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KAM theory in configuration space

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Abstract. A new approach to the Kolmogorov–Arnold–Moser theory concerning the existence of invariant tori having prescribed frequencies is presented. It is based on the Lagrangian formalism in configuration space instead of the Hamiltonian formalism in phase space used in earlier approaches. In particular, the construction of the invariant tori avoids the composition of infinitely many coordinate transformations. The regularity results obtained are applied to invariant curves of monotone twist maps. The Lagrangian approach has been prompted by a recent study of minimal foliations for variational problems on a torus by J. Moser.

1. Introduction and results

In this paper we shall prove existence and regularity results of invariant tori having prescribed frequencies. For this purpose we use the Lagrangian formalism instead of the Hamiltonian formalism previously used. This leads to a considerable simplification of the existence proofs both from a conceptual and from a technical point of view. As outlined in the next paragraph the construction of invariant tori avoids in particular the familiar technique of infinitely many coordinate transformations. Instead a nonlinear functional equation is solved in a family of linear spaces. Moreover, we point out that an annoying analyticity assumption required so far for the unperturbed equation is removed by using a new technical device. On the other hand, our approach requires slightly more derivatives for the functional to start with. This is due to the fact that not the full algebraic structure of the problem is taken into account.

The Lagrangian approach has been prompted by the recent work of J. Moser [21], [23] on minimal solutions of variational problems on a torus which can be viewed as an extension of the Aubry–Mather theory [4], [7], [12], [13], [14], [16] to partial differential equations.

In order to describe the results we start with the Hamiltonian system

$$\dot{z} = J\,\nabla H(z) \tag{1.1}$$

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on $\mathbb{T}^n \times \mathbb{R}^n$ where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the n-torus and J denotes the skew symmetric matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

In the covering space we denote $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and assume that H is periodic with period 1 in the x-variables so that $H(x + j, y) = H(x, y)$ for $j \in \mathbb{R}^n$. The aim is to construct invariant tori for (1.1) with prescribed frequencies. To be more precise, for a given frequency vector $\omega \in \mathbb{R}^n$ we are trying to find an embedding.

$$w = (u, v) : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n \tag{1.2}$$

which maps the constant vector field $\dot{\xi} = \omega$ on the torus $\mathbb{T}^n$ into the given Hamiltonian vector field (1.1). This means that the solutions

$$\xi(t) = \psi^t(\xi) = \xi + \omega t$$

of $\dot{\xi} = \omega$ are mapped into solutions

$$z(t) = \varphi^t(w(\xi)) = w(\xi + \omega t) \tag{1.3}$$

of the Hamiltonian system (1.1). We also assume that u is a diffeomorphism of $\mathbb{T}^n$ satisfying $u(\xi + j) = u(\xi) + j$ for $j \in \mathbb{Z}^n$. In particular, the embedded torus $w(\mathbb{T}^n)$ is a graph in $\mathbb{T}^n \times \mathbb{R}^n$ and consists of quasiperiodic solutions of (1.1). Differentiating the identity (1.3) we obtain the nonlinear partial differential equation

$$Dw = J\,\nabla H \circ w \tag{1.4}$$

where D denotes the following linear first order partial differential operator with constant coefficients

$$D = \sum_{j=1}^{n} \omega_j\, \partial/\partial\xi_j \tag{1.5}$$

Hence $Dw = dw \cdot \omega$ represents the derivative of w in the direction of the frequency vector $\omega \in \mathbb{R}^n$. As a side remark we point out that (1.4) is the Euler equation of the variational principle defined by the functional

$$I(w) = \int_{\mathbb{T}^n} \left(\tfrac{1}{2}\langle -JDw, w\rangle - H \circ w\right) d\xi \tag{1.6}$$

for the embedding w. Indeed, one verifies readily that the gradient of I with respect to the L^2 inner product is given by

$$\nabla I(w) = -JDw - \nabla H \circ w \tag{1.7}$$

so that the critical points of I are precisely the solutions of (1.4). This variational principle is however highly degenerate and has so far not been used for existence proofs quite in contrast to the analogous variational principle for periodic solutions for which we refer to [6] and [8]. In fact, without further conditions on H global critical points of I cannot be expected.

The breakthrough in the existence problem for invariant tori came in the sixties with the development of the famous KAM theory which considers the local perturbation problem for equation (1.4). More precisely, we assume that the Hamiltonian differential equation (1.1) is close to an integrable system which in our case means that

$$H(x, y) = H^0(y) + \varepsilon H^1(x, y; \varepsilon)$$

with a small parameter ε and that the unperturbed system satisfies the nondegeneracy condition

$$\det H^0_{yy} = 0.$$

Under these hypotheses KAM theory asserts that for ε sufficiently small and H sufficiently smooth there exists an abundance of invariant tori corresponding to those frequency vectors $\omega \in \mathbb{R}^n$ which are rationally independent and satisfy, in addition, the Diophantine conditions

$$|j \cdot \omega| \geq \gamma\, |j|^{-\tau}, \qquad 0 \neq j \in \mathbb{Z}^n, \tag{1.8}$$

for two fixed constants $\gamma > 0$ and $\tau \geq n - 1$. These invariant tori continue the ones which for $\varepsilon = 0$ are given by the trivial embeddings

$$w(\xi) = (u(\xi), v(\xi)) = (\xi, y)$$

where $\omega = \nabla H^0(y)$ is the prescribed frequency vector for the corresponding torus. For this theory we refer to Kolmogorov [10], [11], Arnold [1], [2], [3] and Moser [18], [19]. Subsequently their work gave rise to many papers and results among which we mention [5], [9], [26], [27], [28], [29], [30], [31].

We point out that a perturbation theory for invariant tori having rationally independent frequency vectors which do not meet the diophantine conditions (1.8) cannot be expected in the differentiable case. Indeed, quite recently J. Mather [17] proved for the special case of monotone twist maps that every invariant curve whose rotation number is only of Liouville type can be destroyed by a C^∞ perturbation which is arbitrarily small. For rational rotation numbers this was already known to Poincaré. We also recall that a differentiable solution of (1.4) necessarily requires an excessive number of derivatives for H. For this subtle phenomenon we refer to M. Herman [9] and the literature therein. Finally, examples in [15] show that for sufficiently large ε the smooth tori might disappear too.

In order to reformulate the existence problem for a single invariant torus in the Lagrangian framework we make use of the well known fact that under the nondegeneracy condition

$$\det H_{yy} = 0$$

the Hamiltonian differential equation (1.1) can be transformed into the variational problem

$$\int F(x, \dot{x})\, dt$$

where the Lagrangian $F(x, p)$ on $\mathbb{T}^n \times \mathbb{R}^n$ is related to the Hamiltonian $H(x, y)$ by

$$H(x, y) + F(x, p) = y \cdot p, \qquad p = H_y(x, y), \qquad y = F_p(x, p).$$

It is, of course, well known that every solution $z(t) = (x(t), y(t))$ of the Hamiltonian differential equation (1.1) corresponds to a solution $x(t)$ of the Euler–Lagrange equations

$$d/dt F_p(x, \dot{x}) = F_x(x, \dot{x}). \tag{1.9}$$

In particular, it follows that if the embedding w given by (1.2) satisfies (1.4) then for every $\xi \in \mathbb{R}^n$ the function $x(t) = u(\xi + \omega t)$ is a solution of (1.9). Thus we are looking for a diffeomorphism $u:\mathbb{T}^n \to \mathbb{T}^n$ such that $u(\xi)-\xi$ is of period 1 in all variables and the following nonlinear partial differential equation is satisfied

$$DF_p(u, Du) = F_x(u, Du). \tag{1.10}$$

Conversely, every solution u of (1.10) determines an embedding $w=(u, v)$ which satisfies (1.4) if we define

$$v = F_p(u, Du).$$

We point out that (1.10) is the Euler equation of a variational principle for diffeomorphisms of $\mathbb{T}^n$. Namely defining the functional

$$I(u) = I_F(u) = \int_{\mathbb{T}^n} F(u, Du)\, d\xi \tag{1.11}$$

for diffeomorphisms u one verifies readily that the L^2-gradient of I is given by $\nabla I(u) = -E(F, u)$ where

$$E(F, u) = DF_p(u, Du) - F_x(u, Du) \tag{1.12}$$

so that the solutions of (1.10) are indeed the extremals of the variational problem (1.11). This variational problem for invariant tori was first mentioned by Percival [24], [25] and has been used for numerical purposes. In the case of two degrees of freedom and under the additional Legendre condition

$$F_{pp} > 0 \tag{1.13}$$

such a variational principle can be used for the existence theory of Mather sets [7], [12], [13]. So far it has however not been used for existence proofs in the higher dimensional case $n \geq 2$ where there is indeed no global existence theory for equation (1.10). In the following we do not impose condition (1.13).

Our aim is to solve the Euler equation (1.10) as a perturbation problem. Assuming the existence of a reference solution $E(F^0, u^0)=0$ we are looking for a solution u of $E(F, u)=0$ for a given Lagrangian F near F^0 and a fixed frequency vector $\omega \in \mathbb{R}^n$. This requires a stability condition on the pair F^0, u^0.

DEFINITION. The pair (F, u) is called *stable* if the matrix function $a(\xi)$ on $\mathbb{T}^n$ defined by

$$a(\xi) = U^T F_{pp}(u, Du) U, \qquad U = u_\xi,$$

satisfies

$$\det a(\xi) \neq 0, \qquad \xi \in \mathbb{R}^n,$$

$$\det \int_{\mathbb{T}^n} a(\xi)^{-1}\, d\xi \neq 0.$$

In the following the frequency vector $\omega \in \mathbb{R}^n$ entering the definition of the operator D is fixed and assumed to satisfy the Diophantine conditions (1.8). Moreover, we shall denote by C^l for $l \notin \mathbb{Z}$ the space of Hölder functions.

THEOREM. *Let (F^0, u^0) be a stable pair satisfying $E(F^0, u^0) = 0$ and suppose that $F^0 \in C^l$ and $u^0 \in C^{l+1}$ where $l = 4\tau + 2 + \mu$ for some constant $\mu > 0$ and $l - 2\tau - 2$ is not an integer. Moreover, let us define the neighbourhood*

$$B_\delta = \{F \in C^l \mid |F - F^0|_{C^l} \leq \delta\}.$$

Then there exists a constant

$$\delta = \delta(\gamma, \tau, a^0, |F^0|_{C^l}, |u^0|_{C^{l+1}}) > 0$$

such that for every $F \in B_\delta$ there exists a $C^{l-2\tau-2}$ diffeomorphism u of $\mathbb{T}^n$ solving the equation

$$E(F, u) = 0.$$

Moreover u, Du, $D^2u \in C^{2\tau+\mu}$ and the solution is (in this class) locally unique up to translation in $\mathbb{T}^n$. It depends continuously on F in the C^s topology for $s < l - 2\tau - 2$. Moreover, if F is of class C^m with $m \geq l$ and $m - 2\tau - 2$ is not an integer then u is of class $C^{m-2\tau-2}$. In particular, $F \in B_\delta \cap C^\infty$ implies $u \in C^\infty$. Finally, if $F \in B_\delta$ is real analytic then u is real analytic.

For a quantitative version of a stronger result we refer to section 4 and section 5. Moreover, note that the reference solution $E(F^0, u^0) = 0$ need not come from an integrable system and that neither F^0 nor u^0 are assumed to be real analytic.

It is an immediate consequence of the above theorem (in the formulation of section 4) that the invariant torus represented by u^0 is not isolated. It is a cluster point of other invariant tori for F^0 corresponding to frequencies which are close to ω and satisfy the same Diophantine conditions (1.8).

We illustrate the theorem by the simple example

$$F(x, p) = f(p) + \varepsilon V(x),$$

where V is of period 1 in all variables. For $\varepsilon = 0$ we have $E(F^0, id) = 0$ and, moreover, the pair (F^0, id) is stable provided that

$$\det f_{pp}(\omega) \neq 0.$$

Therefore, if $V \in C^l$ with $l > 4\tau + 2$ and if ε is sufficiently small then the equation

$$D\, \nabla f(Du) = \varepsilon\, \nabla V(u) \tag{1.14}$$

has a solution $u \in C^{l-2\tau-2}$ which is a diffeomorphism of $\mathbb{T}^n$ close to the identity. Moreover, up to translation the solution u is locally unique.

The above existence theorem together with a local uniqueness result will be used in section 5 in order to derive the following regularity theorem for invariant tori.

THEOREM. *Let (F, u) be a stable pair satisfying $E(F, u) = 0$ and suppose that u is of class C^l with $l > 4\tau + 3$. Then $F \in C^\infty$ implies $u \in C^\infty$. Moreover, if F is real analytic then so is u.*

Observe that the smoothness assumption on u agrees with the smoothness required for the continuation of u under a perturbation of F.

As an application of the above theorem we shall derive in section 6 a regularity result for invariant curves

$$\psi : S^1 \to S^1 \times \mathbb{R}$$

of an exact symplectic C^∞ diffeomorphism

$$\Phi : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}.$$

We shall assume that ψ is a solution of the nonlinear difference equation

$$\psi(\xi + \alpha) = \Phi \circ \psi(\xi)$$

with an irrational rotation number α satisfying the Diophantine inequalities

$$|p + \alpha q| \geq \gamma\, |q|^{-\tau}$$

for all integers $p \in \mathbb{Z}$ and $q > 0$ and some constants $\gamma > 0$ and $\tau = 1 + \varepsilon$ with $\varepsilon \geq 0$.

We shall prove that if $\psi \in C^l$ with $l > 7 + 4\varepsilon$ and $\Phi \in C^\infty$ satisfies a monotone twist condition then ψ must be of class C^∞.

The proof makes use of the observation due to Moser [19] that Φ can be interpolated by a smooth Hamiltonian vector field depending periodically on time and satisfying in addition the Legendre condition

$$H_{yy}(t, x, y) > 0.$$

In section 6 we will in fact prove a stronger statement where the number of derivatives needed is reduced to

$$l > 5 + 2\varepsilon.$$

This, however, also requires a stronger existence and uniqueness theorem for invariant tori with only $l > 2\tau + 3$ derivatives. Such a result can so far only be proved by using the Hamiltonian approach in phase space involving the composition of infinitely many coordinate transformations [20], [26], [27], [30].

2. Outline of the proofs

The existence proof is based upon a quantitative iteration technique of Newton type in a family of linear spaces. The method is familiar in KAM theory and was invented in order to overcome the socalled small divisor difficulty. We rely on the analytic smoothing technique introduced by Moser [19], [20] and abstracted in [31] in order to prove the existence result in the differentiable case. It turns out that the functional equation meets the assumptions of the abstract implicit function theorem in [31]. This theorem will however not be applied directly since a minor but very crucial technical modification already used in [30] allows to weaken the smoothness requirements for the unperturbed equation and this plays a central role in the proof of the regularity theorem.

