Classifying spaces with injective mod p cohomology.

Autor(en): **Henn, Hans-Werner**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **64 (1989)**

PDF erstellt am: **19.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-48941

Comment. Math. Helvetici 64 (1989) 200–206 0010–2571/89/020200–07$1.50 + 0.20/0

Classifying spaces with injective mod p cohomology

HANS-WERNER HENN

0. Introduction

Let G be a compact Lie group with classifying space BG and let p be a prime. We consider the mod p cohomology $H^*(BG; \mathbb{Z}/p)$ as an object of $\mathcal{U}_p$, the category of unstable left modules over the Steenrod algebra A_p.

Recall that a left A_p module is called unstable, if

$$Sq^n x = 0 \quad \text{whenever} \quad n > |x|, \quad \text{if} \quad p = 2$$

$$\beta^e P^n x = 0 \quad \text{whenever} \quad 2n + e > |x|, \qquad e = 0, 1, \quad \text{if} \quad p > 2.$$

Here $|x|$ denotes the degree of x.

THEOREM. *Let G be a compact Lie group. Then the following statements are equivalent:*

(1) *$H^*(BG; \mathbb{Z}/p)$ is an injective object of $\mathcal{U}_p$.*

(2) *$p \cdot H^n(BG; \mathbb{Z}_{(p)}) = 0$ for $n \gg 0$ (Here $\mathbb{Z}_{(p)}$ denotes localization of $\mathbb{Z}$ at the prime p).*

(3) *G is finite and its p Sylow subgroup G_p is elementary abelian, i.e. $G_p \cong (\mathbb{Z}/p)^k$ for some k.*

Remarks. a) The implication "(1) $\Rightarrow$ (3)" was conjectured by Lannes and Zarati.

b) "(3) $\Rightarrow$ (1)" is immediate from the facts that $H^*(B(\mathbb{Z}/p)^k; \mathbb{Z}/p)$ is injective in $\mathcal{U}_p$ ([C], [Mi], [L–Z]) and that the transfer being induced by a stable map (cp. chap. IV of [A]) provides an A_p – linear splitting of the restriction map $H^*(BG; \mathbb{Z}/p) \rightarrowtail H^*(BG_p; \mathbb{Z}/p)$. In this paper we will prove "(1) $\Rightarrow$ (2)" and "(2) $\Rightarrow$ (3)".

c) If $H^*(BG; \mathbb{Z}/p)$ is injective then the theorem together with a result of Swan [S] gives that $H^*(BG; \mathbb{Z}/p)$ is isomorphic to $H^*(B(\mathbb{Z}/p)^k; \mathbb{Z}/p)^W$ where $(\mathbb{Z}/p)^k \cong G_p$ and W is the "Weyl group" of G_p, i.e. $W = N(G_p)/G_p$, the quotient of the normalizer of G_p by G_p.

d) The injectivity of $H^*(B(\mathbb{Z}/p)^k; \mathbb{Z}/p)$ has been crucial in most of the recent progress concerning the homotopy theory of classifying spaces of finite groups, e.g. H. Millers solution of the Sullivan conjecture [Mi] and J. Lannes recent work on classifying spaces of elementary abelian groups [La]. Their work motivated these investigations.

e) The theorem implies that the following statements are equivalent for a finite group G and $l = 1$.

(1) $p^l \cdot H^n(BG; \mathbb{Z}_{(p)}) = 0$ for all $n \gg 0$

(2) $p^l \cdot H^n(BG; \mathbb{Z}_{(p)}) = 0$ for all $n > 0$

(1) and (2) are also equivalent if $l = 0$, e.g. by the Evens–Venkov–Quillen Theorem ([Q, sect. 2]).

Question. Are (1) and (2) equivalent for all $l \geqq 0$?

The paper is organized as follows. In section 1 we prove that (1) implies (2) and in section 2 we show that (2) implies (3). In the latter section we need part of Lewis' computation [Le] of $H^*(G_3; \mathbb{Z})$ where G_3 is a nonabelian group of exponent p and order p^3. The full computation of this cohomology ring is quite involved and for the convenience of the reader we give an elementary argument for the part that we need in an appendix.

Acknowledgements

I would like to thank J. Lannes and S. Zarati for their interest and several discussions.

