_0^a\left[\frac{\Pi^2}{a^2}\cos^2\frac{\Pi s}{a}+\left(c-\frac{2\Pi^2}{a^2}\right)\sin^2\frac{\Pi s}{a}\right]u\circ\gamma\, ds\leqq 0.$$

As $u\circ\gamma$ is positive, taking $a$ large enough we obtain a contradiction. So $M$ is compact.

The following is the main result of this section.

THEOREM 5. *Let $M$ be a orientable complete surface with constant mean curvature in $R^3$. Then $M$ is stable (as surface with constant mean curvature) if and only if $M$ is a plane or a sphere.*

*Proof.* It is clear that the plane and the sphere are stable.

Let $L=\Delta+|\sigma|^2$. Then we have easily that

$$1+\mathrm{Ind}_0(M)\geqq\mathrm{Ind}(L) \tag{2.2}$$

So from the hypothesis $L$ has finite index on $M$, then from Theorem 3 there exists a positive solution of the equation $Lu=0$ outside of a certain compact set $C$ in $M$. The Gauss equation says that $\frac{1}{2}|\sigma|^2=2H^2-K$, so

$$0=Lu=\Delta u-Ku+(2H^2+\tfrac{1}{2}|\sigma|^2)u\geqq\Delta u-Ku+2H^2u,$$

on $M-C$. If the mean curvature is non-zero it follows by Theorem 4 that $M$ is compact. In this case from the result of Barbosa and do Carmo [1] we conclude that $M$ is a sphere.

If $M$ is minimal by (2.2) we have the following alternative:

i) $\mathrm{Ind}(L)=0$, and so $M$ is a plane, [4] or [2].

ii) $\mathrm{Ind}(L)=1$, and from Theorem 2, $M$ is the Catenoid or the Enneper's surface.

We conclude the proof of Theorem 5 taking into account the following lemmas.

LEMMA 6. *The Catenoid is unstable (as surface with constant mean curvature).*

*Proof.* The Catenoid is given by

$$X(u, v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad v \in R, \qquad u \in [0, 2\Pi]$$

So the induced metric and the Gaussian curvature are given by

$$ds^2 = \cosh^2 v(du^2 + dv^2), \qquad K = -1/\cosh^4 v$$

If the Catenoid were stable, then for any Lipschitz function $f(u, v) = f(v)$ with compact support on $R$, satisfying

$$\int_{-\infty}^{\infty} f(v) \cosh^2 v \, dv = 0, \tag{2.3}$$

we will have

$$\int_{-\infty}^{\infty} f'(v)^2 \, dv \geqq \int_{-\infty}^{\infty} (2/\cosh^2 v) f(v)^2 \, dv \tag{2.4}$$

So the Lemma is proved if we can construct a function $f$, which satisfies (2.3) but not (2.4). Given $0 < a < b$ we consider the function $f_{a,b}: R \to R$ defined by

$$f_{a,b}(v) = 1 \quad \text{if} \quad |v| \leqq a,$$
$$f_{a,b}(v) = (|v| - b)/(a - b) \quad \text{if} \quad a \leqq |v| \leqq b$$
$$f_{a,b}(v) = 0 \quad \text{if} \quad b \leqq |v|$$

Observe that $|f'_{a,b}(v)| \leqq 1/(b-a)$. For $k \geqq 0$ we take $f = 2f_{a,b} - f_{a+k,b+k}$. If $k = 0$ or $k = \infty$, the integral in (2.3) is positive or $-\infty$ respectively. So there exists $k$ such that $f$ verifies (2.3). We call this function $\bar{f}_{a,b}$.

Then we have easily that $\bar{f}_{a,b}(v) = 1$ if $|v| \leqq a$, $|\bar{f}'_{a,b}| \leqq 2/(b-a)$, and the subset of $R$ on which $\bar{f}'_{a,b} \neq 0$ has length not greater than $4(b-a)$.

Putting $\bar{f}_{1,b}$ in (2.4) and taking limit when $b$ goes to $+\infty$ we obtain a contradiction. Q.E.D.

LEMMA 7. *The Enneper's surface is unstable (as surface with constant mean curvature).*

*Proof.* Via the Weierstrass representation, the Enneper's surface is parametrized by the whole complex plane.

