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On trace forms of hilbertian fields

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1. Two conjectures

Let L/K be a finite separable field extension. With it we associate the *trace form* which is the following symmetric bilinear form over K:

$$L \times L \to K, \qquad (x, y) \to \mathrm{Tr}_{L/K}(xy).$$

This form will be denoted by $\mathrm{Tr}_{L/K}\langle 1\rangle$ or by $\langle L\rangle$. If P is an ordering of K, then it is well known that $\mathrm{sign}_P(\langle L\rangle) \geqq 0$. More precisely, this signature is the number of extensions of the ordering P to L. Given now a *positive* symmetric bilinear form φ – that is $\mathrm{sign}_P(\varphi) \geqq 0$ for all orderings P of K – one may ask: Does there exist an extension L/K such that $\varphi \sim \langle L\rangle$. Here $\sim$ denotes Witt equivalence. If there exists such an L we call φ *algebraic*. In view of the methods used in [6], [7] it seems natural to conjecture the following:

CONJECTURE 1. *Let K be an hilbertian field* (that is, a field for which Hilbert's irreducibility theorem holds). *Then every positive form over K is algebraic.*
A special – perhaps more accessible – case of this conjecture is the following:

CONJECTURE 2. *Let K be a function field* (that is a finitely generated extension of transcendence degree >0 of an arbitrary field k). *Then every positive form over K is algebraic.* (Note that K is hilbertian [4].)

The main result of [6] is the proof that conjecture 1 is true for algebraic number fields. The proof proceeds in two steps: one is an application of Hilbert's irreducibility theorem, the other one considers field extensions given by irreducible trinomials $f(T) = T^m + \alpha T + \beta$.

The purpose of this paper is to show that these methods are sufficient to prove the conjectures in some interesting special cases, namely for

– hilbertian fields which are not formally real

- function fields of transcendence degree 1 over $\mathbb{R}$ (or any real closed field)
- forms of dimension 1 and 2
- forms $\langle \alpha_1, \ldots, \alpha_n \rangle$ with all α_i totally positive.

In addition we will see that it is sufficient to prove the conjectures for odd-dimensional forms.

It will turn out that one can combine the methods mentioned above in various ways to get the desired results. Since our subject matter seems to be still developing we include some of these possible alternatives.

We are very grateful to L. Bröcker, A. Prestel, and W. Waterhouse for useful remarks. In particular, Bröcker pointed out to us that the crucial lemma 4 holds for hilbertian fields. This allowed to prove corollary 1 in the present generality. Exactly the same remark was made by Waterhouse. Prestel showed us his work on real function fields [5]. Whereas it seems that our proof of his main result (our corollary 5) is simpler than his, it should be noted that he gets more precise results in this case.

2. Five lemmas

The purpose of this section is to construct explicitly a supply of "elementary" algebraic Witt classes. The necessary field extensions L/K will always be given by irreducible polynomials. To find these polynomials we will either use Hilbert's irreducibility theorem [4] or Eisenstein's criterion. In the next sections we will combine these elementary constructions to get our main results. One may hope that it will be possible to realize more Witt classes by use of more sophisticated explicit constructions (e.g. using polynomials with more nonzero coefficients).

We will consider only fields of characteristic not 2.

LEMMA 1. *Let K be a field of characteristic $p \neq 0$, 2 admitting a non-trivial discrete valuation. Then for every $\lambda \neq 0$ there exists an extension L/K such that $\langle \lambda \rangle \sim \mathrm{Tr}_{L/K} \langle 1 \rangle$.*

Proof. (See [3] page 7.) Let π be a uniformizing parameter. Consider the irreducible separable (Eisenstein) polynomial

$$f(T) = T^p + \lambda \pi^2 T + \pi$$

and let $L = K[T]/f(T)$. Computation shows that the matrix of the trace form

$\mathrm{Tr}_{L/K}\langle 1\rangle$ is the following (using the basis $1, T, \ldots, T^{p-1}$):

$$\begin{pmatrix} 0 & \cdots & 0\lambda\pi^2 \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ 0 & \ddots & * \\ \lambda\pi^2 & & \end{pmatrix}$$

This shows $\mathrm{Tr}_{L/K}\langle 1\rangle \sim \langle\lambda\rangle$. □

Note that the upper left hand zero in the matrix is $p \in K$, this makes the proof work. We also remark that we will not use this lemma in what follows. It may, however, replace the more difficult theorem 2 in the treatment of function fields of finite characteristic.

