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Some links with non-trivial polynomials and their crossing-numbers

W. B. R. Lickorish and M. B. Thistlethwaite

One of the main applications of the Jones polynomial invariant of oriented links has been in understanding links with (reduced, connected) *alternating* diagrams [2], [8], [9]. The Jones polynomial for such a link is never trivial, and the number of crossings in such a diagram is the crossing-number of the link (that is, no diagram of the link has fewer crossings). Here, the non-triviality of the Jones polynomial is established for a wider class of links that includes most pretzel links and the Whitehead double of any alternating knot. The idea of a *semi-alternating* link is defined and for such a link the crossing-number is determined. Finally, the crossing-number of any Montesinos link is found. That determination uses both the polynomial of Jones and its two-variable semi-oriented generalisation (due to Kauffman); it is shown that the latter polynomial is never trivial for a Montesinos knot.

§1. The breadth of the bracket polynomial

The bracket polynomial of Kauffman for a planar diagram D of an unoriented link is an element $\langle D\rangle \in \mathbb{Z}[A^{\pm 1}]$ defined as follows. Let $c_1, c_2, \dots, c_n$ denote the crossings of D. A *state* for D is a function $s:\{i:1\le i\le n\}\to\{-1,1\}$. Let sD be D with all of its crossings nullified according to the rule that a crossing $\times$ labelled c_i changes to $\asymp$ if $s(i)=1$ and to $)($ if $s(i)=-1$. Let $|sD|$ be the number of components (they are disjoint simple closed curves) of sD. Then

$$\langle D\rangle = \sum_{s\in 2^n}\langle D\mid s\rangle \quad\text{where}\quad \langle D\mid s\rangle = A^{\Sigma s(i)}(-A^{-2}-A^{2})^{|sD|-1}.$$

Recall that if D is given an orientation so that it has writhe $w(D)$, then the Jones polynomial $V_D(t)$ is $(-A)^{-3w(D)}\langle D\rangle$ when $A=t^{-1/4}$.

Now let s_+ be the state for which $s_+(i)=1$ for every i, and let s_- be the state for which $s_-(i)=-1$ for every i.

DEFINITION. A link diagram D is said to be *adequate* if $|s_+D| > |sD|$ for every state s for which $\sum_i s(i) = n-2$ and $|s_-D| > |sD|$ for every state s for which $\sum_i s(i) = 2-n$. This is equivalent to saying that when s_+D is created from D by changing $\times$ to $\asymp$ at each crossing, the two segments in this new piece of diagram belong to different components of s_+D; also a similar condition applies to s_-D.

The standard example of adequacy is that any alternating diagram with no nugatory (or removable) crossing is adequate. Now, $\langle D \rangle$ is an element of $\mathbb{Z}[A^{\pm 1}]$. In what follows, $\mathfrak{M}\langle D \rangle$ and $m\langle D \rangle$ will denote the maximum and minimum exponents of A that occur in $\langle D \rangle$, and $\mathfrak{B}\langle D \rangle$ will denote the breadth of $\langle D \rangle$ namely $\mathfrak{M}\langle D \rangle - m\langle D \rangle$.

PROPOSITION 1. *Let D be an adequate diagram with n crossings of an unoriented link. Then the terms of highest and lowest degree in $\langle D \rangle$ are*

$$(-1)^{|s_+(D)|-1}A^{n+2|s_+D|-2} \quad \textit{and} \quad (-1)^{|s_-D|-1}A^{-n-2|s_-D|+2}$$

respectively, so that $\mathfrak{B}\langle D \rangle = 2n + 2(|s_+D| + |s_-D|) - 4$.