In order to describe the method in more detail let us first recall the idea of the Newton iteration. Assuming $E(u) = E(F, u)$ to be small one seeks a correction term v such that $u + v$ is a better approximation of the desired solution and $E(u + v)$ is closer to zero. From the Taylor formula

$$E(u + v) = E(u) + dE(u)v + R(u; v),$$

with

$$dE(u)v = d/d\varepsilon E(u + \varepsilon v)|_{\varepsilon=0},$$

and the remainder term satisfying a quadratic estimate in v one concludes that

$$|E(u+v)| = O(|E(u)|^2)$$

provided that the linear equation

$$E(u) + dE(u)v = 0 \tag{2.1}$$

can be solved for v with suitable estimates. Now the linearized operator is computed to be

$$dE(u)v = D(F_{pp}Dv) + (F_{px} - F_{xp})Dv + (DF_{px} - F_{xx})v \tag{2.2}$$

where $F_{pp} = F_{pp}(u, Du)$ is to be understood as a function of ξ, the argument of u; similarly for the other terms.

One of the difficulties in solving equation (2.1) comes from the operator D. Due to the small divisors, entering the representation of this operator with respect to the Fourier expansion of functions on $\mathbb{T}^n$, its inverse is unbounded. More precisely, we have the following well known estimate in the space W_r of real holomorphic functions in the complex strip $|\operatorname{Im} x| \le r$, $x \in \mathbb{C}^n$, which are periodic with period 1 in all variables. The norm is denoted by

$$|f|_r = \sup\{|f(x)| \mid x \in \mathbb{C}^n, |\operatorname{Im} x| \le r\}.$$

LEMMA 1. *Let $\omega \in \mathbb{R}^n$ satisfy the Diophantine conditions*

$$|j \cdot \omega| \ge \gamma\, |j|^{-\tau}, \qquad 0 \ne j \in \mathbb{Z}^n,$$

for some constants $\gamma > 0$ and $\tau \ge n-1$. Then for every $g \in W_r$ with mean value zero the equation $Du = g$ has a unique solution $u \in W_\rho$ for $\rho < r$ with mean value zero. Moreover, u satisfies the estimate

$$|u|_\rho \le c_0 (r-\rho)^{-\tau}\, |Du|_r, \qquad \rho < r, \tag{2.3}$$

with $c_0 = c/\gamma$ and a suitable constant $c = c(\tau, n) > 0$.

Existence and uniqueness for Hölder functions: assume $g \in C^l(\mathbb{T}^n)$ with $l > \tau$ has mean value zero. Then there exists a unique $u \in L^2(\mathbb{T}^n)$ having mean value zero and satisfying

$$-\int_{\mathbb{T}^n} u \cdot D\varphi \, d\xi = \int_{\mathbb{T}^n} g \cdot \varphi \, d\xi, \qquad \varphi \in C^\infty(\mathbb{T}^n).$$

Moreover, $u \in C^{l-\tau}(\mathbb{T}^n)$ *and*

$$|u|_{C^{l-\tau}} \leq c\ |g|_{C^l} \tag{2.4}$$

with a constant $c = c(n, \tau, \gamma, l) > 0$ *provided that* $l - \tau$ *is not an integer.*

For the proof of the first statement we refer to Rüssmann [29]. The second statement is readily deduced approximating Hölder functions by holomorphic functions using Lemma 3 and 4 below.

Lemma 1 shows that there are two more obstacles for solving equation (2.1). First we have to eliminate the terms of order zero and one in equation (2.2) and then the remaining second order partial differential equation requires a compatibility condition, namely that the inhomogeneous term in the equation be of mean value zero. In order to overcome these difficulties let us first consider the very special case that $u = id$ is an approximate solution of $E(F, id) = 0$. Then it follows from direct considerations that the coefficient matrices of v and Dv in (2.2) are small and can be neglected. In fact the coefficient matrix of v is precisely the Jacobian of $E(F, id)$ and for the Dv-term we refer to statement (ii) in Lemma 2 below in connection with the estimate in Lemma 1. Therefore equation (2.1) can be replaced by

$$D(F_{pp}Dv) = -E(F, id)$$

and, by Lemma 1, this equation can indeed be solved since $E(F, id)$ must always be of mean value zero. We conclude that if $u = id$ is an approximate solution of (1.7) then (2.1) admits an approximate solution v so that the first step of the Newton iteration can be performed in this case. Now the following observation allows us to reduce the general case to the one where u is the identity on $\mathbb{T}^n$.

Abstractly speaking, the group of diffeomorphisms u of the n-torus $\mathbb{T}^n$ acts contravariantly on the space of Lagrangians F by means of the operation

$$u^*F(x, p) = F(u(x), U(x)p)$$

where $U(x) \in \mathbb{R}^{n\times n}$ denotes the Jacobian of u and is therefore of period 1 in all variables. One verifies easily that indeed

$$(u \circ v)^*F = v^*u^*F, \qquad id^*F = F$$

for two diffeomorphisms u and v. The functional (1.11) is compatible with this group action in the sense that

$$I_F(u \circ v) = I_{u^*F}(v) \tag{2.5}$$

and, moreover, it is invariant under the subgroup of translations. Differentiating equation (2.5) with respect to v and recalling that the L^2 gradient of I_F is given by

$$\nabla I_F(u) = -E(F, u)$$

one finds that

$$U^T \circ vE(F, u \circ v) = E(u^*F, v). \tag{2.6}$$

In order to reduce equation (2.1) to the case of the identity transformation on $\mathbb{T}^n$ we again differentiate equation (2.6) with respect to v in the direction of a tangent vector $w: \mathbb{T}^n \to \mathbb{R}^n$ to the group of torus diffeomorphisms at $v = id$ and obtain

$$U^T dE(F, u)Uw = dE(u^*F, id)w - (dU \cdot w)^T E(F, u). \tag{2.7}$$

Moreover, note that (2.6) with $v = id$ reduces to the identity

$$U^T E(F, u) = E(u^*F, id) \tag{2.8}$$

which will be frequently used later on. Of course, this equation can also be verified directly by an easy computation.

It now follows from (2.8) that whenever u is an approximate solution of $E(F, u) = 0$ then also $E(u^*F, id)$ is small and hence the above considerations about the case $u = id$ show that there exists an approximate solution w of the equation $dE(u^*F, id)w = -E(u^*F, id)$. Combining this observation with the identities (2.7) and (2.8) we conclude that equation (2.1) indeed has an approximate solution $v = Uw$ in the sense that errors of quadratic order are ignored. In this context we point out that multiplication with the Jacobian of u naturally transforms a tangent vector w to the group of torus diffeomorphisms at the identity into a tangent vector at u. Abstractly speaking we have used the Lie group structure of the torus diffeomorphisms Diff $(\mathbb{T}^n)$ and transformed equation (2.1) from the tangent space at u to the tangent space at the identity element namely the Lie algebra $X(\mathbb{T}^n)$ of vectorfields on $\mathbb{T}^n$.

In order to derive precise estimates for the approximate solution of the linearized equation (2.1) we summarize some consequences of the above considerations in the lemma below which is crucial for our approach. There we make use of the abbreviations

$$F_{p\xi} = \partial/\partial\xi F_p(u(\xi), Du(\xi)),$$
$$E_\xi = \partial/\partial\xi E(F, u),$$
$$U = u_\xi(\xi).$$

LEMMA 2

(i) $U^T dE(F, u)Uw = D(aDw) + bDw + cw$

where a, b and c are the following $n \times n$ matrix valued functions on $\mathbb{T}^n$

$$a = U^T F_{pp} U, \qquad b = U^T F_{p\xi} - F_{p\xi}^T U, \qquad c = U^T E_\xi.$$

(ii) $Db = c - c^T$.

(iii) $\int_{\mathbb{T}^n} b \, d\xi = 0.$

(iv) $\int_{\mathbb{T}^n} U^T E(F, u) \, d\xi = 0.$

Proof. Statement (i) follows from equation (2.7) by inserting the expression (2.2) with F and v replaced by u^*F and w, respectively.

Statement (ii) can be verified by a direct computation which we leave to the reader. Observe, however, that (ii) expresses the fact – well known in variational theory – that the operator

$$Lw = D(aDw) + bDw + cw$$

which represents the Hessian of the functional (1.11) is formally self adjoint. Indeed, since $a^T = a$ and $b^T = -b$, the formal adjoint operator of L given by

$$\begin{aligned} L^*w &= D(a^T Dw) - D(b^T w) + c^T w \\ &= D(aDw) + bDw + (c^T + Db)w \end{aligned}$$

so that $L^* = L$ if and only if $c^T + Db = c$ as claimed.

Statement (iv) simply reflects the fact that the functional $I(u)$ defined by (1.11) is invariant under the subgroup of translations of the n-torus $\mathbb{T}^n$. Indeed, this implies that $\nabla I(id) = -E(F, id)$ is orthogonal to the corresponding subalgebra $X^0(\mathbb{T}^n)$ of constant vectorfields for any Lagrangian F. Choosing the Lagrangian u^*F the statement follows from (2.8). Alternatively, statement (iii) can be verified directly using partial integration

$$\begin{aligned}\int_{\mathbb{T}^n} U^T E(F, u)\, d\xi &= \int_{\mathbb{T}^n} (U^T DF_p - U^T F_x)\, d\xi \\ &= -\int_{\mathbb{T}^n} (DU^T F_p + U^T F_x)\, d\xi \\ &= -\int_{\mathbb{T}^n} \partial/\partial\xi F(u, Du)\, d\xi \\ &= 0.\end{aligned}$$

A similar argument can be used to establish statement (iii). We will however give an interpretation of this identity in terms of the embedding $w = (u, v)$ of $\mathbb{T}^n$ into $\mathbb{T}^n \times \mathbb{R}^n$ defined by $v = F_p(u, Du)$. Observing that

$$b = u_\xi^T v_\xi - v_\xi^T u_\xi$$

we obtain that the pullback of the standard exact symplectic 2-form $d\lambda$ on $\mathbb{T}^n \times \mathbb{R}^n$ with

$$\lambda = \sum_{j=1}^{n} y_j\, dx_j$$

is given by

$$w^*(d\lambda) = \sum_{i,j=1}^{n} b_{ij}\, d\xi_i \wedge d\xi_j.$$

This 2-form is exact since $w^*(d\lambda) = d(w^*\lambda)$ which implies that

$$b_{ij}(\xi) = \partial f_i/\partial\xi_j - \partial f_j/\partial\xi_i$$

for some function $f: \mathbb{T}^n \to \mathbb{R}^n$ so that indeed the integral of b over $\mathbb{T}^n$ vanishes. This finishes the proof of Lemma 2. □

It follows from Lemma 2 that if u is a solution of $E(F, u) = 0$ and the frequency vector $\omega \in \mathbb{R}^n$ is rationally independent then $c = 0$ and $b = 0$. In fact, since $Db = 0$ the function $b(\xi)$ is constant along the dense line $\xi = \omega t$. Hence it is constant on $\mathbb{T}^n$ and it follows from Lemma 2(iii) that $b = 0$. In view of the remarks in the proof of Lemma 2 the condition $b = 0$ reflects the fact that the embedded torus $w(\mathbb{T}^n)$ is Lagrangian which means that $w^* \, d\lambda = 0$. Thus we have reestablished the well known fact that every solution w of (1.4) defines a Lagrangian invariant torus if the frequencies are rationally independent. As a particular consequence the linearized operator is given by

$$U^T dE(F, u)Uw = D(aDw)$$

in the case $E(F, u) = 0$. As we shall see later this operator can be inverted provided that the pair (F, u) is stable.

Returning to the Newton iteration we shall now replace the linearized equation (2.1) by

$$U^T E(u) + D(aDw) = 0$$

ignoring the terms of order zero and one. As a consequence of Lemma 1 and Lemma 2 this equation has a unique solution w of mean value zero provided that (F, u) is a stable pair as defined in the introduction. Moreover, in the analytic case w satisfies an estimate of the form

$$|w|_\rho \le K(r - \rho)^{-2\tau} \, |E(u)|_r$$

with a constant K which is independent of ρ, r and u. Multiplying the Taylor formula for E with U^T and inserting $v = Uw$ one finds that

$$U^T E(u + v) = bDw + cw + U^T R(u; Uw).$$

Now it follows from Lemma 2 and Lemma 1 that both c and b can be estimated by $|E(u)|$. Combining this with the above inequality for w one concludes that $E(u + v)$ satisfies a quadratic estimate

$$|E(u + v)|_\rho \le K(r - \rho)^{-4\tau} \, |E(u)|_r^2.$$

This suggests a modified Newton iteration in a family of spaces

$$W_{r_\nu}, \qquad r_\nu = r(1 + 2^{-\nu})/2.$$

Starting with an approximate solution $u_0 \in W_r$ such that

$$|E(u_0)|_r \leq \delta r^{4\tau} \tag{2.9}$$

one constructs recursively a sequence $u_\nu \in W_{r_\nu}$ by

$$u_{\nu+1} = u_\nu + U_\nu w, \qquad D(a_\nu D w) = -U_\nu^T E(u_\nu).$$

Then it follows from (2.9) in connection with the quadratic estimate above that the functions u_ν converge in the region $|\mathrm{Im}\, \xi| \leq r/2$ to a solution u of $E(u) = 0$ provided that $\delta > 0$ has been chosen sufficiently small. Moreover, this solution satisfies the estimate

$$|u - u_0|_{r/2} \leq c r^{-2\tau} |E(u_0)|_r \tag{2.10}$$

with a constant c which is independent of r. This summarizes the existence proof in the analytic case. It will be carried out in detail in section 3.

In order to prove the existence result in the differentiable case we shall apply an analytic smoothing technique invented by Moser [19], [20]. It is based on the observation that the Hölder spaces $C^l(\mathbb{R}^n)$ can be characterized in terms of their approximation properties by holomorphic functions. More precisely, for $l = k + \mu$ with k an integer and $0 < \mu < 1$ we denote by $C^l(\mathbb{R}^n)$ the space of k-times continuously differentiable functions f with

$$|f|_{C^l} = |f|_{C^k} + |f|_{k,\mu} < \infty$$

where

$$|f|_{k,\mu} = \sup |\partial^\alpha f(x) - \partial^\alpha f(y)| / |x - y|^\mu$$

the supremum being taken over all $\alpha \in \mathbb{Z}^n$ with $|\alpha| = k$ and all $x, y \in \mathbb{R}^n$ with $0 < |x - y| \leq 1$.

