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On the linearization of actions of linearly reductive groups

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Let G be a linearly reductive algebraic group over an algeabraically closed field k. Assume G acts on k^n by a morphism $A: G \times k^n \to k^n$. Then A may be viewed as a polynomial in n variables with coefficients in $\mathcal{O}(G)^n$. We assume

1° k^n has a fixed point, say the origin O, under the action A,

2° the action is of degree ≤ 2 with respect to k^n, i.e. the polynomial considered above is of degree ≤ 2.

Our aim, roughly speaking, is to prove that under the above assumptions the action is linear in some coordinate system (see the Theorem below). The case of $G = k^*$ and char $k \neq 2, 3$, has been studied in [J] by the Author, who thanks H. Kraft for important suggestions concerning the present paper. A well-known conjecture states that every action of a linearly reductive group on an affine space is linearizable. For other results and references see e.g. [B-B], [K, P], [Ka, R], [Ko, R], [P] and [P, R]). Notice for example that by Lemma 3.2 of [ibid] the assumption 1° is satisfied for all commutative groups of order $\leq 2^2 \cdot 3^2 \cdot 5^2 - 1 = 899$.

Let End (k^n) denote the set of morphisms $k^n \to k^n$. A map $f: G \to \text{End}(k^n)$ is called algebraic if the corresponding map $\bar{f}: G \times k^n \to k^n$ is a morphism. Then $f \in R[X_1, \ldots, X_n]$, where $R = \mathcal{O}(G)^n$. The Reynolds operator, i.e. the canonical G-equivariant projection $\mathcal{O}(G) \to k$ induces a G-equivariant projection (sending X_i to X_i)

$$\int := \int_G : R[X_1, \ldots, X_n] \to k^n[X_1, \ldots, X_n] = \text{End}(k^n),$$

so that $\int_G f$, the *mean value* of f is an endomorphism of k^n. Using the Reynolds operator corresponding to the group $G \times G$ we can consider $\int_{s \in G} \int_{t \in G} f(s, t)$, for an algebraic map $G \times G \to \text{End}(k^n)$ as well. We will use the following property of the mean value operator.

For $F \in \text{End}(k^n)$ and an algebraic map $f: G \to \text{End}(k^n)$ we have $\int_{s \in G} (f(s) \circ F) = (\int f(s)) \circ F$, and if F is linear, also $\int (F \circ f(s)) = F \circ (\int f(s))$. (1)

Assume $A: G \times k^n \to k^n$ is any group action with the point 0 fixed. Let $t \mapsto A(t)$ denote the corresponding homomorphism $G \to \operatorname{Aut}(k^n) \subseteq \operatorname{End} k^n$. We have $A(t) = L(t) + C(t) + \cdots$, where $t \mapsto L(t)$(resp $C(t)$) is the morphism from G to the space of linear maps (resp. quadratic maps) $k^n \to k^n$. From

$$A(st) = A(s)A(t) \tag{2}$$

it follows easily that $L(st) = L(s)L(t)$, i.e. L is a linear representation, and

$$L(s)C(t) + C(s)L(t) = C(st). \tag{3}$$

For $F: k^n \to k^n$ let $(\ )\circ F$ and $F \circ (\ )$ denote the respective right and left composition operator.

PROPOSITION. *Assume the condition* 1° *satisfied. Then there exists a unique quadratic map* $Q: k^n \to k^n$ *(independent of* t*) such that*

a) $C(t) = L(t) \circ Q - Q \circ L(t)$,

b) $\int L(t^{-1}) \circ Q \circ L(t) = 0$.