LEMMA 2. *If δ is not a square in K then $\mathrm{Tr}_{L/K}\langle \frac{1}{2}\alpha\rangle = \langle \alpha, \alpha\delta\rangle$ for $\alpha \in K^{\cdot}$ and $L = K(\sqrt{\delta})$. In particular, $\langle 2, 2\delta\rangle$ is algebraic.*

Proof. Trivial computation. □

LEMMA 3. *Let K be an arbitrary field, $f(T) = T^5 + \alpha T + \alpha\beta$ an irreducible polynomial over K and $L = K[T]/(f(T))$. Then*

$$\langle L\rangle \sim \langle 1, 1, -\delta\rangle$$

where $\delta = \alpha^4(4^4\alpha + 5^5\beta^4)$ is the discriminant.

Proof. (See [1] VI.2.) We put $\alpha\beta = \gamma$ and consider $f(T) = T^5 + \alpha T + \gamma$. Let t be a root of $f(T)$ in L. Computation shows

$$\mathrm{Tr}(1) = 5, \quad \mathrm{Tr}(t) = 0, \quad \mathrm{Tr}(t^2) = 0, \quad \mathrm{Tr}(t^3) = 0$$
$$\mathrm{Tr}(t^4) = -4\alpha, \quad \mathrm{Tr}(t^5) = -5\gamma, \quad \mathrm{Tr}(t^6) = 0, \quad \mathrm{Tr}(t^7) = 0$$
$$\mathrm{Tr}(t^8) = 4\alpha^2.$$

Thus the matrix of $\mathrm{Tr}\langle 1\rangle$ is the following (Hankel) matrix

$$\begin{pmatrix} 5 & 0 & 0 & 0 & -4\alpha \\ 0 & 0 & 0 & -4\alpha & -5\gamma \\ 0 & 0 & -4\alpha & -5\gamma & 0 \\ 0 & -4\alpha & -5\gamma & 0 & 0 \\ -4\alpha & -5\gamma & 0 & 0 & 4\alpha^2 \end{pmatrix}.$$

Computation shows that the determinant of this matrix is $\delta = 4^4\alpha^5 + 5^5\gamma^4$. The form given by this matrix contains $\langle 5, -4\alpha\rangle = \langle 5, -\alpha\rangle$ as a subform with isotropic orthogonal complement. Therefore

$$\mathrm{Tr}\,\langle 1\rangle \sim \langle 5, -\alpha, 5\alpha\delta\rangle.$$

Using the above formula for δ we see that δ is represented by $\langle \alpha, 5\rangle$, that is $\langle \alpha, 5\rangle \cong \langle \delta, 5\alpha\delta\rangle$. Therefore

$$\mathrm{Tr}\,\langle 1\rangle \sim \langle 5, -\alpha, \alpha, 5, -\delta\rangle \sim \langle 1, 1, -\delta\rangle. \quad \square$$

LEMMA 4. *Let K be an hilbertian field and let $\delta \in K^\cdot$. Then there exist $\alpha, \beta \in K$ such that $f(T) = T^5 + \alpha T + \alpha\beta$ is irreducible of discriminant δ. In particular $\langle L\rangle \sim \langle 1, 1, -\delta\rangle$ and all forms $\langle 1, 1, -\delta\rangle$ are algebraic.*

Proof. First let β be an indeterminate X and put $\alpha = 4^{-4}(\delta - 5^5X^4)$. Then

$$f(T, X) = T^5 + \alpha(X)T + \alpha(X)X \in K(X)[T]$$

has discriminant δ (up to squares). Moreover it is irreducible since it is an Eisenstein polynomial with respect to an irreducible factor of the separable polynomial $\alpha(X)$. Since K is hilbertian we can choose $\beta \in K$ such that $f(T) = f(T, \beta)$ is irreducible. Now we apply lemma 3. $\square$