1. Proof of "(1) $\Rightarrow$ (2)"

For an A_p module M we define

$$H_n(M; \beta) = \text{Kern}\,(M_n \xrightarrow{\beta} M_{n+1})/\text{Im}\,(M_{n-1} \xrightarrow{\beta} M_n).$$

If A is an abelian group we denote by $\text{Tors}\,A$ its torsion subgroup. The proof of "(1) $\Rightarrow$ (2)" follows from

LEMMA 1. *Let $H^n(X; \mathbb{Z}_{(p)})$ be finitely generated over $\mathbb{Z}_{(p)}$. Then the following statements are equivalent*

(a) $p \cdot H^n(X; \mathbb{Z}_{(p)}) = 0$ *and* $p \cdot \text{Tors}\, H^{n+1}(X; \mathbb{Z}_{(p)}) = 0$

(b) $H_n(H^*(X; \mathbb{Z}/p); \beta) = 0$.

LEMMA 2. *If M is injective in $\mathcal{U}_p$, then $H_n(M;\beta)=0$ for all $n>1$.*

Proof of Lemma 1. This is straightforward using the fact that β is the first differential in the Bockstein exact couple (see [N, §4] e.g.). □

Proof of Lemma 2. (I owe the following elegant argument to J. Lannes.) We give the proof in case $p=2$. The case of an odd prime is analogous.

Let $F(n)$ be the free unstable module generated by an element i_n of degree n. $F(n)$ is characterized by the fact that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{U}_2}(F(n), M) &\to M_n \\ \varphi &\mapsto \varphi(i_n)\end{aligned} \tag{*}$$

is a natural isomorphism. $F(n)$ has an additive basis given by $\mathrm{Sq}^I i_n$ with I admissible and excess $e(I)\leqq n$ [S–E]. Using this basis it is easy to check that the sequence

$$F(n+1)\xrightarrow{d_n} F(n)\xrightarrow{d_{n-1}} F(n-1),$$

with d_n given by $d_n(i_{n+1})=\beta i_n$, is exact if $n>1$. Because M is injective it follows that

$$\mathrm{Hom}_{\mathcal{U}_2}(F(n+1), M)\xleftarrow{(d_n)^*}\mathrm{Hom}_{\mathcal{U}_2}(F(n), M)\xleftarrow{(d_{n-1})^*}\mathrm{Hom}_{\mathcal{U}_2}(F(n-1), M)$$

is exact if $n>1$. Via (*) this sequence may be identified with

$$M_{n+1}\xleftarrow{\beta} M_n\xleftarrow{\beta} M_{n-1}$$

and the Lemma is proved. □

2. Proof of "(2) ⇒ (3)"

Let H be a closed subgroup of G. Then the Evens–Venkov–Quillen Theorem (cp. [Q, sect. 2]) says that the restriction map makes $H^*(BH;\mathbb{Z}_{(p)})$ into a finitely generated module over $H^*(BG;\mathbb{Z}_{(p)})$. Consequently the property (*) "$p\cdot H^n(BG;\mathbb{Z}_{(p)})=0$ for $n\gg 0$" is inherited by all closed subgroups. In

particular, G cannot contain a torus, i.e. G has to be finite. Furthermore, (*) is inherited by the p-Sylow subgroup G_p and therefore it suffices to prove the following.

LEMMA 3. *Let G be a finite p-group such that G satisfies property (*). Then G is elementary abelian.*

Proof. (By induction on the order of G.) If G is nontrivial, then there is a short exact sequence

$$1 \to K \to G \to \mathbb{Z}/p \to 1.$$

K satisfies property (*), so by induction hypothesis $K \cong (\mathbb{Z}/p)^k$ for some k.

Because (*) is inherited by all subgroups, G cannot contain a subgroup $\mathbb{Z}/p^2$. Therefore each element of G is of order p. If $p = 2$ this implies already that G is elementary abelian and if p is odd we derive that the exact sequence above splits and G is completely determined by the conjugation action of a preimage c of a generator of $\mathbb{Z}/p$. We have to show that this action is trivial. Then G is abelian and the Lemma follows.

So suppose that c acts nontrivially. Because K is a $\mathbb{Z}/p$-vector space we have $(c - id)^p = c^p - id = 0$. Now, since the action is nontrivial, $\operatorname{Ker}(c - id)$ is strictly included in $\operatorname{Ker}(c - id)^2$, so there exists a two dimensional invariant subspace K_2 of K with basis $\{e_1, e_2\}$ and $ce_1 = e_1$, $ce_2 = e_1 + e_2$.