LEMMA 3. *There is a family of convolution operators*

$$S_r f = r^{-n} \int_{\mathbb{R}^n} K(r^{-1}(x - y)) f(y)\, dy, \qquad 0 < r \leq 1,$$

from $C^0(\mathbb{R}^n)$ into the linear space of entire functions on $\mathbb{C}^n$ such that for every $l > 0$ there exist a constant $c = c(l) > 0$ with the following properties. If $f \in C^l(\mathbb{R}^n)$ then *for* $|\alpha| \leq l$ *and* $|\operatorname{Im} x| \leq r$

$$\left|\partial^\alpha S_r f(x) - \sum_{|\beta| \leq l-|\alpha|} \partial^{\alpha+\beta} f(\operatorname{Re} x)(i \operatorname{Im} x)^\beta/\beta!\right| \leq c\, |f|_{C^l} r^{l-|\alpha|} \tag{2.11}$$

and in particular for $\rho \leq r$

$$|\partial^\alpha S_r f - \partial^\alpha S_\rho f|_\rho \leq c\, |f|_{C^l} r^{l-|\alpha|}. \tag{2.12}$$

Moreover, in the real case

$$|S_r f - f|_{C^s} \leq c\, |f|_{C^l} r^{l-s}, \qquad s \leq l,$$
$$|S_r f|_{C^s} \leq c\, |f|_{C^l} r^{l-s}, \qquad l \leq s. \tag{2.13}$$

Finally, if f is periodic in some variables then so are the approximating functions $S_r f$ *in the same variables.*

For the proof of this result as well as Lemma 4 we refer to [19], [30], [31]. Moreover we point out that from (2.13) one can easily deduce the following well known convexity estimates which will be frequently used later on

$$|f|_{C^m}^{l-k} \leq c\, |f|_{C^k}^{l-m} |f|_{C^l}^{m-k}, \qquad k \leq m \leq l, \tag{2.14}$$
$$|f \cdot g|_{C^s} \leq c(|f|_{C^s} |f|_{C^0} + |f|_{C^0} |g|_{C^s}), \qquad s \geq 0. \tag{2.15}$$

As a partial converse of Lemma 3 we shall need

LEMMA 4. *Assume that* $f: \mathbb{R}^n \to \mathbb{R}$ *is the limit of a sequence of real analytic functions* $f_\nu(x)$ *defined in the complex strip* $|\operatorname{Im} x| \leq r_\nu = 2^{-\nu} r_0$, $x \in \mathbb{C}^n$, *with* $0 < r_0 \leq 1$ *and*

$$|f_\nu(x) - f_{\nu-1}(x)| \leq A r_\nu^l, \qquad |\operatorname{Im} x| \leq r_\nu.$$

Then $f \in C^s(\mathbb{R}^n)$ *for every* $s \leq l$ *which is not an integer and moreover*

$$|f - f_0|_{C^s} \leq cA(\theta(1-\theta))^{-1} r_0^{l-s}$$

for $0 < \theta = s - [s] < 1$ *and a suitable constant* $c = c(l, n) > 0$.

The existence result of invariant tori for differentiable Lagrangians as formulated in the introduction can now be proved as follows.

We first observe that the algebraic identity (2.8) allows us to reduce the existence theorem to the case $u^0 = id$. For a given function $F \in C^l(\mathbb{T}^n \times \mathbb{R}^n)$ with $l = 4\tau + 2 + \mu$ we may therefore assume that $E(F, id)$ is sufficiently small. In order to solve the equation $E(F, u) = 0$ we shall now proceed as follows.

Using Lemma 3 we approximate F by a sequence of real analytic Lagrangians

$$F_\nu \to F.$$

The functions F_ν are chosen as to satisfy the estimates of Lemma 3 in the decreasing complex strips $|\mathrm{Im}\, x| \le 8r_\nu$, $|\mathrm{Im}\, p| \le 8r_\nu$, where

$$r_\nu = 2^{-\nu} r_0 \tag{2.16}$$

and the small number $r_0 > 0$ is defined by

$$|E(F, id)|_{C^0} = r_0^{4\tau+\mu}. \tag{2.17}$$

We then construct inductively a sequence of real analytic diffeomorphisms u_ν of $\mathbb{T}^n$ in $|\mathrm{Im}\, \xi| \le r_\nu$ solving

$$E(F_\nu, u_\nu) = 0 \tag{2.18}$$

and, in addition, satisfying suitable estimates. In the first step of the iteration, which is crucial for the regularity statement, we make use of (2.11) together with (2.14) and the definition of r_0 in (2.17) in order to show that the pair (F_0, id) satisfies the following estimate in the complex strip $|\mathrm{Im}\, x| \le 2r_0$

$$|E(F_0, id)|_{2r_0} \le c\, |F|_{C^l}\, r_0^\mu (2r_0)^{4\tau} \tag{2.19}$$

The point now is to choose $E(F, id)$ so small that the width r_0 defined by (2.17) satisfies $c\, |F|_{C^l}\, r_0^\mu \le \delta$ and hence the assumption (2.9) of the analytic existence theorem is satisfied in the strip $|\mathrm{Im}\, x| \le 2r_0$. We therefore find an analytic solution u_0 in $|\mathrm{Im}\, \xi| \le r_0$ of $E(F_0, u_0) = 0$. Having constructed a solution u_ν of (2.18) in $|\mathrm{Im}\, \xi| \le r_\nu$ we will then use the estimate (2.12) for $F_{\nu+1} - F_\nu$ in order to verify that $E(F_{\nu+1}, u_\nu)$ is sufficiently close to $E(F_\nu, u_\nu) = 0$ so that the pair $(F_{\nu+1}, u_\nu)$ meets the requirements of the analytic existence theorem in the complex strip $|\mathrm{Im}\, \xi| \le r_\nu$. This guarantees the existence of a solution $u_{\nu+1}$ of (2.18) in $|\mathrm{Im}\, \xi| \le r_\nu/2 = r_{\nu+1}$. In addition the inequality (2.10) of the analytic theorem shows that $u_{\nu+1}$ satisfies an estimate of the form

$$\begin{aligned}|u_{\nu+1} - u_\nu|_{r_{\nu+1}} &\le c r_\nu^{-2\tau} |E(F_{\nu+1}, u_\nu)|_{r_\nu} \\ &\le c r_\nu^{-2\tau-2} |F_{\nu+1} - F_\nu|_{4r_\nu} \\ &\le c r_\nu^{l-2\tau-2} |F|_{C^l}\end{aligned}$$

with a generic constant $c > 0$ which is independent of ν. Hence it follows from Lemma 4 that the analytic functions u_ν converge on $\mathbb{R}^n$ to a solution $u \in C^{l-2\tau-2}$ of

$$E(F, u) = 0.$$

The details of this argument will be carried out in section 4.

3. The analytic case

This section is devoted to the proof of the following quantitative existence result for invariant tori for analytic Lagrangians. We make use of the abbreviating notation

$$\|u\|_r = |u|_r + |Du|_r + |D^2u|_r$$

for bounded, real analytic functions $u(\xi)$ in $|\text{Im}\, \xi| \le r$.

THEOREM 1. *Let $\omega \in \mathbb{R}^n$ satisfy $|\omega| \le M$ and*

$$|j \cdot \omega| \ge \gamma\, |j|^{-\tau}, \qquad 0 \ne j \in \mathbb{Z}^n,$$

for some constants $M \ge 1$, $\gamma > 0$, $\tau \ge n - 1$ and let $F(x, p)$ be a real analytic function in the domain $|\text{Im}\, x| \le 2\lambda r$, $|\text{Im}\, p| \le 2\lambda r$ which is of period 1 in the x-variables and satisfies

$$|\partial^\alpha F|_{2\lambda r} \le M, \qquad |\alpha| \le 4, \tag{3.1}$$

with suitable constants $0 < r \le 1$ and $\lambda \ge 1$. Moreover, let $x = u_0(\xi)$ be a real analytic diffeomorphism of the n-torus defined in the region $|\text{Im}\, \xi| \le r$ such that $u_0(\xi) - \xi$ is of period 1 and

$$\|u_0 - id\|_r + \|U_0\|_r \le \lambda. \tag{3.2}$$

We assume that u_0 is a stable approximate solution in the sense that

$$|DF_p(u_0, Du_0) - F_x(u_0, Du_0)|_r \le \delta r^{4\tau},$$
$$|U_0^T F_{pp}(u_0, Du_0)U_0 - a|_r \le \delta \tag{3.3}$$

where $a(\xi)$ is an invertible $n \times n$ matrix in $|\mathrm{Im}\, \xi| \le r$ (not necessarily analytic) which is of period 1 and satisfies

$$|a(\xi)^{-1}| \le M, \qquad \left|\left(\int_{\mathbb{T}^n} a(\xi)^{-1}\, d\xi\right)^{-1}\right| \le M. \tag{3.4}$$

for $|\mathrm{Im}\, \xi| \le r$.

Then there exist constants $\delta^* = \delta^*(\gamma, \tau, M, \lambda, n) > 0$ and $c = c(\gamma, \tau, M, \lambda, n) \ge 8M^3$ such that $c\delta^* \le 1$ and the following statement holds. If $\delta \le \delta^*$ then there exists a real analytic torus diffeomorphism $x = u(\xi)$ mapping the strip $|\mathrm{Im}\, \xi| \le r/2$ into $|\mathrm{Im}\, u(\xi)| \le 2\lambda r$, $|\mathrm{Im}\, Du(\xi)| \le 2\lambda r$ such that $u(\xi) - \xi$ is of period 1 and

$$DF_p(u, Du) = F_x(u, Du). \tag{3.5}$$

Moreover, the pair (F, u) is stable and satisfies the estimates

$$\|u - u_0\|_{r/2} \le c\delta r^{2\tau},$$
$$\|U - U_0\|_{r/2} \le c\delta r^{2\tau - 1},$$
$$|U^T F_{pp}(u, Du)U - a|_{r/2} \le c\delta/4M^3. \tag{3.6}$$

Remark. If T and S are complex $n \times n$ matrices such that T is nonsingular and $|S - T|\,|T^{-1}| < 1$ then S is nonsingular and

$$|S^{-1} - T^{-1}| \le |T^{-1}|^2\, |S - T|\, (1 - |T^{-1}|\, |S - T|)^{-1}.$$

Hence it follows from (3.4) and (3.6) that the matrix $A(\xi) = U^T F_{pp}(u, Du)U$ satisfies the inequalities

$$|A^{-1} - a^{-1}|_{r/2} \le (c\delta/4M)(1 - c\delta/4M^2)^{-1} \le c\delta/2M,$$
$$\left|\left(\int_{\mathbb{T}^n} A^{-1}\, d\xi\right)^{-1} - \left(\int_{\mathbb{T}^n} a^{-1}\, d\xi\right)^{-1}\right| \le (Mc\delta/2)(1 - c\delta/2)^{-1} \le c\delta M.$$

In order to prove Theorem 1 we shall need the following two Lemmata.

LEMMA 5. *Let $\omega \in \mathbb{R}^n$ satisfy the Diophantine conditions of Theorem* 1 *and let $a(\xi) \in \mathbb{C}^{n\times n}$ and $g(\xi) \in \mathbb{C}^n$ be real analytic functions defined in the strip $|\mathrm{Im}\,\xi| < r \leq 1$ which are of period* 1 *in all variables. Moreover, assume that g is of mean value zero and that a satisfies*

$$|a(\xi)^{-1}| \leq M, \qquad \left|\left(\int_{\mathbb{T}^n} a(\xi)^{-1}\,d\xi\right)^{-1}\right| \leq M$$

for $|\mathrm{Im}\,\xi| < r$ with some constant $M \geq 1$.

Then there exists a unique real analytic function $w(\xi) \in \mathbb{C}^n$ in $|\mathrm{Im}\,\xi| < r$ which is of period 1 *with mean value zero and solves*

$$D(aDw) = g.$$

Moreover, w satisfies the estimate

$$|w|_\rho + |Dw|_\rho\,\gamma^{-1}(r-\rho)^{-\tau} \leq c_0\gamma^{-2}M^3(r-\rho)^{-2\tau}\,|g|_r$$

for $0 < \rho < r$ with a suitable constant $c_0 = c_0(\tau, n) > 0$.

Proof. By Lemma 1 there exists a unique real analytic solution $f(\xi) \in \mathbb{C}^n$, $|\mathrm{Im}\,\xi| < r$, of $Df = g$ which is of period 1 and mean value zero. Now we choose $\alpha \in \mathbb{R}^n$ such that $a^{-1}(f - \alpha)$ is of mean value zero and define the real analytic function $w(\xi) \in \mathbb{C}^n$ (again of period 1 and mean value zero) to be the unique solution of $Dw = a^{-1}(f - \alpha)$ so that $D(aDw) = g$. In order to derive the estimate for w we denote by $c = c(\tau, n)$ the constant of Lemma 1 and obtain

$$|f|_{(r+\rho)/2} \leq c\gamma^{-1}2^\tau(r-\rho)^{-\tau}\,|g|_r.$$

Now the identity

$$\int_{\mathbb{T}^n} a(\xi)^{-1}\,d\xi\,\alpha = \int_{\mathbb{T}^n} a(\xi)^{-1}f(\xi)\,d\xi$$

shows that

$$|\alpha| \leq c\gamma^{-1}M^2 2^\tau(r-\rho)^{-\tau}\,|g|_r$$

and hence

$$\begin{aligned}|a^{-1}(f-\alpha)|_{(r+\rho)/2} &\leq c\gamma^{-1}M(M^2+1)2^\tau(r-\rho)^{-\tau}\,|g|_r \\ &\leq c\gamma^{-1}M^3 2^{\tau+1}(r-\rho)^{-\tau}\,|g|_r.\end{aligned}$$

We conclude that w satisfies the estimate

$$\begin{aligned}|w|_\rho &\le c\gamma^{-1}2^\tau(r-\rho)^{-\tau}\,|Dw|_{(r+\rho)/2}\\ &\le c^2\gamma^{-2}M^3 2^{2\tau+1}(r-\rho)^{-2\tau}\,|g|_r\end{aligned}$$

and this proves Lemma 5. □

LEMMA 6. *Let $F(x, p)$ be a real analytic function defined in the region $|\mathrm{Im}\, x| \le R$, $x \in \mathbb{C}^n$, $|\mathrm{Im}\, p| \le R$, $p \in \mathbb{C}^n$, and satisfying*

$$|\partial^\alpha F|_R \le M, \qquad |\alpha| \le 4.$$

Moreover, let u and v be real analytic mappings of $\mathbb{C}^n$ such that $|\mathrm{Im}\, u(\xi)| + |\mathrm{Im}\, v(\xi)| \le R$ and $|\mathrm{Im}\, Du(\xi)| + |\mathrm{Im}\, Dv(\xi)| \le R$ for $|\mathrm{Im}\, \xi| \le r$. Then there exists a constant $c = c(R, M) > 0$ such that

$$|E(u+v) - E(u) - dE(u)v|_r \le c\,|v|_r^2\,(1 + |u|_r)$$

where $E(u) = E(F, u) = DF_p(u, Du) - F_x(u, Du)$.