Remark. In order to prove lemma 4 one can avoid Hilbert's irreducibility theorem if one is only interested in algebraic number or algebraic function fields. The lemma is proved for $K = \mathbb{Q}$ in [1] VI.2.8, but it is pointed out in [2] lemma 1 that the proof carries over to an arbitrary algebraic number field. In the function field case one may argue as follows:

We consider first the special case where $\delta = X$ is transcendental over k and $K = k(X)$. We choose an arbitrary $\beta \in k^\cdot$ and $\alpha = X - 5^5 4^{-4}\beta^4$. By lemma 3 the discriminant of $f(T)$ is $\delta = X$ in $K^\cdot/K^{\cdot 2}$ and $f(T)$ is an Eisenstein polynomial with respect to the prime $\alpha(X) = X - 5^5 4^{-4}\beta^4$ of $k[X]$. Since L is ramified exactly at the primes $\alpha(X)$ and X of $k(X)$ we get in fact infinitely many suitable field extensions L/K (at least if k is infinite).

We consider now the special case $\delta \in k^\cdot$ and $K = k(X)$ for some transcendental element X. We choose an arbitrary linear polynomial $\beta = \beta(X) = X - \beta_0$ and take $\alpha = \alpha(X)$ so that

$$\delta = 4^4\alpha(X) + 5^5\beta(X)^4.$$

By lemma 3 the discriminant of $f(T)$ is δ in $K^{\cdot}/K^{\cdot 2}$. Since $\alpha(X)$ has only distinct roots, $f(T)$ is an Eisenstein polynomial with respect to a prime divisor of $\alpha(X)$. Again we get infinitely many suitable L.

In the general function field case we choose an intermediate field $K_0 = k(X)$, $k \subset K_0 \subset K$ having the properties of the field K above. Since K/K_0 is finitely generated, one of the infinitely many fields L/K_0 constructed above will be linearly disjoint from K. Then $f(T)$ is irreducible over K. □

LEMMA 5. *Let L/K be a separable extension of odd degree m. For any $\delta \in K^{\cdot}$ there exists a $\lambda \in L$ such that*

$$\mathrm{Tr}_{L/K}\langle\lambda\rangle \sim \langle\delta\rangle.$$

Proof. This is well-known: Choose a primitive element x and define a "trace" $s: L \to K$ by $s(1) = \cdots s(x^{m-2}) = 0$, $s(x^{m-1}) = \delta$. Then $s\langle 1\rangle$ is given by the matrix

$$\begin{pmatrix} 0 & & \delta \\ & \cdot^{\cdot^{\cdot}} & \\ \delta & & * \end{pmatrix} \sim \langle\delta\rangle$$

and $s\langle 1\rangle \cong \mathrm{Tr}\,\langle\lambda\rangle$ for a suitable λ. □

3. Two theorems

Most of the results mentioned in the introduction follow from two basic theorems. The first has been proved independently by Waterhouse [7] and the second author [6].

THEOREM 1. *Let K be an hilbertian field and φ an arbitrary form over K. Then there exists a separable extension L/K and a $\lambda \in L^{\cdot}$ such that $\varphi \cong \mathrm{Tr}_{L/K}\langle\lambda\rangle$.* □

Separability is not explicitly mentioned in [6], but it is clear from the proof that one can construct a separable extension L/K. As long as we are interested only in the Witt class of φ we may alternatively replace φ by a Witt equivalent form whose dimension is not a multiple of the characteristic. Then L/K is automatically separable.

THEOREM 2. *Let K be an hilbertian field and let $\delta \in K^{\cdot}$ be totally positive. Then $\langle\delta\rangle$ is algebraic.*

Proof. We shall first show that for any n there is an extension L_n/K such that

$$\langle L_n\rangle \sim 2^n \times \langle 1\rangle \perp (2^n - 1)\times\langle -\delta\rangle.$$