Therefore the corresponding semidirect product G_3 of K_2 and $\mathbb{Z}/p$ is a subgroup of G. However, by Lewis' computation [Le] we know that $p \cdot H^n(BG_3; \mathbb{Z}_{(p)}) \neq 0$ whenever $2p$ divides n. So G_3 does not satisfy (*), hence it cannot be a subgroup of G and we arrive at a contradiction. □

3. Appendix

PROPOSITION (cp. Cor. 6.27 in [Le]). *Let p be an odd prime and G be the semidirect product of $K = \mathbb{Z}/p \oplus \mathbb{Z}/p$ and $\mathbb{Z}/p \cdot c$ with c acting on K via the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then there is an element $v \in H^{2p}(BG; \mathbb{Z})$ such that $pv^n \neq 0$ for all n.*
(Note that for a p-group one always has $H^*(BG; \mathbb{Z}) \cong H^*(BG; \mathbb{Z}_{(p)})$ if $* > 0$.)

Proof. The Bockstein exact couple shows that it suffices to construct $v \in H^{2p}(BG; \mathbb{Z})$ with the following property: if $\rho : H^*(\ ; \mathbb{Z}) \to H^*(\ ; \mathbb{Z}/p)$ denotes mod p reduction, then $\rho v^n \notin \operatorname{Im} \beta$. Furthermore the restriction homomorphism $i^* : H^*(BG; \mathbb{Z}/p) \to H^*(BK; \mathbb{Z}/p) = \mathbb{Z}/p[y_1, y_2] \otimes E(x_1, x_2)$ with $|y_i| = 2$, $|x_i| = 1$,

$\beta x_i = y_i$, factors through the invariants $H^*(BK; \mathbb{Z}/p)^{\mathbb{Z}/p \cdot c}$ where c acts on $H^*(BK; \mathbb{Z}/p)$ via the algebra homomorphism determined by

$$c(x_1) = x_1, \qquad c(x_2) = x_1 + x_2, \qquad c(y_1) = y_1, \qquad c(y_2) = y_1 + y_2.$$

So it suffices to construct v such that

$$i^* \rho v^n \notin \operatorname{Im} \beta \mid H^*(BK; \mathbb{Z}/p)^{\mathbb{Z}/p \cdot c}.$$

To obtain v we take a one dimensional complex representation γ of K whose first Chern class $c_1(\gamma)$ satisfies $\rho c_1(\gamma) = y_2$. Let $\gamma^!$ be the induced representation of G and define $v = c_p(\gamma^!)$. Then $i^* \gamma^! \cong \bigoplus_{\lambda=0}^{p-1} \gamma^{c^\lambda}$ by Mackey decomposition (cp. Chap. V, 16.10 in [H]), hence

$$\begin{aligned} i^* \rho v &= \rho i^* v = \rho(c_p(i^* \gamma^!)) \\ &= \rho\left(c_p\left(\bigoplus_{\lambda=0}^{p-1} \gamma^{c^\lambda}\right)\right) = \rho\left(\prod_{\lambda=0}^{p-1} c_1(\gamma^{c^\lambda})\right) \\ &= \prod_{\lambda=0}^{p-1} c^\lambda(y_2) = \prod_{\lambda=0}^{p-1} (y_2 + \lambda y_1), \end{aligned}$$

i.e.

$$i^* \rho v^n = z_p^n \quad \text{with} \quad z_p := \prod_{\lambda=0}^{p-1} (y_2 + \lambda y_1).$$

So it is enough to show that $z_p^n \notin \operatorname{Im} \beta \mid H^*(BK; \mathbb{Z}/p)^{\mathbb{Z}/p \cdot c}$. This is an immediate consequence of part (b) of the following.

LEMMA 4. a) $\mathbb{Z}/p[y_1, y_2]^{\mathbb{Z}/p \cdot c}$ *is the polynomial subalgebra* $\mathbb{Z}/p[y_1, z_p]$.

b) $H^*(BK; \mathbb{Z}/p)^{\mathbb{Z}/p \cdot c}$ *is a free module over* $\mathbb{Z}/p[y_1, z_p]$ *with generators* 1, x_1, $x_1 x_2$ *and* $y_2 x_1 - y_1 x_2$.