Proof. This is essentially Kauffman's proof for alternating links. By definition, $\langle D \mid s_+ \rangle = A^n(-A^{-2}-A^2)^{|s_+D|-1}$, so that $\mathfrak{M}\langle D \mid s_+ \rangle = n + 2|s_+D| - 2$. Now, if s is *any* other state there is a sequence of states, $s_+ = s_0, s_1, s_2, \ldots, s_k = s$, such that s_{r-1} and s_r agree on all but one element i_r of $\{i : 1 \le i \le n\}$ and $s_{r-1}(i_r) = 1$, $s_r(i_r) = -1$. Thus, for each $r \le k$, $\sum_i s_r(i) = n - 2r$ and $|s_rD| = |s_{r-1}D| \pm 1$. Hence, as r increases from r to $r+1$, $\mathfrak{M}\langle D \mid s_r \rangle$ either decreases by 4 or stays the same. However the fact that D is adequate means that $\mathfrak{M}\langle D \mid s_r \rangle$ decreases at the first stage when r increases from 0 to 1, thus $\mathfrak{M}\langle D \mid s_+ \rangle > \mathfrak{M}\langle D \mid s_r \rangle$ for all $r \ge 1$. This means that $\mathfrak{M}\langle D \mid s_+ \rangle > \mathfrak{M}\langle D \mid s \rangle$ for every state $s \ne s_+$, so that $\mathfrak{M}\langle D \rangle = \mathfrak{M}\langle D \mid s_+ \rangle = n + 2|s_+D| - 2$. The analysis for $m\langle D \mid s \rangle$ uses s_- in exactly the same manner. □

Note that in any particular case $|s_+D|$ and $|s_-D|$ are readily determined from the presentation.

COROLLARY 1.1 (Kauffman). *If D is an n-crossing connected alternating diagram with no nugatory crossing, then $\mathfrak{B}\langle D \rangle = 4n$. (This is because in this case $|s_+D| + |s_-D|$ is the number of regions of the link diagram.)* □

COROLLARY 1.2. *If D is an adequate diagram of a knot (with at least one crossing) then $\mathfrak{B}\langle D \rangle > 1$ and so $V_D(t) \ne 1$.*

Proof. The result follows at once from the fact that the adequacy of D implies that $|s_\pm D| \ge 2$. □

Note that this result provides a rapid visual test for knottedness.

EXAMPLE. Let $D(a_1, \ldots, a_r; b_1, \ldots, b_s)$ be the diagram of a pretzel link shown in Fig. 1, where the a_i and b_i are positive integers denoting numbers of half-twists in the directions shown (after mutation, any pretzel link is of this form).

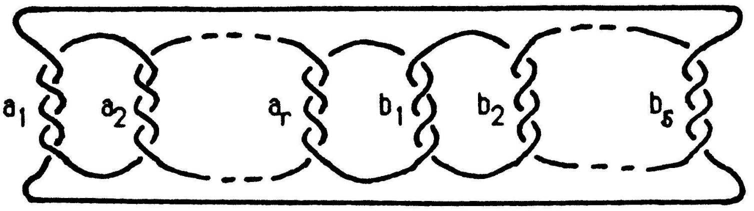
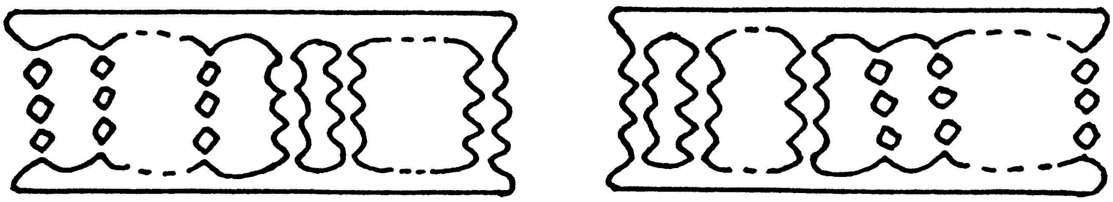


Figure 1

Then s_+D and s_-D are of the form shown in Fig. 2.



s_+D s_-D

Figure 2

It is apparent that $D(a_1, \ldots, a_r; b_1, \ldots, b_s)$ is adequate provided that each $a_j \geq 2$, each $b_j \geq 2$, $r \geq 2$ and $s \geq 2$. Thus for this presentation

$$\mathfrak{M}\langle D\rangle = 3\sum a_j + \sum b_j + 2(s-r) - 2$$

$$m\langle D\rangle = -3\sum b_j - \sum a_j - 2(r-s) + 2$$

so that $\mathfrak{B}\langle D\rangle = 4n - 4$ where n is the total number of crossings.

