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Autor(en): **Cantwell, John / Conlon, Lawrence**

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Leafwise hyperbolicity of proper foliations

JOHN CANTWELL[1] and LAWRENCE CONLON[2]

Introduction

A foliated manifold $(M, \mathcal{F})$ is said to be *proper* if every leaf of $\mathcal{F}$ is proper. A leaf L is proper if the relative topology of L in M coincides with the manifold topology of L. This is equivalent to requiring that each point $x \in L$ lies in a foliation chart $U \subset M$ such that $L \cap U$ is a single plaque. Equivalently, the leaf L is not asymptotic to itself.

Proper foliated manifolds have been studied by various authors. For example, in arbitrary codimension, Millett [Mi] has organized the leaves of such foliations into a countable ordinal hierarchy that is completely analogous to the Epstein hierarchy for foliations with all leaves compact [Ep].

In codimension one, with smoothness class at least C^2, there is a more rigid hierarchy by integral *levels* [C–C1]. Leaves at a given level wind in on those at lower levels in a way reminiscent of the Poincaré–Bendixson theorem. In [C–C2], this hierarchy is combined with Millett's to prove that C^2-smoothness for proper foliated manifolds of codimension one implies C^∞-smoothability.

In the work of Gabai [Ga1], [Ga2], [Ga3], proper foliated 3-manifolds occur that are Reebless and of *finite depth* (which means that there is an upper bound on the levels). Generally, these foliations may only be of class C^0, although frequently they are smooth.

In what follows, $(M, \mathcal{F})$ will denote a C^2-foliated 3-manifold, where M is compact and orientable and $\mathcal{F}$ is of codimension one and transversely orientable.

An interesting geometric problem is to find a Riemannian metric on M relative to which each leaf of $\mathcal{F}$ is *hyperbolic* (i.e., has constant curvature -1). We say that such a metric is *leafwise hyperbolic*.

MAIN THEOREM. *Let $(M, \mathcal{F})$ be proper and assume that each component of ∂M is a leaf of $\mathcal{F}$. Then $\exists$ a leafwise hyperbolic metric on M if and only if no leaf of $\mathcal{F}$ is a torus or a sphere.*

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[1] Work partially supported by N.S.F. Contract 8420322.
[2] Work partially supported by N.S.F. Contract 8420956.

COROLLARY *Let $(M, \mathscr{F})$ be proper and assume that each component of ∂M is either a leaf or is transverse to $\mathscr{F}$. If no compact leaf of $\mathscr{F}$ is a torus, a sphere, a disk, or an annulus, then $\exists$ a leafwise hyperbolic metric on M.*

This corollary is immediate upon doubling $(M, \mathscr{F})$ along the transverse boundary.

Gabai's proper foliations of knot complements $M = S^3 \backslash N^0(k)$ are taut (that is, every leaf meets a closed transversal nontrivially) and admit a minimal genus spanning surface of the knot k as a leaf [Ga1, Theorem 5.5]. Together with the fact that $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$, this implies that, for knots of genus greater than one, no compact leaf is a torus, a sphere, an annulus, or a disk. If the foliation can be chosen to be smooth, the corollary implies that it admits a leafwise hyperbolic metric.

The "only if" part of the Main Theorem is evident. The proof of the "if" part uses the Poincaré–Bendixson theory of totally proper leaves [C–C1, §6] and the smooth parametrization of hyperbolic structures by Fenchel–Nielsen coordinates [Th, § 5].

Remark. The hypothesis in [C–C1] that $\partial M = \varnothing$ can be replaced with the hypothesis that ∂M is a union of leaves without affecting any of the theorems or proofs in that paper. *We fix this hypothesis throughout.*

1. Constructing a hyperbolic skeleton

In this section, unless otherwise specified, $(M, \mathscr{F})$ need not be proper.

We fix a smooth, one dimensional foliation $\mathscr{F}^\perp$ of M that is everywhere transverse to $\mathscr{F}$. There corresponds a decomposition

$$T(M) = T(\mathscr{F}) \oplus T(\mathscr{F}^\perp)$$

into the respective sub-bundles of tangents to the two foliations. We will fix a Riemannian metric on $T(\mathscr{F}^\perp)$. All metrics constructed on $T(M)$ will be understood to induce this fixed metric on $T(\mathscr{F}^\perp)$ and to make the two summands orthogonal. Accordingly, in this and the following sections, the Riemannian metrics that we will be constructing will be metrics on $T(\mathscr{F})$.