This will be shown by induction on n. The case $n = 1$ follows from lemma 4. Let us assume we have constructed L_{n-1}. By lemma 5 there exists $\lambda \in L_{n-1}$ such that $\mathrm{Tr}_{L_{n-1}/K}\langle\lambda\rangle \sim \langle -\delta\rangle$. By lemma 4 there exists an extension L_n/L_{n-1} such that $\langle L_n\rangle_{L_{n-1}} \sim \langle 1, 1, \lambda\rangle$. Then

$$\begin{aligned}\langle L_n\rangle_K &\sim 2\times\langle L_{n-1}\rangle \perp \mathrm{Tr}_{L_{n-1}/K}\langle\lambda\rangle\\ &\sim 2^n\times\langle 1\rangle \perp (2\cdot(2^{n-1}-1)\times\langle -\delta\rangle.\end{aligned}$$

By the Artin–Schreier theorem in the formally real case and trivially otherwise there exists an n such that $2^n \times\langle 1\rangle$ represents δ. Since $2^n\times\langle 1\rangle$ is a Pfister form we have $2^n\times\langle 1\rangle \cong 2^n\times\langle\delta\rangle$. Therefore $\langle L_n\rangle \sim \langle\delta\rangle$. □

Theorem 2 was first proved for algebraic number fields by Conner [2]. His proof required some number theory, in particular the Estes–Hurrelbrink–Perlis norm theorem. Our proof uses no number theory except that which is required to prove Hilbert's irreducibility theorem for algebraic number fields. Since theorem 2 is crucial for the proof of conjecture 1 for algebraic number fields, our simplification of the proof of theorem 2 leads also to a simpler proof in the number field case.

4. Five corollaries

The following corollary is the main result of this paper.

COROLLARY 1. *Let K be an hilbertian field which is not formally real. Then every form φ over K is algebraic.*

Proof. Using theorem 1 we write $\varphi \cong \mathrm{Tr}_{L/K}\langle\lambda\rangle$ for a suitable L. By [4] L is hilbertian. Since L is non-real λ is totally positive and by theorem 2 there is an extension M/L such that $\mathrm{Tr}_{M/L}\langle 1\rangle \sim \langle\lambda\rangle$. Therefore

$$\mathrm{Tr}_{M/K}\langle 1\rangle = \mathrm{Tr}_{L/K}\langle 1\rangle \sim \varphi. \quad \square$$

COROLLARY 2. *Let K be an hilbertian field and $\varphi = \langle\alpha_1, \ldots, \alpha_n\rangle$ with all α_i totally positive. Then φ is algebraic.*

Proof. Using theorem 1 we write $\varphi \cong \mathrm{Tr}_{L/K}\langle\lambda\rangle$. If P is an ordering of K it has at most $n = [L:K]$ extensions Q to L. Using our assumption $\mathrm{sign}_P(\varphi) = n$ and the well-known formula

$$\mathrm{sign}_P(\varphi) = \sum_{Q \mid P} \mathrm{sign}_Q(\lambda)$$

we see that P has in fact exactly n extensions and that λ is positive for all of them. Now we apply theorem 2 in the same way as in the proof of corollary 1. □

COROLLARY 3. *Let K be an hilbertian field and let $\varphi = \langle\alpha, \beta\rangle$ be positive. Then φ is algebraic.*

Proof. By corollary 2 we may assume that $\delta = \alpha\beta$ is not a square. For $L = K(\sqrt{\delta})$ we have $\varphi \cong \mathrm{Tr}_{L/K}\langle\frac{1}{2}\alpha\rangle$. Take an ordering Q of L, then δ must be positive with respect to Q. Hence α and β must have the same sign, necessarily positive with respect to Q. Therefore $\frac{1}{2}\alpha$ is totally positive in L and we apply theorem 2 in the same way as in the proof of corollary 1. □

COROLLARY 4. *Let K be any field. If every odd-dimensional positive form is algebraic, then every even-dimensional form $\varphi \nsim \langle -1, -1\rangle$ is also algebraic.*

Proof. By assumption $\psi = \langle 2\rangle\varphi \perp \langle 1\rangle$ is algebraic. Therefore write $\psi \sim \mathrm{Tr}_{L/K}\langle 1\rangle$. By lemma 5 we can choose $\lambda \in L^{\cdot}$ such that $\mathrm{Tr}_{L/K}\langle\lambda\rangle \sim \langle -1\rangle$. The element λ cannot be a square because then one would get $\langle -1\rangle \sim \mathrm{Tr}_{L/K}\langle\lambda\rangle \cong \mathrm{Tr}_{L/K}\langle 1\rangle \sim \psi$ and therefore $\varphi \sim \langle 2\rangle\langle -1, -1\rangle \cong \langle -1, -1\rangle$. For $M = L(\sqrt{\lambda})$ we get by lemma 2

$$\mathrm{Tr}_{M/K}\langle 1\rangle \cong \mathrm{Tr}_{L/K}\langle 2, 2\lambda\rangle \sim \langle 2\rangle\psi \perp \langle -2\rangle \sim \varphi. \quad \square$$