§2. Paralleled links and doubled knots

In what now follows we explore to some extent the interaction of 'adequacy' with the methods of constructing links as satellites and as sums of 4-end tangles. If D is an n-crossing link diagram let D^r denote the result of replacing every link-component of D by r components all parallel in the plane. Thus to each crossing of D there correspond r^2 crossings of D^r. In creating $s_+(D^r)$ these crossings are changed in a way that parallels the change that creates s_+D from D. For $r=2$ this is shown in Fig. 3.

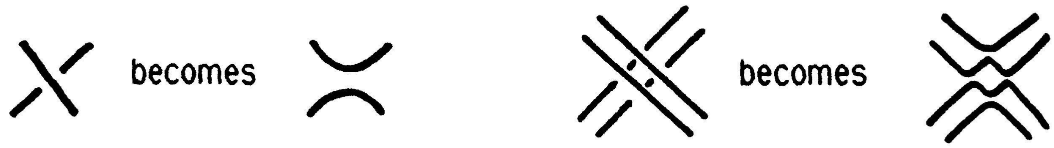


Figure 3

Thus $s_+(D^r)=(s_+D)^r$ and similarly $s_-(D^r)=(s_-D)^r$.

PROPOSITION 2. *If D is an adequate n-crossing diagram, D^r is adequate and* $\mathfrak{M}\langle D^r\rangle = nr^2+2r\,|s_+D|-2$, $m\langle D^r\rangle = -nr^2-2r\,|s_-D|+2$.

Proof. When a crossing of D^r is nullified to create $s_+(D^r)$ the two segments created either belong to different parallels (in $(s_+D)^r$) of one component of s_+D, or to parallels of different components of s_+D. Thus D^r is adequate and the values of the exponents follow immediately from Proposition 1 and the above remarks. □

COROLLARY 2.1. *If D is a connected alternating n-crossing diagram with no nugatory crossing, then* $\mathfrak{B}\langle D^r\rangle = 2nr^2+2nr+4r-4$. □

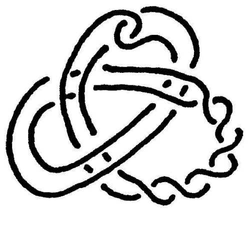
THEOREM 3. *Let $\mathfrak{W}K$ be an untwisted Whitehead double of a non-trivial knot K that has an adequate diagram D. Then* $V_{\mathfrak{W}K}(t)\neq 1$.

Proof. Consider a standard diagram $\mathfrak{W}D$ of $\mathfrak{W}K$ that 'doubles' the adequate diagram D. This new diagram has a pair of crossings, called "the clasp", such that if one of them, c, is switched the unknot is created (up to isotopy). Application of the crossing switching formula for the Jones polynomial shows that, if $V_{\mathfrak{W}K}(t)=1$, then the Jones polynomial of the link created by nullifying c would be equal to that polynomial for the trivial 2-component link. The link in question is isotopic to D^2 modified by having τ positive half-twists, $\tau=-2w(D)$, inserted between the two components so that their linking number is zero. Let D^{2*} denote this diagram (see Fig. 5 for when D is the left-hand trefoil). It may be assumed, after reflection of D if necessary, that $w(D)\leq 0$ so that $\tau\geq 0$. If $\tau=0$ then $D^{2*}=D^2$ and an easy induction on τ shows that

$$\langle D^{2*}\rangle = A^{\tau}\langle D^2\rangle + A^{\tau-2}(1-A^{-4}+A^{-8}-\cdots-(-1)^{\tau}A^{-4\tau+4}).$$

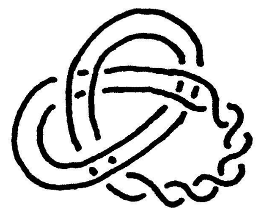
Now, by Proposition 2, if D has n crossings, $\mathfrak{M}\langle D^2\rangle = 4n+4\,|s_+D|-2$. Thus