Let $O(\mathscr{F})$ denote the family of open, $\mathscr{F}$-saturated subsets of M and, for $U \in O(\mathscr{F})$, let $\hat{U}$ denote the completion of U in any Riemannian metric inherited from a Riemannian metric on M. As is well known [Di], [C–C1], [H–H], [Go], $\hat{U}$ is a manifold with boundary and the inclusion $i: U \hookrightarrow M$ induces a C^2 immersion $\hat{\imath}: \hat{U} \to M$ that may identify some components of $\partial \hat{U}$ pairwise. Then we obtain a

C^2 foliation, $\hat{\imath}^{-1}(\mathscr{F}) = \hat{\mathscr{F}}$, of $\hat{U}$ that is tangent to $\partial\hat{U}$, while $\hat{\imath}^{-1}(\mathscr{F}^\perp) = \hat{\mathscr{F}}^\perp$ is a smooth, one dimensional foliation of $\hat{U}$, everywhere transverse to $\hat{\mathscr{F}}$.

Of special interest is the case in which $\hat{U}$ is diffeomorphic to $L \times [0, 1]$ in such a way that the leaves of $\hat{\mathscr{F}}^\perp$ are identified with the intervals $\{x\} \times [0, 1]$, $\forall x \in L$. Following our usage elsewhere (e.g., [C–C1]), we will say that U is a *foliated product.* We will also identify L *with the leaf* $\hat{\imath}(L \times \{0\})$ of $\mathscr{F}$, which may or may not be the same as the leaf $\hat{\imath}(L \times \{1\})$.

DEFINITION. A closed subset $X \subset M$ that is a finite union of leaves will be called a skeleton of $(M, \mathscr{F})$ if each component of $M\backslash X$ is a foliated product.

(1.1) LEMMA. *If* $(M, \mathscr{F})$ *is proper, it has a skeleton.*

Proof. Choose a countable family $\{L_i\}_{i=1}^{\infty}$ of leaves of $\mathscr{F}$ such that $\bigcup_{i=1}^{\infty} L_i$ is everywhere dense in M. Let $X_k = \bigcup_{i=1}^{k} L_i$, $\forall k \geq 1$. Since $(M, \mathscr{F})$ is proper, the closure $\bar{X}_k$ of X_k is a finite union of leaves [C–C1, (4.7)]. Since $\bigcup_{k=1}^{\infty} \bar{X}_k$ is everywhere dense in M, the components of $M\backslash\bar{X}_k$ will all be foliated products, for k sufficiently large [Di, Proposition 1].

Our proof of the Main Theorem will proceed by finding, first, a metric making all leaves of the skeleton X hyperbolic and, then, by modifying the metric in each of the complementary foliated products. For the first step, the only property of the skeleton X that will be needed is that it is a compact union of finitely many leaves.

Let X be a compact, nonempty, union of finitely many leaves of $\mathscr{F}$, these being, of necessity, proper whether or not the foliation is proper (the closure of a nonproper leaf contains uncountably many leaves by the results in [C–C1, § 4 and § 5]). There will be an integer $n \geq 0$ such that each leaf in X has level at most n and, for each of the integers $k = 0, 1, \ldots, n$, at least one leaf in X has level k. The leaves at level 0 are exactly the compact ones, and the leaves at level $k > 0$ are asymptotic to some of the leaves at each level strictly less than k, but to no others. For the general theory of levels, see [C–C1].

Let $L \subset X$ be a leaf at level $k > 0$. The Poincaré–Bendixson theory of totally proper leaves applies [C–C1, §6] and gives a decomposition

$$L = A \cup B^1 \cup \cdots \cup B^r,$$

together with projections

$$p^j : B^j \to L^j,$$

that are *spirals* onto leaves $L^j \subset X$ at lower levels. Each p^j is a projection along leaves of $\mathcal{F}^\perp$ and is a semi-covering with covering semigroup the positive integers $\mathbb{Z}^+$. Indeed, if $y_0 \in B^j$ and $p^j(y_0) = y \in L^j$, there is a compact subarc of a leaf of $\mathcal{F}^\perp$, having endpoints y_0 and y, which meets B^j in a sequence of points $\{y_i\}_{i=0}^\infty$ that converges monotonically to y. The semigroup action of $\mathbb{Z}^+$ on B^j is generated by the proper imbedding $\pi: B^j \to B^j$ defined by $\pi(y_0) = y_1$. Finally, the submanifold A is compact and connected, $A \cap B^j = \partial B^j$ is also a component of ∂A, $1 \le j \le r$, and $B^i \cap B^j = \varnothing$ if $i \ne j$.