Proof. The statement of the Lemma is an immediate consequence of Taylor's formula

$$\begin{aligned}
&E(u+v) - E(u) - dE(u)v\\
&\quad= DF_p(u+v, Du+Dv) - F_x(u+v, Du+Dv) - DF_p(u, Du) + F_x(u, Du)\\
&\qquad - D(F_{pp}(u, Du)Dv) - (F_{px}(u, Du) - F_{xp}(u, Du))Dv\\
&\qquad - (DF_{px}(u, Du) - F_{xx}(u, Du))v\\
&\quad= (F_{pp}(u+v, Du+Dv) - F_{pp}(u, Du))D^2v\\
&\qquad + (F_{px}(u+v, Du+Dv) - F_{px}(u, Du))Dv\\
&\qquad + (F_{pp}(u+v, Du+Dv) - F_{pp}(u, Du) - F_{ppp}(u, Du)Dv - F_{ppx}(u, Du)v)D^2u\\
&\qquad + (F_{px}(u+v, Du+Dv) - F_{px}(u, Du) - F_{pxp}(u, Du)Dv - F_{pxx}(u, Du)v)Du\\
&\qquad - (F_x(u+v, Du+Dv) - F_x(u, Du) - F_{xp}(u, Du)Dv - F_{xx}(u, Du)v). \quad \square
\end{aligned}$$

Proof of Theorem 1. We shall construct a sequence of real analytic torus diffeomorphisms $x = u_\nu(\xi)$, $\nu \in \mathbb{N}$, defined in the complex strip $|\mathrm{Im}\, \xi| \le r_\nu$ with

$$r_\nu = r(1 + 2^{-\nu})/2, \qquad r_0 = r,$$

such that $E(u_\nu) = E(F, u_\nu)$ converges to zero. These transformations are defined inductively by

$$u_{\nu+1} = u_\nu + U_\nu w$$

where $w(\xi)$ is of period 1 with mean value zero and satisfies

$$D(a_\nu Dw) = -U_\nu^T E(u_\nu), \qquad a_\nu = U_\nu^T F_{pp}(u_\nu, Du_\nu) U_\nu. \tag{3.7}$$

In each step of the iteration we shall prove that the pair (F, u_ν) is stable and satisfies the estimates

$$\|u_\nu - u_0\|_{r_\nu} \le c_1 \delta r^{2\tau} \le \lambda, \tag{3.8}$$

$$\|U_\nu - U_0\|_{r_\nu} \le c_1 \delta r^{2\tau - 1} \le \lambda, \tag{3.9}$$

$$|a_\nu - a|_{r_\nu} \le c_1 \delta / 4M^3, \tag{3.10}$$

with a suitable constant $c_1 = c_1(\gamma, \tau, M, \lambda, n) \ge 8M^3$. The remark after Theorem 1 shows that (3.10) implies

$$|a_\nu^{-1} - a^{-1}|_{r_\nu} \le c_1 \delta / 2M \le M,$$

$$\left| \left(\int_{\mathbb{T}^n} a_\nu(\xi)^{-1} \, d\xi \right)^{-1} - \left(\int_{\mathbb{T}^n} a(\xi)^{-1} \, d\xi \right)^{-1} \right| \le c_1 \delta M \le M,$$

provided that $c_1 \delta \le 1$. Combining these inequalities with (3.4) and (3.8–9) with (3.2) we obtain

$$\|u_\nu - id\|_{r_\nu} \le 2\lambda, \qquad \|U_\nu\|_{r_\nu} \le 2\lambda,$$

$$|a_\nu^{-1}|_{r_\nu} \le 2M, \qquad \left| \left(\int_{\mathbb{T}^n} a_\nu(\xi)^{-1} \, d\xi \right)^{-1} \right| \le 2M. \tag{3.11}$$

Observe that these inequalities are, by assumption, satisfied for $\nu = 0$ provided that $c \ge 8M^3$ and $c\delta \le 1$. Moreover, it follows from (3.11) that the transformation $x = u_\nu(\xi)$ maps the strip $|\mathrm{Im}\,\xi| \le r_\nu$ into $|\mathrm{Im}\, u_\nu(\xi)| \le 2\lambda r$. $|\mathrm{Im}\, Du_\nu(\xi)| \le 2\lambda r$ so that the expression $E(u_\nu)$ is well defined in this region. The convergence proof is based on the quadratic error estimate

$$|E(u_{\nu+1})|_{r_{\nu+1}} \le c_2 (r_\nu - r_{\nu+1})^{-4\tau} |E(u_\nu)|_{r_\nu}^2 \tag{3.12}$$

in connection with the inequalities

$$\|u_{\nu+1}-u_\nu\|_{r_{\nu+1}} \le c_2(r_\nu - r_{\nu+1})^{-2\tau}\,|E(u_\nu)|_{r_\nu}, \tag{3.13}$$

$$\|U_{\nu+1}-U_\nu\|_{r_{\nu+1}} \le c_2(r_\nu - r_{\nu+1})^{-2\tau-1}\,|E(u_\nu)|_{r_\nu}, \tag{3.14}$$

with a sufficiently large constant $c_2 = c_2(\gamma, \tau, M, \lambda, n) \ge 1$. The constants δ, c_1 and c_2 will be determined in the course of the iteration.

Let us now fix an integer $N \ge 0$ and assume that the torus diffeomorphisms $u_\nu(\xi)$ have been constructed for $\nu = 0, \ldots, N$ such that the inequalities (3.8–11) are satisfied for $0 \le \nu \le N$ and (3.12–14) for $0 \le \nu \le N-1$.

In order to construct the next approximant $u_{N+1} = u_N + U_N w$ we recall from Lemma 2 that the right hand side of (3.7) is of period 1 and mean value zero. Moreover, the inequality (3.11) shows that the matrix function $a_N(\xi)$ satisfies the requirements of Lemma 5. Hence there exists a unique solution $w(\xi)$ of (3.7) with $\nu = N$ which is of period 1 and mean value zero and satisfies the estimate

$$|w|_\rho + |Dw|_\rho\,(r_N - r_{N+1})^{-\tau} \le c(r_N - r_{N+1})^{-2\tau}\,|E(u_N)|_{r_N} \tag{3.15}$$

with $\rho = (r_N + r_{N+1})/2$. Here – and in the following – we denote by $c > 0$ a generic constant depending only on γ, τ, M, λ and n. Using Cauchy's estimate we obtain

$$|D^2 w|_\rho \le c(r_N - r_{N+1})^{-\tau-1}\,|E(u_N)|_{r_N}$$

with $\rho = (r_N + 3r_{N+1})/4$ and hence

$$\begin{aligned}
|U_N w|_\rho &\le 2\lambda\,|w|_\rho \\
&\le c(r_N - r_{N+1})^{-2\tau}\,|E(u_N)|_{r_N}, \\
|D(U_N w)|_\rho &= |(DU_N)w + U_N(Dw)|_\rho \\
&\le 2\lambda(|w|_\rho + |Dw|_\rho) \\
&\le c(r_N - r_{N+1})^{-2\tau}\,|E(u_N)|_{r_N}, \\
|D^2(U_N w)|_\rho &= |(D^2U_N)w + 2(DU_N)(Dw) + U_N(D^2 w)|_\rho \\
&\le 4\lambda(|w|_\rho + |Dw|_\rho + |D^2 w|_\rho) \\
&\le c(r_N - r_{N+1})^{-2\tau}\,|E(u_N)|_{r_N}.
\end{aligned}$$

Thus we have established (3.13) for $\nu = N$ in the region $|\mathrm{Im}\,\xi| \le \rho = (r_N + 3r_{N+1})/4$ and hence (3.14) follows from Cauchy's estimate. We point out that we have only used (3.11) in order to derive (3.13) and (3.14) so that the constant c_2 is independent of the previous steps in the iteration.

In order to derive the quadratic estimate (3.12) for $\nu = N$ we make use of the hypothesis that it holds already for $0 \le \nu \le N-1$. We define $\varepsilon_\nu = |E(u_\nu)|_{r_\nu}$ for $\nu = 0, \ldots, N$ so that (3.12) can be written in the form

$$\varepsilon_{\nu+1} \le c_2(r_\nu - r_{\nu+1})^{-4\tau}\varepsilon_\nu^2 = \alpha\beta^\nu\varepsilon_\nu^2$$

with $\alpha = c_2(4/r)^{4\tau}$ and $\beta = 2^{4\tau}$. Defining

$$\delta_\nu = \alpha\beta^{\nu+1}\varepsilon_\nu = c_2(2/(r_\nu - r_{\nu+1}))^{4\tau}\varepsilon_\nu \tag{3.16}$$

for $0 \le \nu \le N$ we obtain

$$\delta_{\nu+1} = \alpha\beta^{\nu+2}\varepsilon_{\nu+1} \le \alpha^2\beta^{2\nu+2}\varepsilon_\nu^2 = \delta_\nu^2$$

for $0 \le \nu \le N-1$ so that δ_ν will converge to zero provided that $\delta_0 < 1$. In fact, we will choose δ^* so small that

$$c_1\delta^* \le 1, \qquad c_1 = c_2 8^{4\tau+2}\lambda^2 M^4 \ge 8M^3. \tag{3.17}$$

In view of $\varepsilon_0 \le r^{4\tau}\delta$ and $\delta \le \delta^*$ this leads to the inequality

$$\delta_0 = c_2(8/r)^{4\tau}\varepsilon_0 \le c_2 8^{4\tau}\delta \le 1/64\lambda^2 M^4 \tag{3.18}$$

and in particular we have $\delta_0 \le 1/2$ so that $\delta_\nu \le 2^{-\nu}\delta_0$ for $\nu \le N$.

Now (3.13) has already been established for $\nu = N$. In combination with (3.16) this leads to the estimate

$$\begin{aligned}
\|u_{\nu+1} - u_\nu\|_{r_{\nu+1}} &\le c_2(r_\nu - r_{\nu+1})^{-2\tau}\varepsilon_\nu \\
&\le c_2 r^{2\tau}(2/(r_\nu - r_{\nu+1}))^{4\tau}\varepsilon_\nu \\
&= r^{2\tau}\delta_\nu \\
&\le r^{2\tau}2^{-\nu}\delta_0
\end{aligned}$$

for $\nu = 0, \ldots, N$. Likewise we obtain from (3.14) that

$$\|U_{\nu+1} - U_\nu\|_{r_{\nu+1}} \le r^{2\tau-1}\delta_\nu \le r^{2\tau-1}2^{-\nu}\delta_0.$$

these two inequalities in connection with (3.18) show that

$$\|u_{N+1}-u_0\|_{r_{N+1}} \le r^{2\tau}2\delta_0 \le r^{2\tau}2\cdot 8^{4\tau}c_2\delta \le \lambda \tag{3.19}$$

and

$$\begin{aligned}\|U_{N+1}-U_0\|_{r_{N+1}} &\le \|U_{N+1}-U_N\|_{r_{N+1}} + \|U_N-U_0\|_{r_{N+1}}\\ &\le r^{2\tau-1}2\delta_0\\ &\le r^{2\tau-1}2\cdot 8^{4\tau}c_2\delta\\ &\le \lambda.\end{aligned} \tag{3.20}$$

Moreover, it follows from (3.17–20) that

$$\begin{aligned}&|a-a_{N+1}|_{r_{N+1}}\\ &\quad\le |a-a_0|_{r_{N+1}} + |a_0-a_{N+1}|_{r_{N+1}}\\ &\quad\le \delta + |U_0^TF_{pp}(u_0, Du_0)U_0 - U_{N+1}^TF_{pp}(u_{N+1}, Du_{N+1})U_{N+1}|_{r_{N+1}}\\ &\quad\le \delta + |U_0|_{r_{N+1}}^2\, |F_{pp}(u_0, Du_0) - F_{pp}(u_{N+1}, Du_{N+1})|_{r_{N+1}}\\ &\qquad + |U_0^TF_{pp}(u_{N+1}, Du_{N+1})|_{r_{N+1}}\, |U_0 - U_{N+1}|_{r_{N+1}}\\ &\qquad + |U_0-U_{N+1}|_{r_{N+1}}\, |F_{pp}(u_{N+1}, Du_{N+1})U_{N+1}|_{r_{N+1}}\\ &\quad\le \delta + \lambda^2M\, \|u_0-u_{N+1}\|_{r_{N+1}} + 3\lambda M\, |U_0-U_{N+1}|_{r_{N+1}}\\ &\quad\le \delta + 2\lambda^2M\,\delta_0 + 6\lambda M\,\delta_0\\ &\quad\le \delta + 8\lambda^2Mc_28^{4\tau}\delta\\ &\quad= \delta + c_1\delta/8M^3\\ &\quad\le c_1\delta/4M^3.\end{aligned}$$

Thus we have established the inequalities (3.8–10) for $\nu = N+1$ and we have already seen that (3.11) follows from (3.10). Therefore it remains to prove the crucial estimate (3.12) for $\nu = N$. For this purpose we recall from Lemma 2 and (3.7) that

$$\begin{aligned}&U_N^T\,dE(u_N)U_Nw + U_N^TE(u_N)\\ &\quad= D(a_NDw) + b_NDw + c_Nw - U_N^TE(u_N)\\ &\quad= b_NDw + c_Nw\end{aligned} \tag{3.21}$$

where $c_N = U_N^TE(u_N)_\xi$. In view of (3.11) the matrix c_N satisfies the inequality

$$|c_N|_\rho \le 2\lambda\, |E(u_N)_\xi|_\rho \le 4\lambda(r_N - r_{N+1})^{-1}\, |E(u_N)|_{r_N} \tag{3.22}$$

with $\rho = (r_N + r_{N+1})/2$. Since b_N is of period 1 with mean value zero and satisfies $Db_N = c_N - c_N^T$ (Lemma 2) we obtain from Lemma 1 that

$$\begin{aligned} |b_N|_{r_{N+1}} &\le 2^\tau c_0 (r_N - r_{N+1})^{-\tau} |c_N - c_N^T|_\rho \\ &\le c(r_N - r_{N+1})^{-\tau-1} |E(u_N)|_{r_N}. \end{aligned} \tag{3.23}$$

Here $c \ge 1$ again denotes a generic constant depending only on γ, τ, M, λ and n. Moreover, observe that by (3.1) and (3.11)

$$|U_N^{-1}|_{r_N} = |a_N^{-1} U_N^T F_{pp}(u_N, Du_N)|_{r_N} \le 4\lambda M^2$$

and hence it follows from (3.21) in combination with the estimates (3.15) and (3.22–23) that

$$\begin{aligned} &|dE(u_N)U_N w + E(u_N)|_{r_{N+1}} \\ &\quad = |(U_N^T)^{-1}(b_N Dw + c_N w)|_{r_{N+1}} \\ &\quad \le c(r_N - r_{N+1})^{-2\tau-1} |E(u_N)|_{r_N}^2. \end{aligned} \tag{3.24}$$

Recalling from (3.20) and (3.2) that

$$\|U_N\|_{r_{N+1}} + \|U_{N+1} - U_N\|_{r_{N+1}} \le 2\lambda \tag{3.25}$$

we observe that the transformations $u = u_N$ and $v = u_{N+1} - u_N$ meet the requirement of Lemma 6 with $R = 2\lambda r$. Hence it follows from (3.24) and Lemma 6 that

$$\begin{aligned} |E(u_{N+1})|_{r_{N+1}} &\le |E(u_{N+1}) - E(u_N) - dE(u_N)(u_{N+1} - u_N)|_{r_{N+1}} \\ &\quad + |E(u_N) + dE(u_N)U_N w|_{r_{N+1}} \\ &\le c\, \|u_{N+1} - u_N\|_{r_{N+1}}^2 + c(r_N - r_{N+1})^{-2\tau-1} |E(u_N)|_{r_N}^2 \\ &\le c(r_N - r_{N+1})^{-4\tau} |E(u_N)|_{r_N}^2 \end{aligned}$$

and thus we have established (3.12) for $v = N$. We point out that the last inequality is based on (3.13) with $v = N$ so that the constant c_2 from (3.13) and (3.14) has to be enlarged in order to derive (3.12). However, all three estimates (3.12–14) have been obtained from (3.11) and (3.25) only so that the constant c_2 is independent of the constant c_1 defined by (3.17). This finishes the induction.