$\mathfrak{M}\langle D^{2*}\rangle = 4n + 4\,|s_+D| - 2 + \tau$. If now D^{2*} is oriented in any way, then $w(D^{2*}) = 2w(D) = -\tau$. Hence the Jones polynomial for D^{2*} (when $A = t^{-1/4}$) is $(-A)^{3\tau}\langle D^{2*}\rangle$. In this expression the maximum exponent of A is $4n + 4\,|s_+D| - 2 + 4\tau$, and this certainly exceeds 2 which would be its values were D^{2*} to have the same Jones polynomial as the trivial link. □



$\mathfrak{W}D$

Figure 4



D^{2*}

Figure 5

The motivation for considering Whitehead doubles was the search for a non-trivial knot K for which the oriented polynomial $P_K = 1$. The oriented polynomial P_K specialises to the Alexander polynomial Δ_K and to the Jones polynomial V_K. If K is any knot $\Delta_{\mathfrak{W}K} = 1$ so there is a chance that $P_{\mathfrak{W}K}$ might be 1. The theorem shows that is not the case for knots K with adequate diagrams.

DEFINITION. A tangle diagram with four ends is *adequate* if each of the two closures of its depicted in Fig. 6 is an adequate diagram of some link. An example is when each of those closures is a connected alternating diagram with no nugatory crossing and in that case the tangle diagram will be called *strongly alternating*.

Figure 6

PROPOSITION 4. *The partial sum of two adequate tangle diagrams each with four ends is adequate.* (*Here the 'partial sum' refers to the tangle diagram resulting from the insertion of the two diagrams into the shaded discs of Fig.* 7.)

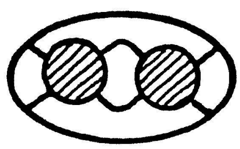


Figure 7

Proof. Suppose that the partial sum of the two tangle diagrams is closed off in one of the two ways of Fig. 6 to give a link diagram D. If D is inadequate then, in creating $s_{\pm}D$, when some crossing c of D is nullified the two segments formed near c lie in the same component of $s_{\pm}D$. But c is in one of the original tangles and the appropriate closure X of that tangle can be chosen so that the two segments lie in the same component of $s_{\pm}X$, contradicting the adequacy of X. □

The consequence of this proposition is that adequate 4-ended tangles may be added together in any tree-like pattern and the result is always adequate. Then either closure gives an adequate link whose bracket polynomial can be analysed in the above way. In particular many (but not all) algebraic links occur in this way. Any knot so constructed has non-trivial Jones polynomial as does its Whitehead double.

DEFINITION. A *semi-alternating* diagram D is a non-alternating diagram obtained by summing two strongly alternating 4-end tangles T_1 and T_2 by taking their partial sum using the format of Fig. 7 and taking its closure as in the right-hand part of Fig. 6.

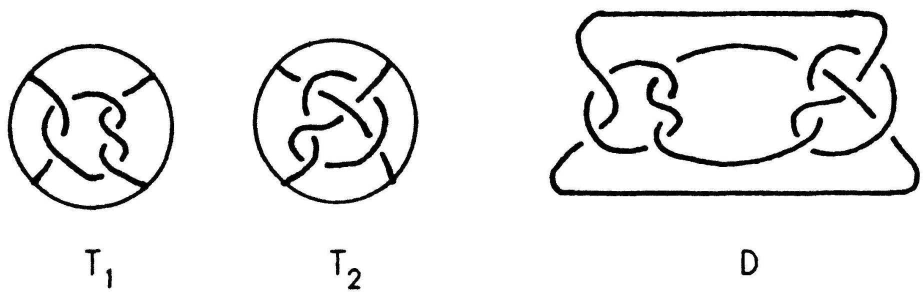


Figure 8

Of course, a semi-alternating diagram is adequate. As an example, Fig. 8 shows how the Kinoshita–Terasaka knot with trivial Alexander polynomial is constructed in this way; its mutant discovered by J. H. Conway is evidently also semi-alternating. Another example is furnished by the pretzel diagram of Fig. 1, so long as $r \geq 2$, $s \geq 2$ and, for all j, $a_j \geq 2$ and $b_j \geq 2$. A more detailed analysis of the bracket polynomial for semi-alternating diagrams follows in the next section.