The above action of $\mathbb{Z}^+$ corresponds to an element of contracting holonomy of the leaf L^j. This element of holonomy determines a *compactly supported* cohomology class $\alpha \in \mathrm{H}_c^1(L^j; \mathbb{Z})$. The proof that α is compactly supported makes essential use of the hypothesis that $(M, \mathcal{F})$ is of class C^2. The Poincaré dual of α can be represented by a compact, connected, nonseparating submanifold $N^j \subset L^j$ of codimension one, called the *juncture* of the spiral p^j. In our case, $\dim(L^j) = 2$, so N^j is always a circle.

The geometric interpretation of the juncture N^j is as follows. The manifold B^j will be an "infinite repetition" of segments $\{B_i^j\}_{i=0}^\infty$, each diffeomorphic to a copy of the manifold L_*^j with two boundary components that is obtained by cutting L^j along N^j. If $k > 1$, then B_i^j and B_{i+k}^j will be disjoint, $\forall i \ge 0$, while $B_i^j \cap B_{i+1}^j = N_{i+1}^j$ will be a common boundary component of these manifolds. We also let N_0^j denote ∂B^j, one of the two components of ∂B_0^j. Finally, the projection p^j will carry N_i^j diffeomorphically onto N^j, $\forall i \ge 0$.

For all of the above, only the homology class of N^j in L^j matters, so we will always take the juncture to be a closed geodesic in a metric relativized from a metric on M.

Finally, it should be remarked that the compact manifold $A \subset L$ can be chosen as large as desired. It follows that, if L is not topologically a plane or a cylinder, then A can be chosen to be neither a disk nor an annulus.

For more details concerning the Poincaré–Bendixson theory of totally proper leaves, the reader should consult [C–C1, § 6].

(1.2) PROPOSITION. *Let X be a compact union of finitely many leaves of $\mathcal{F}$. Then, if no leaf in X is a torus or a sphere, $\exists$ a Riemannian metric g on M and an open neighborhood W of X such that $g \mid W$ is a leafwise hyperbolic for $\mathcal{F} \setminus W$ and such that projection along the leaves of $\mathcal{F}^\perp \mid W$ defines local isometries between the leaves of $\mathcal{F} \mid W$.*

Proof. We proceed by induction on the levels of the leaves in X. The leaves at level 0 are compact and, by the hypothesis, each supports a hyperbolic metric. In standard fashion, one constructs a metric on M that relativizes to the chosen hyperbolic metric on each leaf at level 0 in X. Arrange that, on the union W_0 of suitable disjoint normal neighborhoods of these compact leaves (the normal fibers are open subarcs of leaves of $\mathcal{F}^{\perp}$), the metric is the lift, via the projection, of the hyperbolic metric. Hence, the metric is leafwise hyperbolic in W_0 and projection along the normal fibers defines local isometries between the leaves of $\mathcal{F} \mid W_0$.

Let X_k denote the union of all leaves in X at levels at most k. This is a compact set. Let $W_k \subseteq M$ be an open neighborhood of X_k obtained as the (generally not disjoint) union of sufficiently small normal neighborhoods of the leaves in X_k. Inductively, assume that a metric on M has been found relative to which each leaf of $\mathcal{F} \mid W_k$ has constant curvature -1 and such that the local projections along leaves of $\mathcal{F}^{\perp} \mid W_k$ define local isometries between the leaves of $\mathcal{F} \mid W_k$. Let L be a leaf in X at level $k+1$. Choose a decomposition $L = A \cup B^1 \cup \ldots \cup B^r$ with spirals $p^j : B^j \to L^j$, $1 \leq j \leq r$. By the inductive hypothesis, each of the leaves L^j is hyperbolic and we can arrange, as remarked above, that the juncture N^j be a closed geodesic in L^j. The metric on each leaf L^j is lifted, via p^j, to a hyperbolic metric on B^j in which ∂B^j is a closed geodesic. Furthermore, this metric on $B^j \cap W_k$ agrees with the one already defined.