The trivial example of an algebraically closed field shows that the hypothesis $\varphi \nsim \langle -1, -1\rangle$ cannot be eliminated completely.

Our final application will be concerned with function fields in one variable over the reals. We will use a few standard facts from the theory of quadratic forms over such fields (see [7], [8]).

FACTS. *Let $K/\mathbb{R}$ be finitely generated of transcendence degree* 1.

(1) *A quadratic form of dimension $\geqq 3$ is isotropic if and only if it is indefinite with respect to all orderings.*

(2) *Quadratic forms over K are classified by dimension, signatures, and determinant.*

(3) *If X is the space of orderings of K and $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$, X_i open, then there exists $f \in K$ positive at X_1 and negative at X_2* (SAP-*property*).

A consequence of (1) is the following:

(4) *Let φ be positive of dimension $n = 2m$ or $n = 2m - 1$, $m > 1$. Then $\varphi \cong m \times \langle 1 \rangle \perp \psi$.* □

We deduce from these facts the following results stating that every form over K has an essentially unique "dyadic development".

PROPOSITION. *Let $K/\mathbb{R}$ be finitely generated of transcendence degree* 1 *and let φ be an odd-dimensional form over K. Assume* $\dim(\varphi) < 2^n$. *Then there exist $\delta_i \in K^{\cdot}$, $i = 0, \ldots, n-1$ such that*

$$\varphi \sim \mathop{\perp}_{i=0}^{n-1} 2^i \times \langle \delta_i \rangle.$$

The δ_i, $i = 1, \ldots, n-1$ are unique up to totally positive factors and $\delta_0 = \det \varphi$.

Proof. Adding hyperbolic planes we may assume $\dim(\varphi) = 2^n - 1$. Let X be the space of orderings of K and define

$$X_j = \{P \in X \mid \operatorname{sign}_P(\varphi) = 2^n - 1 - 2j\}$$

so that we have a disjoint union of clopen subsets:

$$X = X_0 \cup X_1 \cup \cdots \cup X_{2^n-1}.$$

Note that $P \in X_j$ exactly if φ has j negative and $2^n - 1 - j$ positive coefficients in any diagonalization. There exist δ_i which are positive at X_j if the summand 2^i is not contained in the dyadic development of j and which are negative otherwise. We may choose $\delta_0 = \det(\varphi)$ and then the forms φ and $\perp 2^i \times \langle \delta_i \rangle$ have equal invariants and are therefore isometric. The uniqueness statement follows from the fact that the signatures of the δ_i are uniquely determined everywhere. □

COROLLARY 5. *Let $K/\mathbb{R}$ be finitely generated of transcendence degree* 1. *Then every positive form φ over K is algebraic.*

Proof. By corollary 4 we may assume $\dim(\varphi)$ odd and without loss of generality $\dim(\varphi) = 2^{n+1} - 1$. By fact (4) we have $\varphi \cong 2^n \times \langle 1 \rangle \perp \psi$ where ψ can be written as in the proposition. By lemma 4 there exists an extension L/K such that $\mathrm{Tr}_{L/K} \langle 1 \rangle \sim \langle 1, 1, \delta_{n-1} \rangle$ and by lemma 5 there are $\delta_i' \in L$ such that $\mathrm{Tr}_{L/K} \langle \delta_i' \rangle \sim \langle \delta_i \rangle$. Therefore

$$\varphi \sim \mathrm{Tr}_{L/K}\, \varphi', \qquad \varphi' = 2^{n-1} \times \langle 1 \rangle \perp \mathop{\perp}_{i=0}^{n-2} 2^i \times \langle \delta_i' \rangle$$

and we can repeat this construction with φ'. □

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