Proof. (A more general situation is treated in [Mu] but in our specific case one may give a direct proof as follows.)

a) It is obvious that y_1 and z_p are algebraically independent and that $\mathbb{Z}/p[y_1, z_p]$ is contained in the invariants. Conversely let $q = \lambda y_1^n + y_2 q'$, $\lambda \in \mathbb{Z}/p$, $q' \in \mathbb{Z}/p[y_1, y_2]$ be invariant. Then $y_2 q'$ is invariant and hence divisible by $c^i(y_2)$ for all i. Factoriality of $\mathbb{Z}/p[y_1, y_2]$ implies that $y_2 q' = z_p q''$ and that q'' is also invariant. Finally $|q''| < |q|$ and by induction on the degree we find $q \in \mathbb{Z}/p[y_1, z_p]$.

b) Again it is obvious that the submodule of $H^*(BK; \mathbb{Z}/p)$ described in (b) is free and contained in the invariants.

Conversely let $q = q_0 + q_1x_1 + q_2x_2 + q_3x_1x_2$ be invariant with $q_i \in \mathbb{Z}/p[y_1, y_2]$. Then

$$c(q) = c(q_0) + (c(q_1) + c(q_2))x_1 + c(q_2)x_2 + c(q_3)x_1x_2,$$

which shows that q_0, q_2 and q_3 have to be invariant. Therefore it suffices to show that $q_1x_1 + q_2x_2 = \bar{q}_1x_1 + \bar{q}_2(y_2x_1 - y_1x_2)$ with $\bar{q}_1$, $\bar{q}_2$ invariant.

First we see that

$$q_1y_1 + q_2y_2 = \beta(q_1x_1 + q_2x_2)$$

is invariant, so by (a)

$$q_1y_1 + q_2y_2 = \bar{q}_1y_1 + \lambda z_p^n$$

with $\bar{q}_1 \in \mathbb{Z}/p[y_1, z_p]$, $\lambda \in \mathbb{Z}/p$. Assume $\lambda \neq 0$; then $|q_2| \not\equiv 0 \pmod{2p}$ and invariance of q_2 yields $q_2 = q_2'y_1$, $q_2' \in \mathbb{Z}/p[y_1, z_p]$. It would follow that z_p^n is divisible by y_1 which is absurd. Therefore $q_1y_1 + q_2y_2 = \bar{q}_1y_1$, i.e. $q_1x_1 + q_2x_2 - \bar{q}_1x_1 \in \operatorname{Kern}\beta$. From Lemma 1 we know that $\operatorname{Kern}\beta = \operatorname{Im}\beta$ in $\tilde{H}^*(BK; \mathbb{Z}/p)$ and we deduce that $q_1x_1 + q_2x_2 - \bar{q}_1x_1 = \bar{q}_2(y_2x_1 - y_1x_2)$ with $\bar{q}_2 \in \mathbb{Z}/p[y_1, y_2]$. Now invariance of the left hand side implies, say by comparing coefficients of x_2, invariance of $\bar{q}_2$ and the Lemma is proved. □

REFERENCES

[A] J. F. Adams, *Infinite loop spaces*, Annals of Math. Studies *90* (1978).

[C] G. Carlsson, *G. B. Segal's Burnside ring conjecture for* $(Z/2)^k$, Topology *22* (1983), 83–103.

[H] B. Huppert, *Endliche Gruppen 1*, Berlin-Heidelberg-New York, Springer 1967.

[La] J. Lannes, *Sur la cohomologie modulo p des p-groupes abéliens élémentaires*, Proc. Durham Symposium on Homotopy Theory 1985, L.M.S., Camb. Univ. Press 1987.

[L-Z] J. Lannes and S. Zarati, *Sur les U-injectifs*, Annales de l'E.N.S. *19* (1986), 303–333.

[Le] G. Lewis, *The integral cohomology rings of groups of order* p^3, Trans. Amer. Math. Soc. *132* (1968), 501–529.

[Mi] H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Annals of Math. *120* (1984), 39–87.

[Mu] H. Mui, *Modular invariant theory and the cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo *22* (1975), 319–369.

[N] J. A. Neisendorfer, *Primary homotopy theory*, Memoirs Amer. Math. Soc. *232* (1980).

[Q] D. G. QUILLEN, *The spectrum of an equivariant cohomology ring:* I, Annals of Math. *94* (1971), 549–572.

[S–E] N. E. STEENROD and D. B. A. EPSTEIN, *Cohomology operations*, Annals of Math. Studies *50* (1962).

[S] R. SWAN, *The p-period of a finite group*, Illinois J. Math. *4* (1960), 341–346.

Hans-Werner Henn
Mathematisches Institut der Universität
Im Neuenheimer Feld 288
D-6900 Heidelberg
West-Germany

Received August 19, 1987