The induced metric and the Gauss curvature are given by

$$ds^2 = [\tfrac{1}{2}(1 + |z|^2]^2 \, |dz|^2, \qquad K = -[4/(1 + |z|^2)^2]^2$$

In a similar way than in Lemma 6, we consider functions in polar coordinates $f(r, \theta) = f(r)$. If the Enneper's surface were stable, then

$$\int_0^\infty f'(r)^2 r \, dr \geqq \int_0^\infty \frac{8f(r)^2 r}{(1 + r^2)^2} \, dr$$

for any Lipschitz function, vanishing for large $r$, such that

$$\int_0^\infty f(r)(1 + r^2)^2 r \, dr = 0$$

Using an argument as in Lemma 6, we get a contradiction. Q.E.D.

Now we will prove some simple but relevant results.

PROPOSITION 8. *Let $M$ be a complete orientable surface immersed with constant mean curvature in $R^3$. Then* $\mathrm{Ind}_0(M)$ *is finite if and only if $M$ is either compact or a minimal surface with finite total curvature.*

*Proof.* If $\mathrm{Ind}_0(M)$ is finite, reasoning as in the proof of Theorem 5, we conclude that $M$ is compact or minimal. In the minimal case we have from [3] Theor. 2, that $M$ has finite total curvature.

The nontrivial part of the converse assertion follows also from [3], Theor. 2, and the inequality $\mathrm{Ind}_0(M) \leqq \mathrm{Ind}(M)$, which holds clearly for minimal surfaces. Q.E.D.

The following result is an interesting consequence of Theorem 4.

COROLLARY 9. *Let $M$ be a complete, non compact, Riemannian surface, and $q: M \to R$ a smooth function on $M$ such that $q \geqq c$ everywhere, for some positive constant $c$.*

*Let $u$ be a solution of the equation $\Delta u - Ku + qu = 0$ on $M$. Then $u^{-1}(0)$ is a non-compact set in $M$.*

*Proof.* On $M - u^{-1}(0)$, $|u|$ is positive and satisfies $\Delta |u| - K|u| + c|u| \leqq 0$. If $u^{-1}(0)$ is compact, modifying $|u|$ in a bounded neighborhood of $u^{-1}(0)$ we get a positive function on $M$ which satisfies the above differential inequality outside of a compact set. So theorem 4 gives a contradiction. Q.E.D.

The study of the Gauss map is one of the fundamental problems in the classical theory of surfaces. For surfaces with constant mean curvature we have the following result.

THEOREM 10. *Let $M$ be a complete, non-compact, surface immersed in $R^3$ with non-zero constant mean curvature. Let $N$ be the Gauss map of $M$, $N: M \to S^2(1)$. Then $N^{-1}(S^1)$ is a non-compact set in $M$ for any equator $S^1$ in $S^2(1)$.*

*Proof.* We recall that $\Delta N + |\sigma|^2 N = 0$ on $M$, or equivalently, for any vector $a$ in $S^2(1)$ the function $u = \langle N, a \rangle$ satisfies the equation $\Delta u - Ku + (2H^2 + \frac{1}{2}|\sigma|^2)u = 0$. If $S^1$ is the equator in $S^2(1)$ orthogonal to $a$, then $N^{-1}(S^1) = u^{-1}(0)$. So the result follows from Corollary 9. Q.E.D.

Note that if $X$ is a Killing vector field in $R^3$, the function $u = \langle N, X \rangle$ on surfaces with constant mean curvature satisfies the equation $\Delta u + |\sigma|^2 u = 0$. So for these functions we have the corresponding version of Theorem 10.

Finally, we state the following extension of a result in [9].

THEOREM 11. *Let $M$ be a complete, non-compact, surface immersed with non-zero constant mean curvature in $R^3$, and let $N: M \to S^2(1)$ be its Gauss map. If $N(M - C)$ lies in a closed hemisphere of $S^2(1)$, for some compact set $C$ in $M$, then $M$ is a circular cylinder.*

*Proof.* Suppose that the hemisphere is given by $\{x \in S^2(1)/\langle x, a \rangle \geqq 0\}$, with $a$ in $S^2(1)$, Then the function $u = \langle N, a \rangle$ is non negative and satisfies the equation $\Delta u + |\sigma|^2 u = 0$ on $M - C$. From the maximum principle the function $u$ is constantly zero or everywhere positive on $M - C$. In the first case we conclude using the unique continuation property that $u = 0$ on $M$, and so $M$ is a circular cylinder, see [9].

In the second case, Theorem 4 gives a contradiction. Q.E.D.

*Added in August of 1987.* A. M. da Silveira and B. Palmer have obtained independent proofs of Theorem 5. Also S. Y. Cheng and J. Tysk has announced a proof of Theorem 2 in the embedded case.

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