An easy induction on the level of L (necessarily > 1) shows that this leaf has infinite genus, so the compact submanifold A can be chosen to be neither a disk nor an annulus. Decompose A into "pairs of pants". Each pair of pants has a hyperbolic metric, uniquely specified (up to isometries that are isotopic to the identity) by requiring each boundary circle to be a geodesic of specified length [Po]. Using this, one readily extends the metric on $\bigcup_{j=1}^{r} B^j$ smoothly over A so as to produce a hyperbolic metric on the leaf L. One does this for each leaf $L \subset X$ at level $k+1$. There is a neighborhood $\tilde{W}$ of X_{k+1} of the desired type, obtained by extending W_k via disjoint, normal neighborhoods of the compact sets $L \setminus W_k$, and the metric extends over $\tilde{W}$ to be as desired. Select the new neighborhood W_{k+1} of X_{k+1} by shrinking $\tilde{W}$ slightly in the $\mathcal{F}^{\perp}$ directions and then modify the metric in $\tilde{W} \setminus W_{k+1}$ so that it extends C^2-smoothly to a Riemannian metric on M.

2. The metric on the foliated products

In this section, we assume that $(M, \mathcal{F})$ is proper. By (1.1) and (1.2), we choose a skeleton $X \subset M$ and a Riemannian metric on M that makes each leaf in X hyperbolic.

Let U be a component of $M \setminus X$. Fix the identification $\hat{U} \cong L \times [0, 1]$. Let

$$\hat{\rho} : L \times \{0\} \to L \times \{1\}$$

be projection along the leaves of $\hat{\mathscr{F}}^{\perp}$. That is to say, $\hat{p}(x, 0) = (x, 1)$. Let $L = \hat{\imath}(L \times \{0\})$ and $\tilde{L} = \hat{\imath}(L \times \{1\})$. The projection $\hat{p}$ defines a diffeomorphism

$$\rho : L \to \tilde{L}$$

that is generally not an isometry. For instance, it is quite possible that $L = \tilde{L}$ and then, if L is not compact, it is impossible that ρ be the identity and generally it is impossible that it be an isometry. Consequently, we cannot expect $\hat{\rho}$ to be an isometry either.

Recall the decomposition of L into the compact submanifold A and "arms" B^j, $1 \le j \le r$, together with $p^j : B^j \to L^j$. Let $\tilde{A} = \hat{\imath}(A \times \{1\})$, let $\tilde{B}^j = \hat{\imath}(\tilde{B}^j \times \{1\})$, $1 \le j \le r$, and let p^j denote both of the spirals $B^j \to L^j$ and $\tilde{B}^j \to L^j$. The metrics on B^j and $\tilde{B}^j$ have been lifted by p^j from a hyperbolic metric on L^j, $\forall j$. If $L = \tilde{L}$, then, depending on the side of the leaf L^j on which the arm B^j spirals, either ρ or ρ^{-1} carries B^j into itself and generates the covering semigroup $\mathbb{Z}^+$ on B^j. In any case, ρ carries B^j isometrically onto $\tilde{B}^j$ and it follows that, in $\hat{U}$ and relative to the metric pulled back by $\hat{\imath}$, the projection

$$\hat{\rho} : L \times \{0\} \to L \times \{1\}$$

carries $B^j \times \{0\}$ isometrically onto $B^j \times \{1\}$, $1 \le j \le r$. The problem alluded to in the previous paragraph is that the diffeomorphism $\hat{\rho} \mid (A \times \{0\})$ of $A \times \{0\}$ onto $A \times \{1\}$ is not generally an isometry. It is true, however, that $\hat{\rho}$ is an isometry of the boundary geodesics since these are also boundary geodesics for the arms.

Fix a decomposition of $A \times \{0\}$ into pairs of pants, each bounded by closed geodesics. Then $\hat{\rho}$ carries this to a decomposition of $A \times \{1\}$ into pairs of pants, but the boundary circles that are not already components of $\partial A \times \{1\}$ may not be geodesics. These circles can be replaced with closed geodesics in their free homotopy class, giving a decomposition of $A \times \{1\}$ that is diffeomorphic to the first one. Finally, one smoothly alters $\hat{\mathscr{F}}^{\perp} \mid (A \times [0, 1])$, without changing it near $A \times \{0, 1\}$ nor near $\partial A \times [0, 1]$, so that $\hat{\rho}$ carries the decomposition of $A \times \{0\}$ to the one of $A \times \{1\}$, although *not isometrically*. Remark that $\mathscr{F}^{\perp}$ itself has been slightly modified, but in regions not affecting any preceding construction.