§3. The crossing-number of a semi-alternating link

The results of §1 will now be used to establish the crossing-number of a link with a semi-alternating diagram.

PROPOSITION 5. *Let L be a link admitting a semi-alternating diagram D with n crossings. Then the Jones polynomial $V_L(t)$ is non-alternating, its extreme coefficients are ± 1, and its breadth is $n-1$.*

Proof. Let D be the sum of strongly alternating tangle diagrams T_1 and T_2 with n_1 and n_2 crossings respectively. Let the regions of D be coloured black and white in chequerboard fashion; without loss of generality the crossings of T_1 and T_2 all conform to the pictures in Fig. 10. Also, for $i = 1, 2$ let b_i and w_i be the numbers of black regions and white regions respectively in the 'vignette' of T_i (see Fig. 8).

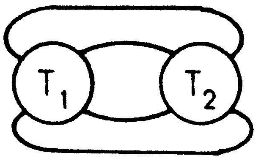


Figure 9

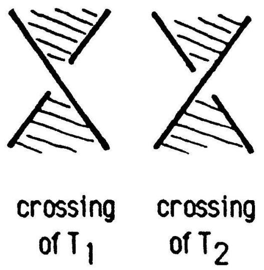


Figure 10

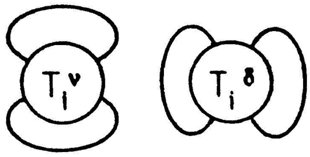


Figure 11

Then D has $b_1 + b_2 - 2$ black regions and $w_1 + w_2 - 2$ white regions. In the diagram s_+D there is a single simple closed curve that contains all four of the arcs connecting T_1 to T_2. Each other component of s_+D encloses one of the $b_1 - 2$ black regions of T_1 not incident upon these connecting arcs or one of the corresponding $w_2 - 2$ white regions of T_2. Thus $|s_+D| = (b_1 - 2) + (w_2 - 2) + 1 = b_1 + w_2 - 3$; $|s_-D| = w_1 + b_2 - 3$. From Proposition 1, the extreme coefficients of $\langle D \rangle$ are ± 1, and the quotient of the term of highest degree in $\langle D \rangle$ by that of lowest degree is

$$(-1)^{|s_+D| - |s_-D|} A^{2|s_+D| + 2|s_-D| + 2n - 4} = (-1)^n A^{4(n-1)}.$$

Hence the breadth of $V_L(t)$ is $n - 1$, and the non-alternating nature of $V_L(t)$ is deduced from the fact that the "end" terms have coefficients of equal sign if and only if this breadth is odd. □

It is intended to use Proposition 5, together with results of [9], to show that a link L with a semi-alternating diagram cannot be projected with fewer crossings than in that diagram. In order to do this it is necessary first to show that L is not a split link, that is, it cannot be separated by a 2-sphere in $S^3 - L$. With this purpose in mind, we consider a simple geometric consequence of Proposition 5. Let (B, T) be a "2-string tangle" as defined and discussed in [4]; T consists of two disjoint arcs properly embedded in a 3-ball B. Recall from [4] that (B, T) is *untangled* if it is pairwise homeomorphic to two straight segments in the standard unit ball.

COROLLARY 5.1. *If (B, T) is a tangle admitting a strongly alternating diagram, then (B, T) is not untangled.*

Proof. Any strongly alternating diagram can be summed with itself-rotated (through $\pi/2$) to produce a semi-alternating diagram of some link L. By Proposition 5, V_L has breadth greater than 1, so L is not a trivial link of one or two components. Also by Proposition 5, V_L is non-alternating. Now any sum of two untangles is a two-bridged link which, if non-trivial, has an alternating Jones polynomial by Theorem 1 of [9]. □