Finally, the inequality (3.13) for $v \in \mathbb{Z}$ shows that

$$\|u_{v+1} - u_v\|_{r_{v+1}} \le c_2 (r_v - r_{v+1})^{-2\tau} |E(u_v)|_{r_v} \le \delta_v \le 2^{-v} \delta_0$$

and hence $u_\nu(\xi)$ is a Cauchy sequence in the domain $|\mathrm{Im}\, \xi| \leq r/2 \leq r_\nu$. It follows from (3.8–10) that the limit function

$$u(\xi) = \lim u_\nu(\xi)$$

satisfies the inequalities (3.6). Moreover, the above estimate shows that $E(u_\nu)$ converges to zero in $|\mathrm{Im}\, \xi| \leq r/2$ so that u is a solution of (3.5). This finishes the proof of Theorem 1. □

4. The differentiable case

In the theorem below we make use of the abbreviating notation

$$\|u\|_{C^s} = |u|_{C^s} + |Du|_{C^s} + |D^2u|_{C^s}.$$

THEOREM 2. *Let $\omega \in \mathbb{R}^n$ satisfy $|\omega| \leq M$ and*

$$|j \cdot \omega| \geq \gamma\, |j|^{-\tau}, \qquad 0 \neq j \in \mathbb{Z}^n,$$

for some constants $\gamma > 0$, $\tau \geq n-1$ and let $F(x, p)$ be a function of class C^l with $l = 4\tau + 2 + \mu$, $\mu > 0$. We assume that F is of period 1 in the x-variables and satisfies

$$\begin{aligned} &|F|_{C^4} \leq M, \\ &|F_{pp}(x, \omega)^{-1}| \leq M, \qquad x \in \mathbb{R}^n, \\ &\left| \left(\int_{\mathbb{T}^n} F_{pp}(x, \omega)^{-1}\, dx \right)^{-1} \right| \leq M. \end{aligned} \tag{4.1}$$

Then there exists a constant $\varepsilon = \varepsilon(\gamma, \tau, \mu, M, n) > 0$ such that if

$$|F|_{C^l}\, |DF_p(x, \omega) - F_x(x, \omega)|^{\mu/(4\tau+\mu)} \leq \varepsilon \tag{4.2}$$

for $x \in \mathbb{R}^n$ then the equation

$$DF_p(u, Du) = F_x(u, Du). \tag{4.3}$$

admits a solution $x = u(\xi)$ such that $u(\xi) - \xi$ is of period 1. If $2\tau + \mu$ is not an integer then u, Du and D^2u are of class $C^{2\tau+\mu}$ and satisfy the estimate

$$\|u - id\|_{C^s} \leq c_s\; |F|_{C^l}\, |E(F, id)|_{C^0}^{(2\tau+\mu-s)/(4\tau+\mu)} \tag{4.4}$$

for $0 < s \le 2\tau + \mu$, $s \notin \mathbb{Z}$, *where* $c_s = c/\theta(1-\theta)$ *and* $0 < \theta = s - [s] < 1$ *with a suitable constant* $c = c(\gamma, \tau, \mu, M, n) \ge 1$. *Moreover, if* $F \in C^m$ *for some* $m \ge l$ *and* $m - 2\tau - 2$ *is not an integer then* u, Du *and* $D^2 u$ *are of class* $C^{m-2\tau-2}$. *So in particular* $F \in C^\infty$ *implies* $u \in C^\infty$. *Finally, if* F *is real analytic then so is* u.

Proof. We approximate $F(x, p)$ by a sequence of real analytic functions $F_\nu(x, p)$, $\nu = 0, 1, \dots$, which are of period 1 in the x-variables and defined in the strip

$$|\mathrm{Im}\, x| \le 8r_\nu, \qquad |\mathrm{Im}\, p| \le 8r_\nu, \qquad r_\nu = 2^{-\nu} r_0,$$

where the small constant $r_0 > 0$ is given by

$$r_0^{4\tau+\mu} = |E(F, id)|_{C^0}.$$

If $\varepsilon > 0$ is sufficiently small then, in view of (4.2), we may assume that $0 < r_0 < 1/2$. By Lemma 3, the function F_0 can be chosen as to satisfy

$$\left| F_{0pp}(x, \omega) - \sum_{|\sigma| \le l-2} \partial_x^\sigma F_{pp}(\mathrm{Re}\, x, \omega)(i\, \mathrm{Im}\, x)^\sigma / \sigma! \right| \le c_0 |F|_{C^l} r_0^{l-2} \tag{4.5}$$

for $|\mathrm{Im}\, x| \le 8r_0$ with a suitable constant $c_0 = c_0(l, n) > 0$. Analogous inequalities hold with F_{pp} replaced by F_{px} or F_x. Moreover, Lemma 3 allows us to assume that

$$|\partial^\alpha F_\nu - \partial^\alpha F_{\nu-1}|_{8r_\nu} \le c_0 |F|_{C^l} r_\nu^{l-|\alpha|}, \qquad |\alpha| \le l, \qquad \nu \ge 1, \tag{4.6}$$

$$|\partial^\alpha F_\nu|_{8r_\nu} \le c_0 |F|_{C^4}, \qquad |\alpha| \le 4, \qquad \nu \ge 0. \tag{4.7}$$

We will then construct inductively a sequence of real analytic functions $x = u_\nu(\xi)$ in the strip $|\mathrm{Im}\, \xi| \le r_\nu$ such that $u_\nu(\xi) - \xi$ is of period 1 and

$$DF_{\nu p}(u_\nu, Du_\nu) = F_{\nu x}(u_\nu, Du_\nu). \tag{4.8}$$

For $\nu = 0$ we shall use the estimates (4.2) and (4.5) in order to verify that the pair (F_0, id) satisfies the requirements of Theorem 1 with $r = 2r_0$ and $\lambda = 2$. First, it follows from (4.1), (4.7) and (4.5) that

$$|\partial^\alpha F_0|_{8r_0} \le c_0 M, \qquad |\alpha| \le 4,$$

$$|F_{0pp}(x, \omega) - F_{pp}(\mathrm{Re}\, x, \omega)| \le c_0 |F|_{C^l} r_0^\mu, \qquad |\mathrm{Im}\, x| \le 4r_0. \tag{4.9}$$

Moreover, denoting by $c_0 > 0$ a generic constant depending only on l, M and n, we obtain from (4.5) that for $|\mathrm{Im}\, x| \le 4r_0$

$$
\begin{aligned}
&|DF_{0p}(x, \omega) - F_{0x}(x, \omega)| \\
&\quad \le \Big| DF_{0p}(x, \omega) - \sum_{|\sigma| \le l-2} \partial_x^\sigma DF_p(\mathrm{Re}\, x, \omega)(i\, \mathrm{Im}\, x)^\sigma/\sigma! \Big| \\
&\qquad + \Big| F_{0x}(x, \omega) - \sum_{|\sigma| \le l-2} \partial_x^\sigma F_x(\mathrm{Re}\, x, \omega)(i\, \mathrm{Im}\, x)^\sigma/\sigma! \Big| \\
&\qquad + \sum_{|\sigma| \le l-2} |\partial_x^\sigma (DF_p(\mathrm{Re}\, x, \omega) - F_x(\mathrm{Re}\, x, \omega))| \cdot |(i\, \mathrm{Im}\, x)^\sigma/\sigma!| \\
&\quad \le c_0 |F|_{C^l} r_0^{l-2} + c_0 \sum_{k \le l-2} r_0^k |E(F, id)|_{C^k} \\
&\quad \le c_0 |F|_{C^l} r_0^{l-2} + c_0 \sum_{k \le l-2} r_0^k |E(F, id)|_{C^0}^{(l-k-2)/(l-2)} |E(F, id)|_{C^{l-2}}^{k/(l-2)} \\
&\quad \le c_0 |F|_{C^l} r_0^{l-2} + c_0 \sum_{k \le l-2} r_0^k r_0^{l-k-2} |F|_{C^l}^{k/(l-2)} \\
&\quad \le c_0 |F|_{C^l} r_0^\mu (2r_0)^{4\tau}.
\end{aligned}
$$

Observe that we have used the interpolation inequality (2.14). Moreover, note that this estimate in connection with (4.1) and (4.9) shows that the pair (F_0, id) satisfies the requirements of Theorem 1 with $r = 2r_0$, $\delta = \delta_0 = c_0 |F|_{C^l} r_0^\mu$ and $a(\xi) = F_{pp}(\mathrm{Re}\, \xi, \omega)$ provided that

$$
\delta_0 = c_0 |F|_{C^l} r_0^\mu = c_0 |F|_{C^l} |E(F, id)|_{C^0}^{\mu/(4\tau+\mu)} \le \delta^*
$$

where $\delta^* = \delta^*(\gamma, \tau, c_0 M, 2, n)$ is the constant of Theorem 1. But this inequality is indeed satisfied if (4.2) holds with $\varepsilon = \delta^*/c_0$. Hence there exists a real analytic diffeomorphism $x = u_0(\xi)$ of the n-torus defined in the region $|\mathrm{Im}\, \xi| \le r_0$ such that $u_0(\xi) - \xi$ is of period 1 and (4.8) holds for $\nu = 0$. Moreover, Theorem 1 yields the estimates

$$
\begin{aligned}
&\|u_0 - id\|_{r_0} \le c_0 c_1 |F|_{C^l} r_0^{2\tau+\mu}, \\
&\|U_0 - I\|_{r_0} \le c_0 c_1 |F|_{C^l} r_0^{2\tau+\mu-1}, \\
&|U_0^T F_{0pp}(u_0, Du_0) U_0 - F_{pp}(\mathrm{Re}\, \xi, \omega)|_{r_0} \le c_0 c_1 |F|_{C^l} r_0^\mu / 4M^3,
\end{aligned} \tag{4.10}
$$

where $c_1 = c_1(\gamma, \tau, c_0, M, 2, n)$ is the constant of Theorem 1 enlarged by the factor $2^{2\tau}$. For later purposes we assume that

$$
c_1 \delta^* \le 1 - 2^{-\mu}
$$

and, moreover, we permit oursleves to enlarge the constant c_0 in the forthcoming estimates, if necessary.

Now suppose that the solutions $x = u_\nu(\xi)$, $|\mathrm{Im}\,\xi| \le r_\nu$, of (4.8) have been constructed for $\nu = 0, \ldots, N$ such that $u_\nu(\xi) - \xi$ is of period 1 and

$$\begin{aligned}
&\|u_\nu - u_{\nu-1}\|_{r_\nu} \le c_0 c_1 \,|F|_{C^l}\, r_\nu^{2\tau+\mu}, \\
&\|U_\nu - U_{\nu-1}\|_{r_\nu} \le c_0 c_1 \,|F|_{C^l}\, r_\nu^{2\tau+\mu-1}, \qquad \nu = 1, \ldots, N, \\
&|a_\nu - a_{\nu-1}|_{r_\nu} \le c_0 c_1 \,|F|_{C^l}\, r_\nu^{\mu}/4M^3,
\end{aligned} \tag{4.11}$$

where $a_\nu(\xi) = U_\nu^T F_{\nu pp}(u_\nu, Du_\nu) U_\nu$ for $|\mathrm{Im}\,\xi| \le r_\nu$ and U_ν denotes the Jacobian of u_ν. In order to make sure that the expression $F_\nu(u_\nu, Du_\nu)$ is well defined in the domain $|\mathrm{Im}\,\xi| \le r_\nu$ we point out that

$$c_0 c_1 \,|F|_{C^l} \sum_{j=0}^{\infty} r_j^\mu = c_0 c_1 \,|F|_{C^l}\, r_0^\mu/(1 - 2^{-\mu}) \le c_1 \delta^*/(1 - 2^{-\mu}) \le 1.$$

It therefore follows from (4.10) and (4.11) that

$$\|u_\nu - id\|_{r_\nu} \le c_0 c_1 \,|F|_{C^l} \sum_{j=0}^{\nu} r_j^{2\tau+\mu} \le r_0^{2\tau} \le 1/2,$$

and likewise

$$\|U_\nu - I\|_{r_\nu} \le r_0^{2\tau-1} \le 1/2$$

so that

$$\|u_\nu - id\|_{r_\nu} + \|U_\nu\|_{r_\nu} \le 2, \qquad \nu = 0, \ldots, N. \tag{4.12}$$

This shows that u_ν maps the strip $|\mathrm{Im}\,\xi| \le r_\nu$ into $|\mathrm{Im}\,u_\nu(\xi)| \le 2r_\nu = 4r_{\nu+1}$, $|\mathrm{Im}\,Du_\nu(\xi)| \le 2r_\nu$ so that both $F_\nu(u_\nu, Du_\nu)$ and $F_{\nu+1}(u_\nu, Du_\nu)$ are well defined in this region. It also follows from (4.10) and (4.11) that

$$|a_\nu(\xi) - F_{pp}(\mathrm{Re}\,\xi, \omega)| \le c_0 c_1 \,|F|_{C^l} \sum_{j=0}^{\nu} r_j^\mu/4M^3 \le 1/4M^3$$

for $\nu = 0, \ldots, N$ and $|\mathrm{Im}\,\xi| \le r_\nu$. Combining this inequality with (4.1) we obtain from the remark after Theorem 1 that

$$\begin{aligned}
&|a_\nu(\xi)^{-1}| \le 2M, \qquad |\mathrm{Im}\,\xi| \le r_\nu, \\
&\left|\left(\int_{\mathbb{T}^n} a_\nu(\xi)^{-1}\, d\xi\right)^{-1}\right| \le 2M, \qquad \nu = 0, \ldots, N.
\end{aligned} \tag{4.13}$$