(2.1) LEMMA. *The Riemannian metric on $A \times \{0, 1\}$ extends to a metric g' on $A \times [0, 1]$ that restricts to a hyperbolic metric g'_t on each level set $A \times \{t\}$. For t sufficiently near* 0 *or* 1, *respectively, the metric g'_t is independent of t. Finally, g'_t makes each component $\Sigma \times \{t\}$ of $\partial A \times \{t\}$ into a geodesic isometric to Σ,* $0 \le t \le 1$.

Proof. The space of isotopy classes of hyperbolic metrics on A, relative to which the components of ∂A are closed geodesics of specified lengths, is smoothly parametrized as a Euclidean space $\mathbb{R}^N$, in a way that depends on the choice of decomposition of A into pairs of pants. These coordinates, due to Fenchel and Nielsen [Th, § 5], are the logarithms of the lengths of the boundary components of the pairs of pants, other than the components of ∂A, together with real numbers that record relative "twisting" along common boundary geodesics. Since $\hat{\rho}$ carries the decomposition of $A \times \{0\}$ to that of $A \times \{1\}$, we can view the metrics on these manifolds as points $(a_1, \ldots, a_N)$ and $(b_1, \ldots, b_N)$ in the same Euclidean space $\mathbb{R}^N$. Choose a smooth path $\sigma: [0, 1] \to \mathbb{R}^N$ that joins these points and is constant near 0 and 1. Then $\sigma(t) = g'_t$ is a hyperbolic metric on $A \times \{t\}$, $0 \leq t \leq 1$, and all assertions follow.

Should it happen that $\hat{\mathscr{F}} \mid (A \times [0, 1])$ is the product foliation, we are done. This is not generally the case. Furthermore, if Σ is a component of ∂A and if F is a leaf of $\hat{\mathscr{F}} \mid (A \times [0, 1])$, the components of $F \cap (\Sigma \times [0, 1])$ need not be closed.

Let $\mathscr{H}'$ denote the product foliation of $A \times [0, 1]$ and let $\mathscr{H}$ denote the foliation of that manifold induced by $\hat{\mathscr{F}}$. Then g' is a Riemannian metric on $T(\mathscr{H}')$. If $v_i \in T_{(x,t)}(\mathscr{H})$, $i = 1, 2$, let $v'_i \in T_{(x,t)}(\mathscr{H}')$ be the unique vector that differs from v_i by an element of $T(\hat{\mathscr{F}}^{\perp})$. Define a Riemannian metric g on $T(\mathscr{H})$ by setting $g(v_1, v_2) = g'(v'_1, v'_2)$.

(2.2) LEMMA. *Under the metric g, each leaf of $\mathscr{H}$ has constant curvature* -1.

Proof. For v_i and v'_i as above, it is evident that (v_1, v_2) is a g-orthonormal frame if and only if (v'_1, v'_2) is g'-orthonormal. It will be enough, therefore, to show that the respective curvature tensors satisfy the identity

$$(R(v_1, v_2)v_2)' = R'(v'_1, v'_2)v'_2.$$

Let $\lambda: A \times [0, 1] \to A$ be the canonical projection. Set $\omega_i = \lambda_*(v_i) = \lambda_*(v'_i)$, $i = 1,2$. Let X_i be a smooth extension of ω_i to a field on A. Let $Y_i \in \Gamma(T(\mathscr{H}))$ and $Y'_i \in \Gamma(T(\mathscr{H}'))$ be the unique elements that are λ-related to X_i, $i = 1,2$. By integrability, $[Y_1, Y_2] \in \Gamma(T(\mathscr{H}))$ and $[Y'_1, Y'_2] \in \Gamma(T(\mathscr{H}'))$, and these fields are λ-related to $[X_1, X_2]$. It follows that $[Y_1, Y_2]' = [Y'_1, Y'_2]$. Let ∇ and ∇' be the respective Levi–Civita connections (along the leaves of $\mathscr{H}$ and $\mathscr{H}'$, respectively) for the metrics g and g'. The well known formula for the Levi–Civita connection [K–N, p. 160] involves only the metric and the bracket operation, hence

$$(\nabla_{Y_1} Y_2)' = \nabla'_{Y'_1} Y'_2$$

and it follows that $(R(v_1, v_2)v_2)' = R'(v'_1, v'_2)v'_2$. $\square$

(2.3) LEMMA. *Let H be a leaf of $\mathscr{H}$. Then each component of $H \cap (\partial A \times [0, 1])$ is a geodesic for the metric $g \mid H$ and λ defines a local isometry of this geodesic onto one of the closed geodesics bounding A.*