PROPOSITION 6. *Any semi-alternating link L is not split.*

Proof. Let L admit a diagram which is the sum of strongly alternating tangle diagrams T_1 and T_2, then there is a 2-sphere S^2 which meets L transversely in four points and which separates the subsets L_1 and L_2 of L projecting to T_1 and T_2 respectively. For each $i \in \{1, 2\}$ let B_i be the closure in S^3 of the component of $S^3 - S^2$ containing L_i. Suppose some 2-sphere F lying in the interior of some B_i separates L; then each closure of the tangle diagram T_i will represent a split link, in contradiction to Theorem 1(a) of [7]. On the other hand, suppose there is a 2-sphere F, separating L, which has non-empty intersection with S^2. We may assume that F meets S^2 transversely in a finite number of disjoint, simple, closed curves. Let C be a component of $F \cap S^2$ which is innermost on F. Then C bounds a disc Δ in F which is properly embedded in some B_i, say B_1 without loss of generality. If Δ does not separate L_1 in B_1, F may be isotoped in $S^3 - L$ so as to reduce the number of components of $F \cap S^2$. Therefore, if the case $F \cap S^2 = \emptyset$ does not apply, we may assume that Δ separates L_1 in B_1; also, if we are not in the previous case, each component of $S^2 - \partial\Delta$ contains two points of $L \cap S^2$. Let the subsets of L_1 in the two components of $B_1 - \Delta$ be λ_1 and λ_2. We may suppose that at least one of the λ_i, say λ_1, is not an unknotted spanning arc of B_1; otherwise (B_1, L_1) will be untangled, contradicting Corollary 5.1. This subset λ_1 will represent a non-trivial connected summand in the alternating link formed from either closure of the tangle T_1, so, by Theorem 1(b) of [7], this subset will appear as a subset τ of the tangle diagram T_1 that is separated from the rest of T_1 by a circle in the projection plane meeting T_1 transversely in two points. Replacing τ by a simple arc in the projection plane will result in a new tangle diagram T_i' which is still strongly alternating, and which represents a tangle (B_1, L_1') separated by a disc in the same way as was (B_1, L_1). Therefore, T_1 can be reduced to a strongly alternating diagram representing an untangle, and this contradicts Corollary 5.1. □

THEOREM 7. *Let L be a link admitting a semi-alternating diagram with n crossings. Then L is non-alternating and its cross-number is n.*

Proof. $V_L(t)$ is non-alternating by Proposition 5, and L is not split by Proposition 6. Thus L admits no alternating diagram by Theorem 1(ii) of [9]. Then, using the breadth $n-1$ of $V_L(t)$ given by Proposition 5, together with Theorem 2(ii) of [9], the crossing-number of L is strictly greater than $n-1$ (the hypothesis of diagrammatic primeness in that theorem is no hindrance here; see the first paragraph of the proof of Theorem 2(ii) of [9]). But L admits an n-crossing diagram by hypothesis, so n is its crossing-number. □

§4. The crossing-number of a Montesinos link

Let L be an arborescent link based on a star-shaped tree, otherwise known as a *Montesinos link.* Then L (or its reflection) admits a diagram D composed of $m \geq 3$ rational [1] tangle diagrams $R_1, R_2, \ldots, R_m$ and $k \geq 0$ half-twists, put together as in Fig. 12 (if $m = 2$, L is 2-bridged). Recall that in [1] it is shown that a standard rational tangle diagram corresponds to any expansion of a rational number p/q as a repeated fraction; choice of the expansion in which all terms have the same sign gives an alternating diagram.