We will now verify that the pair (F_{N+1}, u_N) satisfies the requirements of Theorem 1 with $r = r_N$, $\delta = c_0 |F|_{C^l} r_{N+1}^{\mu}$ and $a(\xi) = a_N(\xi)$. First it follows from (4.1) and (4.7) that

$$|\partial^\alpha F_{N+1}|_{2\lambda r_N} = |\partial^\alpha F_{N+1}|_{8 r_{N+1}} \le c_0 M \tag{4.14}$$

where $\lambda = 2$. Moreover, enlarging the constant c_0 where necessary, we obtain from (4.12) and (4.6) that

$$\begin{aligned} &|U_N^T F_{N+1,pp}(u_N, Du_N) U_N - a_N|_{r_N} \\ &\quad \le |U_N|_{r_N}^2 \, |F_{N+1,pp}(u_N, Du_N) - F_{N,pp}(u_N, Du_N)|_{r_N} \\ &\quad \le c_0 |F|_{C^l} r_{N+1}^{l-2} \\ &\quad \le c_0 |F|_{C^l} r_{N+1}^{\mu} \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} &|DF_{N+1,p}(u_N, Du_N) - F_{N+1,x}(u_N, Du_N)|_{r_N} \\ &\quad \le |F_{N+1,pp}(u_N, Du_N) - F_{N,pp}(u_N, Du_N)|_{r_N} \, |D^2 u_N|_{r_N} \\ &\qquad + |F_{N+1,px}(u_N, Du_N) - F_{N,px}(u_N, Du_N)|_{r_N} \, |Du_N|_{r_N} \\ &\qquad + |F_{N+1,x}(u_N, Du_N) - F_{N,x}(u_N, Du_N)|_{r_N} \\ &\quad \le c_0 |F|_{C^l} r_{N+1}^{l-2} (|D^2 u_N|_{r_N} + |Du_N - \omega|_{r_N} + 1 + |\omega|) \\ &\quad \le c_0 |F|_{C^l} r_{N+1}^{l-2} \\ &\quad \le c_0 |F|_{C^l} r_{N+1}^{\mu} r_N^{4\tau}. \end{aligned} \tag{4.16}$$

The estimates (4.12–16) show that the assumptions of Theorem 1 are indeed satisfied with $F = F_{N+1}$, $u_0 = u_N$, and $\delta = c_0 |F|_{C^l} r_{N+1}^{\mu} \le \delta^*$. Hence there exists a real analytic solution $x = u_{N+1}(\xi)$, $|\mathrm{Im}\, \xi| \le r_N/2 = r_{N+1}$, of (4.8) and this solution satisfies the estimate (4.11) with $\nu = N + 1$. This finishes the induction.

It follows from (4.11) that $u_\nu(\xi)$ is a Cauchy sequence for $\xi \in \mathbb{R}^n$ and, by Lemma 4, the limit function $u(\xi) = \lim u_\nu(\xi)$ is of class $C^{2\tau+\mu}$ provided that $2\tau + \mu \notin \mathbb{Z}$. Moreover, Lemma 4 shows that u satisfies the estimate

$$|u - id|_{C^s} \le (c_2/\theta(1-\theta)) \, |F|_{C^l} r_0^{2\tau+\mu-s}$$

for $0 < s \le 2\tau + \mu$ and $0 < \theta = s - [s] < 1$ with a suitable constant $c_2 = c_2(\gamma, \tau, \mu, M, n) > 0$. This proves (4.4) and it follows from (4.8) that u is a solution of (4.3).

In order to prove higher differentiability of u let us now assume that $F \in C^m$ for some $m \ge l$. Then the inequality (4.6) is satisfied with l replaced by m and c_0

replaced by a larger constant c_m. The same holds for the inequalities (4.15) and (4.16) where in addition μ has to be replaced by $m - 4\tau - 2$. Hence Theorem 1 can be applied in the ν-th step of the iteration with $\delta = c_m\, |F|_{C^m}\, r_\nu^{m-4\tau-2}$ and it follows that

$$|u_\nu - u_{\nu-1}|_{r_\nu} \le c_1 c_m\, |F_{C^m}\, r_\nu^{m-2\tau-2}$$

for $\nu = 1, 2, \ldots$. In connection with Lemma 4 this estimate shows that u, Du and D^2u are of class $C^{m-2\tau-2}$.

Finally, in order to prove that u is real analytic, let us suppose that F is real analytic and recall from Lemma 3 that

$$\left|\partial^\alpha F_N(x, p) - \sum_{|\beta| \le l - |\alpha|} \partial^{\alpha+\beta} F(\operatorname{Re} x, \operatorname{Re} p)(i(\operatorname{Im} x, \operatorname{Im} p))^\beta / \beta!\right| \le c r_N^{l-|\alpha|}$$

for $\alpha \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $|\alpha| \le l$ and $|\operatorname{Im} x| \le 8r_N$, $|\operatorname{Im} p| \le 8r_N$. The same estimate holds with F_N replaced by F if $r_N > 0$ is sufficiently small so that

$$|\partial^\alpha F - \partial^\alpha F_N|_{8r_N} \le c r_N^{l-|\alpha|}, \qquad |\alpha| \le l,$$

where the constant $c > 0$ is independent of N. This shows that the estimates (4.15) and (4.16) can be performed with F_{N+1} replaced by F leading to the inequalities

$$|U_N^T F_{pp}(u_N, Du_N) U_N - a_N|_{r_N} \le c r_N^\mu,$$

$$|DF_p(u_N, Du_N) - F_x(u_N, Du_N)|_{r_N} \le c r_N^\mu r_N^{4\tau}.$$

It follows that for large N the pair (F, u_N) satisfies the requirements of Theorem 1 with $r = r_N$ and $\delta = c r_N^\mu \le \delta^*$. Hence there exists a real analytic diffeomorphism $x = v(\xi)$ of the n-torus defined in the strip $|\operatorname{Im} \xi| \le r_{N+1}$ such that $v(\xi) - \xi$ is of period 1 and $E(F, v) = 0$. Moreover, v satisfies the estimate

$$|v - u_N|_{r_{N+1}} \le c r_N^{2\tau+\mu}.$$

Combining this estimate with (4.11) we obtain from Lemma 4 that

$$\|v - u\|_{C^s} \le (c/\theta(1-\theta)) r_N^{2\tau+\mu-s}$$

for $0 < s \le 2\tau + \mu$ and $0 < \theta = s - [s] < 1$ where the constant $c > 0$ does not depend on our choice of N. Choosing $s \notin \mathbb{Z}$ such that $2\tau < s < 2\tau + \mu$ and assuming that $r_N > 0$ is sufficiently small we can apply the uniqueness result from the next section (Theorem 4) and obtain that $u(\xi)=v(\xi+\xi_0)$ for some constant vector $\xi_0\in\mathbb{R}^n$ so that u is real analytic. This proves Theorem 2. $\square$

We will use the abstract Lie group structure as described in section 2 in order to derive the perturbation theorem for invariant tori in general position represented by a diffeomorphism u_0 which is not necessarily close to the identity map. For this purpose we shall need the following composition estimate.

LEMMA 7. *Let $u\in C^{l+2}(\mathbb{R}^n,\mathbb{R}^n)$ and $v, w\in C^l(\mathbb{R}^n,\mathbb{R}^n)$ be given such that*

$$\|u-id\|_{C^{l+2}}\le M,\qquad \|v-id\|_{C^l}\le M,\qquad \|w-id\|_{C^l}\le M$$

and let $\omega\in\mathbb{R}^n$ satisfy $|\omega|\le M$. Then

$$\|u\circ v-u\circ w\|_{C^l}\le c\,\|v-w\|_{C^l}$$

with a suitable constant $c=c(l,M,n)$. The same statement holds with $\|\cdot\|$ replaced by $|\cdot|$.

Proof. We first prove the result with the standard norm $|\cdot|$ and observe that

$$\begin{aligned}
&u(x)-u(\xi)-u(y)+u(\eta)\\
&=\int_0^1\int_0^1 \partial u/\partial x\cdot(x-\xi-y+\eta)\,ds\,dt\\
&\quad+\int_0^1\int_0^1(\xi-\eta+t(x-\xi-y+\eta))\cdot\partial^2u/\partial x^2\cdot(y-\eta+s(x-\xi-y+\eta))\,ds\,dt\\
&=0(|x-\xi-y+\eta|+|\xi-\eta|\cdot|y-\eta|)
\end{aligned}$$

where $\partial u/\partial x$ and $\partial^2u/\partial x^2$ are to be understood with the argument $\eta+s(\xi-\eta)+t(y-\eta)+st(x-\xi-y+\eta)$. This shows that for $0<|\xi-\eta|\le 1$ we have the estimate

$$\begin{aligned}
&|u(v(\xi))-u(w(\xi))-(u(v(\eta))-u(w(\eta)))|\\
&\quad\le c(|v(\xi)-w(\xi)-v(\eta)+w(\eta)|+|w(\xi)-w(\eta)|\cdot|v(\eta)-w(\eta)|)\\
&\quad\le c\,|\xi-\eta|^\mu\,(|v-w|_{C^\mu}+|w|_{C^\mu}\,|v-w|_{C^0})\\
&\quad\le c\,|\xi-\eta|^\mu\,|v-w|_{C^\mu}
\end{aligned}$$

with a suitable constant c depending on the C^2 norm of $u-id$ and on the C^μ norm of $w-id$. Thus we have proved the statement for $0\le l<1$.

Now we proceed by induction. Assuming that the statement has been shown for $l \geq 0$ we mke use of the identity

$$(u \circ v)_{x_k} - (u \circ w)_{x_k} = \sum_{\nu=1}^{n} (u_{x_\nu} \circ v)(v_{\nu x_k} - w_{\nu x_k}) + \sum_{\nu=1}^{n} (u_{x_\nu} \circ v - u_{x_\nu} \circ w) w_{\nu x_k}$$

and obtain from (2.15) that

$$|(u \circ v)_{x_k} - (u \circ w)_{x_k}|_{C^l} \leq c\, |v - w|_{C^{l+1}}$$

with a constant $c > 0$ depending on n, l, the C^{l+3} norm of $u - id$ and the C^{l+1} norm of $v - id$ and $w - id$. This proves the statement of the lemma in the case of the standard norm $|\cdot|$. In the case of the norm $\|\cdot\|$ the statement follows from the identities

$$\begin{aligned} D(u \circ v) - D(u \circ w) &= (U \circ v)(Dv - Dw) + (U \circ v - U \circ w) Dw \\ D^2(u \circ v) - D^2(u \circ w) &= (U \circ v)(D^2 v - D^2 w) + D(U \circ v)(Dv - Dw) \\ &\quad + (U \circ v - U \circ w) D^2 w + (D(U \circ v) - D(U \circ w)) Dw. \quad \square \end{aligned}$$

THEOREM 3. *Let* $\omega \in \mathbb{R}^n$ *satisfy* $|\omega| \leq M$ *and*

$$|j \cdot \omega| \geq \gamma\, |j|^{-\tau}, \qquad 0 \neq j \in \mathbb{Z}^n,$$

for some constants $\gamma > 0$, $\tau \geq n - 1$. *Moreover, let* $F(x, p)$ *be of period* 1 *in the* x*-variables and let* $x = u_0(\xi)$ *be a diffeomorphism of the* n*-torus such that* $u_0(\xi) - \xi$ *is of period* 1. *We assume that* $F \in C^l$ *and* $u_0 \in C^{l+1}$ *where* $l = 4\tau + 2 + \mu$, $\mu > 0$, *and*

$$|F|_{C^l} \leq M, \qquad |u_0 - id|_{C^{l+1}} \leq M.$$

Finally, suppose that the pair (F, u_0) *is stable and let* M *be such that*

$$|a_0(\xi)^{-1}| \leq M, \qquad \left| \left(\int_{\mathbb{T}^n} a_0(\xi)^{-1}\, d\xi \right)^{-1} \right| \leq M,$$

for $\xi \in \mathbb{R}^n$ *where* $a_0(\xi) = U_0^T F_{pp}(u_0, Du_0) U_0$.

Then there exists a constant $\varepsilon = \varepsilon(\gamma, \tau, \mu, M, n) > 0$ *such that if* $|E(F, u_0)|_{C^0} \leq \varepsilon$ *then the equation* $E(F, u) = 0$ *admits a solution* $x = u(\xi)$ *such that* $u(\xi) - \xi$ *is of period* 1. *If* $l - 2\tau - 2$ *is not an integer then* u, Du *and* $D^2 u$ *are of class* $C^{l-2\tau-2}$

and satisfy the estimate

$$|u - u_0|_{C^s} \leq c_s \, |E(F, u_0)|_{C^0}^{(l-2\tau-2-s)/(l-2)} \tag{4.17}$$

for $0 < s \leq l - 2\tau - 2$, $s \notin \mathbb{Z}$, *where* $c_s = c/\theta(1-\theta)$ *with* $0 < \theta = s - [s] < 1$ *and a suitable constant* $c = c(\gamma, \tau, \mu, M, n) \geq 1$.