Proof. Apply $L(s^{-1}) \circ (\ )$ to (3) and rewrite it in the form

$$C(t) = L(t) \circ L((st)^{-1}) \circ C(st) - L(s^{-1}) \circ C(s) \circ L(t). \tag{4}$$

Here $(s, t) \in G \times G$. Set

$$Q := \int L(s^{-1}) \circ C(s). \tag{5}$$

Applying the operator $\int_{s \in G}$ to (4) one gets the identity a). Apply $L(t^{-1}s^{-1}) \circ (\ )$ to (3). The result may be written in the form

$$L((st)^{-1}) \circ C(st) - L(t^{-1}) \circ C(t) = L(t^{-1}) \circ (L(s^{-1}) \circ C(s)) \circ L(t)$$

Now apply $\int_{s \in G} \int_{t \in G}$ to both sides to get b). Finally the identity (5) follows from a) and b), hence the uniqueness. ∎

REMARK. Suppose 1° and 2° satisfied. Let I stand for the identity on k^n. By (5), $I + Q = \int L(t^{-1}) \circ A(t)$. The expression under the integral may be viewed as the deviation of the action $A(t)$ from its linear part $L(t)$. So $I + Q$ is the mean value of that deviation. This last morphism turns out to be a conjugating automorphism in case of action of degree two:

THEOREM. *Let G be linearly reductive and assume* 1° *and* 2°. *Let Q be the quadratic map defined in the proposition above. Suppose either of the following holds*

a) $\mathrm{char}(k) \neq 2$

b) *G is commutative (hence diagonalizable).*

Then $A(t) = (I - Q)\circ L(t)\circ(I + Q)$ and $I - Q$, $I + Q$ are mutually invers automorphisms of k^n. In particular the action A is linearizable.

Recall, that G is diagonalizable if and only if it is a finite product of multiplicative groups k^* and a finite commutative group of order prime to $\mathrm{char}(k)$ ([B], ch. III, §8).

Proof of the Theorem. Set $S := L(s)$, $T := L(t)$. Then $A(t) = T + TQ - QT$ and the identity (2) reduces to $(SQ - QS)\circ T = (SQ - QS)\circ(T + TQ - QT)$, for all $(s, t) \in G \times G$. Apply $(\ )\circ T^{-1}$;

$$(SQ - QS) = (SQ - QS)\circ(I - Q + TQT^{-1}) \tag{6}$$

Then applying $S^{-1}\circ(\ )$ we get $Q - S^{-1}QS = (Q - S^{-1}QS)\circ(I - Q + TQT^{-1})$. Further, $\int_{s\in G}$ gives $Q = Q\circ(I - Q + TQT^{-1})$, by Prop., b). Then $SQ = SQ\circ(I - Q + TQT^{-1})$ and subtracting (6) we have also $QS = QS\circ(I - Q + TQT^{-1})$. Apply $(\ )\circ S^{-1}$ to get

$$Q = Q\circ(I - SQS^{-1} + (ST)\circ Q\circ(ST)^{-1}). \tag{7}$$

Now assume a). Let $Q' : k^n \times k^n \to k^n$, be the bilinear symmetric map such that $Q(x) = Q'(x, x)$. We have

$$Q = Q(I - SQS^{-1}) + Q\circ(ST)\circ Q\circ(ST)^{-1} + 2Q'(I - SQS^{-1}, (ST)\circ Q\circ(ST)^{-1}). \tag{8}$$

Now apply $\int_{t\in G}$. We have $\int Q\circ(ST)\circ Q\circ(ST)^{-1} = \int QTQT^{-1}$ and by (1) and Prop., b), $\int_{t\in G}$ of the last summand of (8) vanishes. So

$$Q = Q\circ(I - SQS^{-1}) + \int_{s\in G} QSQS^{-1} \tag{9}$$

Extract now the parts of degree 4 with respect to k^n to get

$$0 = QSQS^{-1} + \int QSQS^{-1}. \tag{10}$$

Applying $\frac{1}{2}\int_{s\in G}$ we get

$$\int_{s\in G} QSQS^{-1} = 0. \tag{11}$$

By (9) we have $Q = Q\circ(I - SQS^{-1})$. Since the part of degree 3 vanishes, $Q = Q\circ(I + SQS^{-1})$, too. Apply $(\ )\circ S$ to get $QS = QS\circ(I + Q)$. Now we are ready to conjugate the linear action $(s, x)\mapsto S(x)$:

$(I - Q)\circ S\circ(I + Q) = S + SQ - QS\circ(I + Q) = S + SQ - QS = A(s)$, as required. For $t = 1$ one gets $(I - Q)\circ(I + Q) = I$. Replacing the action $A(t)$ by $A'(t) := (-I)\circ A(t)\circ(-I)$ one gets by an analogous argument that $(I + Q)\circ(I - Q) = I$. This completes the proof in case a).

So we assume char $(k) = 2$, and G diagonalizable. Then the proof alters as follows. Choose a bilinear map $Q\hat{}: k^n \times k^n \to k^n$ such that $Q(x) = Q\hat{}(x, x)$ for all x. Then (8) holds with $2Q'$ replaced by the map $(x, y)\mapsto Q\hat{}(x, y) + Q\hat{}(y, x)$. To obtain (11) we must replace the Reynolds operator by a more precise tool, available for diagonalizable groups:

Denote by $X = X(G)$ the group of characters $G\to k^*$, with the additive notation. Let t^i stand for the value of $i\in X$ at $t\in G$. It follows easily from [B], Ch. III, §8, that

any algebraic map $f: G\to \operatorname{End}(k^n)$ can be written in a unique way as

$$f(t) = \sum_{i\in X} t^i f_i, \text{ for some } f_i \in \operatorname{End}(k^n). \tag{12}$$

Notice that $\int_{t\in G} f$ coincides with f_0 in this case. For $i\in X(G)$ let the morphisms $b_i: k^n\to k^n$ be defined by the expansion $SQS^{-1} = \sum s^i b_i$ introduced in (12), and set $q_{i,j} := Q\hat{}(b_i, b_j)$. Since $b_0 = 0$ we have $q_{0,j} = q_{i,0} = 0$. Now extract the part of degree 4 in (7) to get

$$0 = Q\circ(SQS^{-1} + (ST)Q(ST)^{-1}) = \sum_{i,j\in X} s^{i+j}(t^i - 1)(t^j - 1)q_{i,j}.$$

Apply $\int_{s\in G}$;

$$0 = \sum_{i\in X} (t^i + 1)(t^{-i} + 1)q_{i,-i} = \sum (t^i + t^{-i})q_{i,-i}. \tag{13}$$

Let P be a subset of $X\setminus\{0\}$ such that for all $i \in X\setminus\{0\}$ one and only one of the characters i, $-i$ belongs to P. Hence X is the disjoint union of P, $-P := \{i \mid -i \in P\}$ and $\{0\}$ (indeed, $i+i \neq 0$ for $i \neq 0$ by the condition (char k, $G : G_{\text{conn}}) = 1$). Since $q_{0,0} = 0$ we have by (13) that

$$\sum_{i\in P} (t^i + t^{-i})(q_{i,-i} + q_{-i,i}) = 0.$$

By (12) this implies that

$$\sum_{i\in P} t^i(q_{i,-i} + q_{-i,i}) = 0.$$

For $t = 1$ we obtain

$$0 = \sum_{i\in P} (q_{i,-i} + q_{-i,i}) = \sum_{i\in X} q_{i,-i}.$$

On the other hand

$$\int_{s\in G} QSQS^{-1} = \int_{s\in G} Q \circ \Big(\sum_X s^i b_i\Big) = \int \Big(\sum_{i,j} s^{i+j} q_{i,j}\Big) = \sum_{i\in X} q_{i,-i}.$$

So we get again (11), and the rest of the proof follows as in the part a), obviously simplified. ∎

REMARK. If char $(k) \neq 2$, 3 then exactly as in [J] one proves that the conjugating automorphism $I + Q$ of k^n is triangular in some basis of this vector space.

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