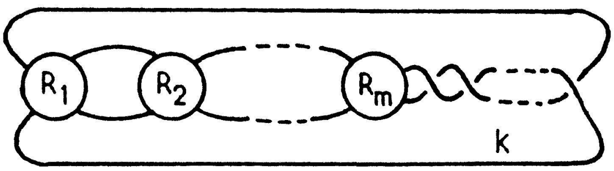


Figure 12

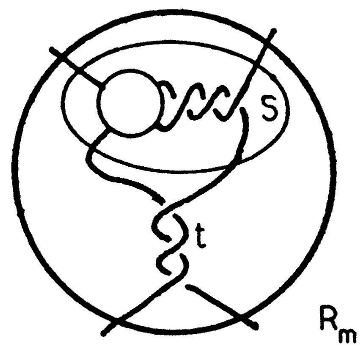


Figure 13

If D is not alternating and $k > 0$ it is easy to change the diagram to another diagram for L of the same form but with fewer crossings and a smaller value for k. Thus, without loss of generality, we may assume (i) that D is alternating or (ii) that $k = 0$ and each R_i is an alternating rational tangle diagram, with at least two crossings, placed in D so that the two lower ends of R_i belong to arcs incident to a common crossing in R_i (as in Fig. 13). If the diagram D of Fig. 12 satisfies one of these two conditions we shall call D a *reduced Montesinos diagram.* The aim of this section is to show, in Theorem 10, that the crossing-number of L is precisely the number of crossings in any reduced Montesinos diagram of L.

If the reduced Montesinos diagram D is alternating, Corollary 1 of [9] applies, so we only need to consider the case when condition (ii) applies. Then, some of the tangle diagrams R_i contain crossings of only the first sort illustrated in Fig. 10 (with respect to some black and white colouring of the regions of D) and the remaining tangle diagrams contain crossings only of the second sort. The polynomial invariants that will be considered are unchanged by mutation, as is the question of link-splitness (a link is split if and only if its double branched cover is reducible; such covers are not changed by mutation). The proof of Theorem 10 depends only on these mutation invariants and on n, the number of crossings of D, so D may be changed by mutations corresponding to permutations of the R_i (which certainly leave n unchanged). Thus it is assumed henceforth that D is a sum of *alternating* tangle diagrams T_1 and T_2, where T_1 is constituted from r tangles $R_1, \ldots, R_r$ containing crossings of only the first sort of Fig. 10, and T_2 is constituted from s tangles $R_{r+1}, \ldots, R_m$ containing crossings of only the second sort. Changing the colouring of the regions if necessary, we can assume that $r \geq 2$ and $s \geq 1$. The special case where L is a pretzel link is illustrated in Fig. 1.

There are two cases to consider, namely $s \geq 2$ and $s = 1$. If $s \geq 2$, D is semi-alternating, and so the conclusion of Theorem 10 follows in this case from Theorem 7. For the case $s = 1$, we need to invoke the results of [10] on Kauffman's invariant of regular isotopy $\Lambda_D(a, z)$. Let the highest exponent of the variable z occurring in any term of $\Lambda_D(a, z)$ be the *z-degree* of $\Lambda_D(a, z)$.

LEMMA 8. *Let D be an n-crossing, non-alternating, reduced Montesinos diagram with $r \geq 2$ and $s = 1$. Then the z-degree of $\Lambda_D(a, z)$ is $n-2$, and the coefficient of z^{n-2} in $\Lambda_D(a, z)$ is either $a^{-2} + 1$ or $1 + a^2$.*

Proof. Repeated use will be made of the crossing change formula

$$\Lambda_{D_+} + \Lambda_{D_-} = z(\Lambda_{D_0} + \Lambda_{D_\infty})$$

applied to the crossing of $T_2 = R_m$ that appears lowest in Fig. 13. We proceed by induction on the number t of half-twists indicated in that figure.

(i) Suppose the sub-tangle S of R_m has no crossings. Then $t \geq 2$, so to provide a basis for the induction we need first to deal with the case where $t = 2$. Switching the 'bottom' crossing of R_m results in a diagram D_- isotopic to one with $n-2$ crossings; hence by Theorem 3 of [10] the z-degree of Λ_{D_-} is less than $n-2$. One of the nullifications of this crossing results in a non-alternating diagram D_0 with $n-1$ crossings and a bridge of length 3, so by Theorem 3 of [10] the z-degree of Λ_{D_0} is at most $n-4$. The other nullification, followed by the removal of a kink, results in a prime $(n-2)$-crossing alternating diagram D_∞ with Conway basic polyhedron 1*. Therefore, from Theorem 4 of [10], the coefficient of z^{n-3} in Λ_{D_∞} is $a^{\pm 1}(a^{-1}+a)$. The desired result then follows from the crossing-change formula for Λ.