Proof. Recall from (2.8) that the function

$$G(x, p) = u_0^* F(x, p) = F(u_0(x), U_0(x)p)$$

satisfies

$$E(G, id) = U_0^T E(F, u_0), \qquad G_{pp}(\xi, \omega) = a_0(\xi).$$

This shows that the pair (G, id) satisfies all the requirements of Theorem 2 provided that the constant $\varepsilon > 0$ has been chosen sufficiently small. Hence there exists a diffeomorphism $x = v(\xi)$ of the n-torus such that $v(\xi) - \xi$ is of period 1 and $E(G, v) = 0$. Moreover, Theorem 2 shows that v is of class $C^{l-2\tau-2}$ and satisfies an estimate of the form

$$|v - id|_{C^s} \leq c_s \, |E(G, id)|_{C^0}^{(l-2\tau-2-s)/(l-2)}$$

with a constant $c_s > 0$ as above. It follows from (2.6) and $\det U_0 \neq 0$ that the function $u = u_0 \circ v$ satisfies $E(F, u) = 0$. Furthermore, Lemma 7 yields the estimate

$$\begin{aligned} |u - u_0|_{C^s} &= |u_0 \circ v - u_0|_{C^s} \\ &\leq c \, |v - id|_{C^s} \\ &\leq c c_s \, |E(G, id)|_{C^0}^{(l-2\tau-2-s)/(l-2)} \\ &\leq c c_s (|U_0^{-1}|_{C^0} \cdot |E(G, id)|_{C^0})^{(l-2\tau-2-s)/(l-2)} \end{aligned}$$

and hence the inequality (4.17) is a consequence of the fact that $U_0^{-1} = a_0^{-1} U_0^T F_{pp}(u_0, Du_0)$ is in norm bounded by M^3. This proves Theorem 3. □

5. Uniqueness and regularity

In order to prove the uniqueness result for invariant tori we need the following differentiable version of Lemma 5. We recall that

$$\|u\|_{C^s} = |u|_{C^s} + |Du|_{C^s} + |D^2u|_{C^s}.$$

LEMMA 8. *Assume $\omega \in \mathbb{R}^n$ satisfies $|\omega| \leq M$ and the Diophantine conditions*

$$|j \cdot \omega| \geq \gamma\, |j|^{-\tau}, \qquad 0 \neq j \in \mathbb{Z}^n,$$

for two constants $\gamma > 0$, $\tau \geq n-1$ where $n \geq 2$. Let $a \in C^{\tau+\mu}(\mathbb{R}^n; \mathbb{R}^{n\times n})$ be an invertible matrix function of period 1 and let the constant M be chosen such that

$$|a^{-1}|_{C^{\tau+\mu}} \leq M, \qquad \left|\left(\int_{\mathbb{T}^n} a(\xi)^{-1}\, d\xi\right)^{-1}\right| \leq M.$$

Moreover, let $g \in C^{2\tau+\mu}(\mathbb{T}^n; \mathbb{R}^n)$ be of mean value zero and suppose that $\mu > 0$ and $\tau + \mu$ are not integers. Then there exists a unique function $u \in C^\mu(\mathbb{T}^n, \mathbb{R}^n)$ of mean value zero such that Du and D^2u are also of class C^μ and

$$\int_{\mathbb{T}^n} u^T D(a^T D\varphi)\, d\xi = \int_{\mathbb{T}^n} g^T \varphi\, d\xi, \qquad \varphi \in C^\infty(\mathbb{T}^n, \mathbb{R}^n). \tag{5.1}$$

Moreover, u satisfies the estimate

$$\|u\|_{C^\mu} \leq c\, |g|_{C^{2\tau+\mu}} \tag{5.2}$$

with a suitable constant $c = c(\gamma, \tau, \mu, M, n)$.

Proof. We proceed as in the proof of Lemma 5 and define $f \in C^{\tau+\mu}(\mathbb{T}^n; \mathbb{R}^n)$ to be the unique function of mean value zero which satisfies $Df = g$ (Lemma 1). Then choose $\alpha \in \mathbb{R}^n$ such that $a^{-1}(f - \alpha) \in C^{\tau+\mu}(\mathbb{T}^n; \mathbb{R}^n)$ is of mean value zero and define $u \in C^\mu(\mathbb{T}^n; \mathbb{R}^n)$, again of mean value zero, to be the unique weak solution of $Du = a^{-1}(f - \alpha)$. Then u of course satisfies (5.1).

Conversely, let $u \in C^\mu(\mathbb{T}^n, \mathbb{R}^n)$ be of mean value zero such that Du and D^2u are also of class C^μ and (5.1) is satisfied. Then the function

$$f(\xi) = a(\xi)Du(\xi) + \alpha, \qquad \alpha = -\int_{\mathbb{T}^n} aDu\, d\xi$$

is of mean value zero, of class C^{μ} and satisfies $Df=g$ in the weak sense. Hence Lemma 1 shows that $f\in C^{\tau+\mu}$ and

$$|f|_{C^{\tau+\mu}}\leq c\;|g|_{C^{2\tau+\mu}}$$

with a suitable constant $c=c(\gamma,\tau,\mu,n)>0$. Moreover, the identity

$$\int_{\mathbb{T}^n} a(\xi)^{-1}\,d\xi\alpha=\int_{\mathbb{T}^n} a(\xi)^{-1}f(\xi)\,d\xi$$

yields the estimate

$$|\alpha|\leq M\left|\int_{\mathbb{T}^n} a(\xi)^{-1}f(\xi)\,d\xi\right|\leq M^2\,|f|_{C^0}\leq M^2c\;|g|_{C^{2\tau+\mu}}$$

and this implies

$$\begin{aligned}
|u|_{C^{\mu}}&\leq c\;|Du|_{C^{\tau+\mu}}=c\;|a^{-1}(f-\alpha)|_{C^{\tau+\mu}}\\
&\leq c(|a^{-1}|_{C^0}\,|f-\alpha|_{C^{\tau+\mu}}+|a^{-1}|_{C^{\tau+\mu}}\,|f-\alpha|_{C^0})\\
&\leq 2Mc(|f|_{C^{\tau+\mu}}+|\alpha|)\\
&\leq 2Mc(c\;|g|_{C^{2\tau+\mu}}+M^2c\;|g|_{C^{\tau+\mu}}).
\end{aligned}$$

In connection with the inequality

$$|D^2u|_{C^{\mu}}\leq M\;|Du|_{C^{1+\mu}}\leq M\;|Du|_{C^{\tau+\mu}}$$

this proves the estimate (5.2). Finally, the uniqueness is an immediate consequence of (5.2). This proves Lemma 8. □

THEOREM 4 (Uniqueness). *Let $\omega\in\mathbb{R}^n$ satisfy $|\omega|\leq M$ and*

$$|j\cdot\omega|\geq\gamma\;|j|^{-\tau},\qquad 0\neq j\in\mathbb{Z}^n,$$

for some constants $\gamma>0$, $\tau\geq n-1$, $M\geq 1$. Moreover, let $F\in C^{2\tau+4+\mu}(\mathbb{T}^n\times\mathbb{R}^n)$ be given with $\mu>0$ and let $x=u(\xi)$ be a $C^{2\tau+5+\mu}$ diffeomorphism of $\mathbb{T}^n$ such that $u(\xi)-\xi$ is of period 1 and $E(F,u)=0$. We assume that the pair (F,u) is stable and satisfies the estimates

$$|a^{-1}|_{C^{\tau+\mu}}\leq M,\qquad \left|\left(\int_{\mathbb{T}^n} a(\xi)^{-1}\,d\xi\right)^{-1}\right|\leq M.$$

$$|F|_{C^{2\tau+4+\mu}}\leq M,\qquad |u-id|_{C^{2\tau+5+\mu}}\leq M,$$

where $a(\xi) = U^T F_{pp}(u, Du)U$. *Finally, let* $x = v(\xi)$ *be another solution of* $E(F, v) = 0$ *such that* v, Dv, *and* D^2v *are of class* $C^{2\tau+\mu}$ *and* $v(\xi) - \xi$ *is of period* 1.

Then there exists a constant $\delta = \delta(\gamma, \tau, \mu, M, n) > 0$ *such that if* $|v - u|_{C^{2\tau+\mu}} \le \delta$ *then* $v(\xi) = u(\xi + \xi_0)$ *for all* $\xi \in \mathbb{R}^n$ *and some constant vector* $\xi_0 \in \mathbb{R}^n$.

Proof. It follows from the periodicity of $u(\xi) - \xi$ that

$$\int_{\mathbb{T}^n} u(\xi + \xi_0)\, d\xi = \int_{\mathbb{T}^n} u(\xi)\, d\xi + \xi_0$$

and hence we can assume without loss of generality that $v(\xi) - u(\xi)$ is of mean value zero. Changing $\mu > 0$ if necessary we can also assume that μ and $\tau + \mu$ are not integers.

We consider first the case $u = id$ and recall from section 2 that under the assumption $E(id) = E(F, id) = 0$ we have

$$dE(id)w = D(aDw).$$

Hence it follows from Lemma 8 and the differentiable version of Lemma 6 (obtained from the remainder formula in the proof of Lemma 6 in connection with the estimate (2.15)) that

$$\begin{aligned} \|v - id\|_{C^\mu} &\le c\, |D(aD(v - id))|_{C^{2\tau+\mu}} \\ &= c\, |E(v) - E(id) - dE(id)(v - id)|_{C^{2\tau+\mu}} \\ &\le c\, \|v - id\|_{C^\mu}\, \|v - id\|_{C^{2\tau+\mu}} \end{aligned}$$

with a generic constant $c = c(\gamma, \tau, \mu, M, n) > 0$. We conclude that if $c\, \|v - id\|_{C^{2\tau+\mu}} < 1$ then $v = id$.

In the general case it follows from (2.6) that

$$E(u^*F, id) = 0, \qquad E(u^*F, u^{-1} \circ v) = 0.$$

Moreover, Lemma 7 shows that

$$\|u^{-1} \circ v - id\|_{C^{2\tau+\mu}} \le c\, \|v - u\|_{C^{2\tau+\mu}}$$

with a constant $c > 0$ depending on $\|u^{-1}\|_{C^{2\tau+2+\mu}}$. But a bound on this norm as well as on the $C^{2\tau+4+\mu}$ norm of u^*F is guaranteed by the assumptions of Theorem 4. This shows that the pair (u^*F, id) satisfies the requirements of the theorem in the case of the identity transformation and therefore $u^{-1} \circ v$ is a translation provided that $\delta > 0$ has been chosen sufficiently small. This proves Theorem 4. $\square$

THEOREM 5 (Regularity). *Let $\omega \in \mathbb{R}^n$ satisfy the conditions of Theorem 4 and let $F \in C^l(\mathbb{T}^n \times \mathbb{R}^n)$ be given with $l > 4\tau + 2$. Moreover, let $x = u(\xi)$ be a C^{l+1} diffeomorphism of $\mathbb{T}^n$ such that $u(\xi) - \xi$ is of period 1. We assume that u is a stable solution of $E(F, u) = 0$.*

If F is of class C^m with $m > l$ and $m - 2\tau - 2$ is not an integer then u, Du and D^2u are of class $C^{m-2\tau-2}$. In particular, $F \in C^\infty$ implies $u \in C^\infty$. Moreover, if F is real analytic then so is u.

Proof. Let us choose $\mu > 0$ such that $4\tau + 2 + 2\mu \le l$ and $2\tau + \mu \notin \mathbb{Z}$. Then Lemma 3 shows that for any $\delta > 0$ we can find a real analytic diffeomorphism $x = v(\xi)$ of $\mathbb{T}^n$ such that $v(\xi) - \xi$ is of period 1 and

$$|v - u|_{C^{4\tau+3+2\mu}} < \delta.$$

Hence we can make $v^*F - u^*F$ in the $C^{4\tau+2+2\mu}$ norm as small as we please. Making use of the fact that the pair (u^*F, id) is stable and satisfies $E(u^*F, id) = 0$ we conclude that with a suitable choice of δ the Lagrangian v^*F satisfies the requirements of Theorem 2 with μ replaced by 2μ. In particular the pair (v^*F, id) is stable. Hence there exists a diffeomorphism $x = w(\xi)$ of $\mathbb{T}^n$ such that $w(\xi) - \xi$ is of period 1 and $E(v^*F, w) = 0$. Since $\det v_\xi \neq 0$ it follows from equation (2.6) that $E(F, v \circ w) = 0$. Moreover, Theorem 2 shows that the transformation w and hence $v \circ w$ has the required regularity properties. It therefore remains to show that $u = v \circ w$ up to a translation.

For this purpose note that, again by Theorem 2, there exists a constant c depending on ω, F and u such that

$$\|w - id\|_{C^{2\tau+\mu}} \le c\, |E(v^*F, id|_{C^0}^{\mu/(4\tau+2\mu)}$$

(replace μ by 2μ in that theorem and choose $s = 2\tau + \mu \notin \mathbb{Z}$). Since, by definition of v, there is an upper bound for

$$\|v - id\|_{C^{2\tau+2+\mu}} \le c\, |v - id|_{C^{4\tau+3+2\mu}}$$

we obtain from Lemma 7 that

$$\begin{aligned}\|u - v \circ w\|_{C^{2\tau+\mu}} &\le \|u - v\|_{C^{2\tau+\mu}} + \|v - v \circ w\|_{C^{2\tau+\mu}} \\ &\le \|u - v\|_{C^{2\tau+\mu}} + c\, \|w - id\|_{C^{2\tau+\mu}} \\ &\le \|u - v\|_{C^{2\tau+\mu}} + c\, |E(v^*F, id)|_{C^0}^{\mu/(4\tau+2\mu)}\end{aligned}$$

with a suitable constant $c>0$. By choice of $\delta>0$ we can make the right hand side of this inequality as small as we want. Moreover, in view of $\tau \geq 1$, we have $F \in C^{2\tau+4+\mu}$ and $u \in C^{2\tau+5+\mu}$. This allows us to apply Theorem 4 and we conclude that

$$u(\xi)=v \circ w(\xi+\xi_0)$$

for all $\xi \in \mathbb{R}^n$ and some fixed vector $\xi_0 \in \mathbb{R}^n$. This proves Theorem 5. □

6. Time dependent Hamiltonians and monotone twist maps

We first point out that all the results obtained so far remain valid for a Lagrangian $F(t, x, p)$ which depends periodically on time t. In this case we are trying to find solutions of the Euler equation which can be written in the form

$$x(t)=u(t_0+t, \xi_0+\omega t)$$

where the function $u(t, \xi)-\xi$ is periodic with period 1 in all variables. Moreover, for every fixed $t \in \mathbb{R}$ we assume that the map $\xi \rightarrow x=u(t, \xi)$ is a diffeomorphism of $\mathbb{T}^n$. This leads to the nonlinear partial differential equation

$$DF_p(t, u, Du)=F_x(t, u, Du) \tag{6.1}$$

for functions $x=u(t, \xi)$ where the first order differential operator D is now given by

$$D=\partial/\partial t+\sum_{j=1}^n \omega_j\, \partial/\partial \xi_j .$$

The frequency vector $\omega \in \mathbb{R}^n$ is required to satisfy the Diophantine conditions

$$|j \cdot \omega-k| \geq \gamma\, |(j, k)|^{-\tau}, \qquad j \in \mathbb{Z}^n,\ k \in \mathbb{Z},\ (j, k) \neq 0, \tag{6.2}$$

for some constants $\gamma>0$ and $\tau \geq n$. Note that (6.1) is the Euler equation for the variational problem defined by the functional

$$I(u)=I_F(u)=\int_{\mathbb{T}^{n+1}} F(t, u, Du)\, dt\, d\xi$$

whose gradient with respect to the L^2 inner product is given by $\nabla I(u) = -E(F, u)$ where

$$E(F, u) = DF_p(t, u, Du) - F_x(t, u, Du).$$

In this context the pair (F, u) is said to be *stable* if the matrix function

$$a(t, \xi) = u_\xi^T F_{pp}(t, u, Du) u_\xi$$

satisfies the conditions

$$\det a(t, \xi) \neq 0, \qquad (t, \xi) \in \mathbb{R}^{n+1},$$

$$\det \int_{\mathbb{T}^{n+1}} a(t, \xi)^{-1}\, dt\, d\xi \neq 0.$$

One verifies readily that with these changes in the notation the existence, uniqueness and regularity statements (Theorems 1–5) remain valid.