Proof. Let J be a subarc of a component of $H \cap (\partial A \times [0, 1])$, small enough that λ carries it one-one into ∂A. Relative to g, let v be a unit tangent field along J. At each point of J, the corresponding vector v' has unit length relative to g'. Since λ is an isometry of each $\partial A \times \{t\}$ (in the g' metric) onto ∂A, it follows that $\lambda_*(v)$ is a unit tangent field along $\lambda(J)$. The field $\lambda_*(v)$ lifts to a field Z on $\lambda^{-1}(J)$ that is everywhere tangent to $\mathscr{H}$. The corresponding field Z', tangent to $\mathscr{H}'$, has unit g'-length and is everywhere tangent to curves that are geodesics in leaves of $\mathscr{H}'$, hence $\nabla'_{Z'}Z' \equiv 0$. Then, as in the proof of (2.2), $(\nabla_Z Z)' = \nabla'_{Z'}Z' \equiv 0$, so $\nabla_Z Z \equiv 0$. In particular, J is a geodesic segment for $g \mid H$. □

Remarks. Let g^j be the metric on $B^j \times [0, 1]$ lifted from B^j via projection. Since $\partial B^j \subset \partial A$ is a closed geodesic, the subarc J in the above proof has exactly the same geodesic structure relative to the metric g^j as it has relative to g.

We now build the metric g on $\hat{U}$ (more precisely, on $T(\hat{\mathscr{F}})$) by using the metric constructed above in $A \times [0, 1]$ and using, in each arm $B^j \times [0, 1]$, the lifted metric g^j. The only problem is to make sure these definitions fit together smoothly (i.e., C^2). For this, it will be enough to verify leafwise smoothness.

(2.4) COROLLARY. *The metric g, as defined above on $\hat{U}$, is of class C^2 and leaf-wise hyperbolic.*

Proof. Let H be a leaf of $\mathscr{H}$ and let J be a small open subarc of $H \cap (\partial A \times [0, 1])$, as in the proof of (2.3). Let W be an open subset of H such that $W \cap (\partial A \times [0, 1]) = J$. Let $W^+ = W \cap (A \times [0, 1])$ and $W^- = W \cap (B^j \times [0, 1])$ and let $\varphi^\pm : W^\pm \to \mathbb{H}^2$ be isometric mappings into the hyperbolic plane $\mathbb{H}^2$. Then $\varphi^-(J)$ and $\varphi^+(J)$ are hyperbolic line segments of the same length and, by standard hyperbolic geometry, there is an isometry $\tau : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\tau(\varphi^-(J)) = \varphi^+(J)$. Thus, the isometric imbeddings $\tau \circ \varphi^- : W^- \to \mathbb{H}^2$ and $\varphi^+ : W^+ \to \mathbb{H}^2$ fit together to define an isometric imbedding $\tilde{\varphi} : W \to \mathbb{H}^2$. □

(2.5) LEMMA. *Outside of a compact subset of $A \times \,]0, 1[$, the metric g is just the lift, via projections along the leaves of $\hat{\mathscr{F}}^\perp$, of the hyperbolic metric on $\partial\hat{U}$ that is pulled back from the skeleton X via $\hat{\imath}$.*

Proof. This is clear in the arms. In $A \times [0, 1]$, it follows from the relation between g and g' and the fact that, for t sufficiently near 0 (respectively, 1), g'_t is independent of t. □

For each component U of $M \backslash X$, we have produced a leafwise hyperbolic metric of class at least C^2. We claim that these metrics, together with the hyperbolic metric on X produced in (1.2), assemble to form the desired C^2 metric on M. Indeed, the metric in (1.2) is of class C^2 and, by (2.5), it agrees in a neighborhood of X with the metric we have just produced. Since the only points where differentiability could fail are the points of X, the proof of the Main Theorem is complete.

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St. Louis University
St. Louis, MO 63103

and

Washington University
St. Louis, MO 63130

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