If $t>2$, we can use the inductive hypothesis to get the coefficient $a^{\pm 1}(a^{-1}+a)$ from Λ_{D_0}, with no contribution from D_- or D_∞.

(ii) If the sub-tangle S is non-trivial, the analysis is similar; for $t=1$ we get the coefficient $a^{\pm 1}(a^{-1}+a)$ from Λ_{D_-} and a contribution $a^{-2}+2+a^2$ from Λ_{D_∞}, and for $t>1$ one proceeds exactly as for the inductive step in the case where S is trivial. □

The motivation behind Lemma 8 is that it tells us, in conjunction with Theorem 3 of [10], that in the case $s=1$ the Montesinos link L cannot be projected with fewer than n crossings unless it admits an $(n-1)$-crossing, prime, alternating diagram. But then, by Kauffman's result (Corollary 1.1 of this paper), $\mathfrak{B}\langle D\rangle$ would have to equal $4(n-1)$; we will show that $\mathfrak{B}\langle D\rangle<4(n-1)$. This can be accomplished by means of the method of Proposition 1, but in this instance D is inadequate. Here it is easier to use instead the *numerator formula* for the bracket polynomial of a link diagram D that is a sum of tangle diagrams T_1 and T_2:

$$-(A^{-4}+1+A^4)\langle D\rangle$$
$$=\langle T_1^\gamma\rangle\langle T_2^\delta\rangle+\langle T_1^\delta\rangle\langle T_2^\gamma\rangle+(A^{-2}+A^2)(\langle T_1^\gamma\rangle\langle T_2^\gamma\rangle+\langle T_1^\delta\rangle\langle T_2^\delta\rangle)$$

where D, T_i^γ and T_i^δ are as in Figs. 9 and 11. This formula, of a type due to Conway [1], may easily be proved by induction on the number of crossings of D, using the recurrence formula for $\langle D\rangle$. The version of this formula for the oriented polynomial invariant $P_L(l, m)$, for oriented tangle diagrams for which inputs and outputs alternate around the periphery of the diagram, is given in [6]. A thorough discussion of such formulae in the context of the dichromatic polynomial of a graph is given in [11].

LEMMA 9. *Let D be as in Lemma* 8. *Then* $\mathfrak{B}\langle D\rangle<4(n-1)$.

Proof. It is clear that the numerator formula can be used to obtain an upper bound on $\mathfrak{B}\langle D\rangle$, using the extreme terms of the bracket polynomials of the alternating diagrams T_i^γ and T_i^δ $(i=1,2)$. From this numerator formula it is readily checked that $\mathfrak{B}\langle D\rangle \leq 4(n-1)$. The required strict inequality arises from the cancellation of terms in the right-hand expression of the numerator formula, on account of the existence of at least one nugatory crossing in T_2^γ. □

Collecting these results, we arrive finally at

THEOREM 10. *If a link L admits an n-crossing, reduced Montesinos diagram, then L cannot be projected with fewer than n crossings.* □

Recall that the semi-oriented polynomial invariant of ambient isotopy (due to Kauffman [3]) is defined by the formula

$$F_L(a, z) = a^{-w(D)}\Lambda_D(a, z)$$

where D is any diagram of an oriented link L and $w(D)$ is the writhe of D. In the absence of any known non-trivial knot with trivial F-polynomial, the following is of some interest.

THEOREM 11. *If K is a Montesinos knot, then $F_K(a, z) \neq 1$.*

Proof. Let D be a reduced Montesinos diagram of K. Using the above notation, if $s = 1$ the z-degree of Λ_D, hence also that of F_K, is greater than zero by Lemma 8. If, on the other hand, $s > 1$, then F_K cannot equal 1 as $V_K(t) \neq 1$, by Corollary 1.2, and $V_K(t) = F_K(t^{-3/4}, -(t^{-1/4} + t^{1/4}))$, as proved in [5]. □

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