We shall use the time dependent version of Theorem 5 in order to prove the regularity theorem for an invariant curve of an area preserving C^∞ diffeomorphism

$$\Phi : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$$

of monotone twist type. In the covering space $\mathbb{R}^2$ this map is given by $\Phi(x, y) = (f(x, y), g(x, y))$ and is assumed to satisfy the following conditions

$$f(x+1, y) = f(x, y) + 1, \qquad g(x+1, y) = f(x, y),$$

the 1-from $\Phi^*\lambda - \lambda$ with $\lambda = y\,dx$ is exact on $S^1 \times \mathbb{R}$,

$$\partial f / \partial y > 0. \tag{6.3}$$

We assume that the invariant curve is given by an embedding

$$\psi = (u, v) : S^1 \to S^1 \times \mathbb{R}$$

such that, in the covering space, $u(\xi) - \xi$ and $v(\xi)$ are of period 1 and

$$u'(\xi) > 0$$

for all $\xi \in \mathbb{R}$. Moreover, ψ solves the nonlinear difference equation

$$\psi(\xi+\alpha)=\Phi \circ \psi(\xi) \tag{6.4}$$

for an irrational rotation number $\alpha \in \mathbb{R}$ which satisfies the Diophantine conditions

$$|p+\alpha q| \geq \gamma\, |q|^{-\tau} \tag{6.5}$$

for all integers $p \in \mathbb{Z}$ and $q>0$ and some constants $\gamma>0$ and $\tau \geq 1$. Note that this condition is equivalent to (6.2) with $\omega=\alpha$.

A result due to J. Moser [22] shows that Φ can be interpolated by a time dependent Hamiltonian differential equation

$$\dot{x}=H_y(t, x, y), \qquad \dot{y}=-H_x(t, x, y), \tag{6.6}$$

where $H(t, x, y)$ is a smooth (C^∞) Hamiltonian vector field which depends periodically on t and x and, in addition, satisfies the Legendre condition

$$H_{yy}(t, x, y)>0. \tag{6.7}$$

The time-1-map of H agrees with the given mapping Φ that is

$$\varphi^1(x, y)=\Phi(x, y) \tag{6.8}$$

where $\varphi^t \in \operatorname{Diff}(S^1 \times \mathbb{R})$ denotes the flow of H defined by

$$d/dt\varphi^t=J\, \nabla H(t, \varphi^t), \qquad \varphi^0=id.$$

We use this flow in order to extend the invariant curve ψ to an invariant torus

$$w(t, \xi)=(u(t, \xi), v(t, \xi))=\varphi^t \circ \psi(\xi-\alpha t) \tag{6.9}$$

for (6.6). This function satisfies the nonlinear differential equation

$$Dw=J\, \nabla H(t, w) \tag{6.10}$$

where $D=\partial/\partial t+\alpha\, \partial/\partial \xi$ is defined as above with $n=1$ and $\omega=\alpha$. Moreover, it follows from (6.4) and (6.8) that $u(t, \xi)-\xi$ and $v(t, \xi)$ are of period 1 in t and ξ.

The Legendre condition (6.7) allows us to transform the Hamiltonian system (6.6) into the Euler equations of the varitional problem corresponding to a C^∞

Lagrangian $F(t, x, p)$ which is of period 1 in t and x and it follows that the function $u(t, \xi)$ solves the Euler equation (6.1). If u meets the assumptions of Theorem 5 in the time dependent case, we can conclude that u and

$$v = F_p(t, u, Du)$$

are C^∞ functions proving our claim. We therefore have to verify that (F, u) is a stable pair as defined above. For this purpose observe that, by (6.7),

$$F_{pp}(t, x, p) > 0$$

so that it remains to show that $u_\xi(t, \xi) > 0$.

LEMMA 9. *Let $H \in C^2(\mathbb{T}^2 \times \mathbb{R})$ satisfy the Legendre condition* (6.7) *and suppose that the embedding $\psi = (u, v): S^1 \to S^1 \times \mathbb{R}$ satisfies $u'(\xi) > 0$ and* (6.4) *with $\Phi = \varphi^1$ and some number $\alpha \in \mathbb{R}$. Moreover, let $u(t, \xi)$ be defined by* (6.9). *Then*

$$u_\xi(t, \xi) > 0.$$

Proof. The proof of this statement is based on an argument due to Moser [22]. We define

$$X(t, \xi) = u_\xi(t, \xi + \alpha t), \qquad Y(t, \xi) = v_\xi(t, \xi + \alpha t),$$

and observe that

$$(X(t, \xi), Y(t, \xi)) = (\varphi^t \circ \psi)'(\xi) = d\varphi^t(\psi(\xi))\psi'(\xi) \neq 0 \tag{6.11}$$

for all t and ξ since $u'(\xi) > 0$. This allows us to introduce the angle

$$\theta(t, \xi) = \arg (X(t, \xi) + iY(t, \xi))$$

as a continuous function of t and ξ. Since $X(0, \xi) > 0$ for all ξ we may choose the function θ as to satisfy

$$-\pi/2 < \theta(0, \xi) < \pi/2. \tag{6.12}$$

Moreover, $X(1, \xi) = X(0, \xi + \alpha) > 0$ and hence

$$2\pi j - \pi/2 < \theta(1, \xi) < 2\pi j + \pi/2 \tag{6.13}$$

for all $\xi \in \mathbb{R}$ and some integer j. We shall prove that $j=0$ and observe for this purpose that $\varphi^t \circ \psi(\xi+\varepsilon) \neq \varphi^t \circ \psi(\xi)$ for all $(t, \xi) \in \mathbb{R}^2$ and all $\varepsilon > 0$. We can therefore define for $\varepsilon > 0$ a continuous function

$$\theta_\varepsilon(t, \xi) = \arg\,(\varphi^t \circ \psi(\xi+\varepsilon) - \varphi^t \circ \psi(\xi))$$

satisfying

$$-\pi/2 < \theta_\varepsilon(0, \xi) < \pi/2. \tag{6.14}$$

for all $\varepsilon > 0$ and all $\xi \in \mathbb{R}$. Observing that

$$\theta_\varepsilon(1, \xi) - \theta_\varepsilon(0, \xi + \alpha) \in 2\pi\mathbb{Z}$$

and, by definition,

$$\theta(t, \xi) = \lim_{\varepsilon \to 0} \theta_\varepsilon(t, \xi)$$

we obtain from (6.13) and (6.14) that

$$2\pi j - \pi/2 < \theta_\varepsilon(1, \xi) < 2\pi j + \pi/2. \tag{6.15}$$

for all $\varepsilon > 0$ and all $\xi \in \mathbb{R}$. Now we make use of the fact that

$$\varphi^t \circ \psi(\xi+1) - \varphi^t(\xi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for all t and ξ and hence

$$\theta_1(t, \xi) = 2\pi k$$

for some fixed integer k. Combining this observation with (6.15) and (6.14) we obtain that $j = k = 0$ and therefore it follows from (6.13) that

$$-\pi/2 < \theta(1, \xi) < \pi/2 \tag{6.16}$$

for all $\xi \in \mathbb{R}$ as claimed.

We shall use (6.16) in order to prove that $\theta(t, \xi) > -\pi/2$ for every $\xi \in \mathbb{R}$ and every $t \in [0, 1]$. We proceed by contradiction and assume that there were a $t \in (0, 1)$ such that $\theta(t, \xi) \leq -\pi/2$ for some $\xi \in \mathbb{R}$. By (6.16), there is a largest such t which we denote by t^* and we choose $\xi^* \in \mathbb{R}$ such that $\theta(t^*, \xi^*) = -\pi/2$. This implies that

$$X(t^*, \xi^*) = 0, \qquad Y(t^*, \xi^*) < 0.$$

It now follows from (6.11) that X and Y, considered as functions of t, satisfy the linearized differential equation

$$\dot{X} = H_{yx}X + H_{yy}Y, \qquad \dot{Y} = -H_{xx}X - H_{xy}Y.$$

In view of the Legendre condition (6.7) this shows that

$$\dot{X}(t^*, \xi^*) < 0$$

and hence $\theta(t, \xi^*) < -\pi/2$ for $t > t^*$ close to t^*. This contradicts the definition of t^* and we conclude that indeed $\theta(t, \xi) > -\pi/2$.

A similar argument using (6.12) shows that $\theta(t, \xi) < \pi/2$ and hence $X(t, \xi) > 0$ for all t and ξ. This proves Lemma 9. $\square$

The stronger regularity statement for ψ promised in the introduction is based on a regularity result for the partial differential equation (6.10) which we describe next.

We consider a function $H(t, x, y)$ on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ which is of period 1 in the t and x variables and define the functional

$$E(H, w) = Dw - J\,\nabla H(t, w)$$

for mappings

$$w = (u, v) : \mathbb{T}^{n+1} \to \mathbb{T}^n \times \mathbb{R}^n$$

such that $u(t, \xi) - \xi$ and $v(t, \xi)$ are of period 1 in all variables.

DEFINITION. The pair (H, w) is said to be *stable* if the matrix $u_\xi(t, \xi)$ is nonsingular for all $(t, \xi) \in \mathbb{R}^{n+1}$ and

$$\det \int_{\mathbb{T}^{n+1}} u_\xi^{-1} H_{yy}(t, u, v)(u_\xi^{-1})^T \, dt \, d\xi \neq 0.$$

The following existence theorem extends a result due to J. Moser [20] to the time dependent case improving, its regularity assumptions on the unperturbed Hamiltonian H^0.

THEOREM 6. *Let* $\omega \in \mathbb{R}^n$ *satisfy* $|\omega| \leq M$ *and the Diophantine conditions* (6.2) *for some constants* $\gamma > 0$, $\tau \geq n$ *and* $M \geq 1$. *Moreover, let* $H^0 \in C^l(\mathbb{T}^{n+1} \times \mathbb{R}^n)$, $u^0 \in C^{l+1}(\mathbb{T}^{n+1}, \mathbb{T}^n)$ *and* $v^0 \in C^l(\mathbb{T}^n, \mathbb{R}^n)$ *be given with* $l = 2\tau + 2 + \mu$, $\mu > 0$, *such that* $w^0 = (u^0, v^0)$ *is a stable solution of* $E(H^0, w^0) = 0$.

Then there exists a constant $\delta > 0$ *such that for every Hamiltonian H of class* C^l *satisfying*

$$|H - H^0|_{C^l} \leq \delta$$

the equation

$$E(H, w) = 0$$

admits a solution $w = (u, v): \mathbb{T}^{n+1} \to \mathbb{T}^n \times \mathbb{R}^n$ *with* $u \in C^{l-2\tau-1}$ *and* $v \circ u^{-1} \in C^{l-\tau-1}$ *provided that* $l - 2\tau - 1$ *and* $l - \tau - 1$ *are not integers. In the space* $C^{s-2\tau-1} \times C^{s-\tau-1}$ *with* $s < 1$ *the pair* $(u, v \circ u^{-1})$ *depends continuously on* $H \in C^l$. *If, in addition,* $H \in C^m$ *with* $m \geq l$ *and* $m - 2\tau - 1$ *and* $m - \tau - 1$ *are not integers then* $u \in C^{m-2\tau-1}$ *and* $v \circ u^{-1} \in C^{m-\tau-1}$. *In particular,* $H \in C^\infty$ *implies* $w \in C^\infty$. *Moreover, if H is real analytic then w is real analytic.*

Combining this existence statement with a corresponding uniqueness result as in section 5 one derives the following regularity theorem.

THEOREM 7. *Let* $\omega \in \mathbb{R}^n$ *satisfy the Diophantine conditions* (6.2) *and let* $H \in C^l(\mathbb{T}^{n+1} \times \mathbb{R}^n)$ *be given with* $l > 2\tau + 2$. *Moreover, let* $w = (u, v): \mathbb{T}^{n+1} \to \mathbb{T}^n \times \mathbb{R}^n$ *be a stable solution of* $E(H, w) = 0$ *such that* $u \in C^{l+1}$ *and* $v \in C^l$. *Then* $H \in C^\infty$ *implies* $w \in C^\infty$ *and if H is real analytic then w is real analytic.*

Note that the smoothness assumption in Theorem 7 again agrees with the smoothness required for the perturbation theorem.

We do not carry out the proofs of Theorem 6 and Theorem 7, they will appear elsewhere. We do, however, point out that in contrast to the method described in sections 2–5 the proof is based on the transformation theory for Hamiltonian systems. The interation procedure requires the composition of infinitely many symplectic transformations. This quite familiar technique is more complicated, allows however to replace the number 4τ in the statement for the Lagrangian F by the number 2τ for the Hamiltonian H. Moreover, equation (6.10) involves only first derivatives of H as opposed to second derivatives of F in (6.1) so that the loss of derivatives in the perturbation theory for (6.10) is only $2\tau + 1$.

We can now apply Theorem 7 to the special case where the Hamiltonian $H \in C^\infty(\mathbb{T}^2 \times \mathbb{R})$ has been obtained by interpolating a monotone twist map

$$\Phi : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$$

and the invariant torus $w : \mathbb{T}^2 \to S^1 \times \mathbb{R}$ is related to an invariant curve

$$\psi = (u, v) : S^1 \to S^1 \times \mathbb{R}$$

via (6.9). It follows from the above considerations that if Φ and ψ satisfy the requirements of Theorem 8 below then H and w satisfy those of Theorem 7. In particular, $\psi \in C^{l+1}$ implies $w \in C^{l+1}$ and Lemma 9 shows that if $u'(\xi) > 0$ for all ξ then the pair (H, w) is stable in the sense of the above definition. This allows us to conclude that $w \in C^\infty$ and hence $\psi \in C^\infty$. Thus we have proved the following regularity result for invariant curves.

THEOREM 8. *Let Φ satisfy the conditions* (6.3) *and suppose that*

$$\psi = (u, v) : S^1 \to S^1 \times \mathbb{R}$$

is a parametrized curve such that, in the covering, $u(\xi) - \xi$ and $v(\xi)$ are of period 1 and $u'(\xi) > 0$ for all $\xi \in \mathbb{R}$. Moreover, assume that ψ satisfies the nonlinear difference equation (6.4) *for an irrational number $\alpha \in \mathbb{R}$ satisfying the Diophantine inequalities* (6.5) *with some constants $\gamma > 0$ and $\tau \geq 1$. Finally, suppose that $\Phi \in C^\infty$ and $\psi \in C^{l+1}$ with $l > 2\tau + 2$. Then $\psi \in C^\infty$.*

We point out that the map Φ in the above theorem is not required to be close to an integrable mapping.

It is an open question whether the regularity of an invariant curve ψ can be concluded under weaker differentiability assumptions on ψ.

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