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Introduction

Let $\pi: E \to B \leftarrow E': \pi'$ be vector bundles over the same base B and $f: SE \to E'$ a fibre-preserving ($\pi' f = \pi$) odd ($f(-x) = -f(x)$) map, where $SE \subset E$ is the sphere-bundle of E. The parametrized Borsuk–Ulam problem asks for the totality Z of solutions of the equation $f(x) = 0$; thus $Z = \{x \in SE \mid f(x) = 0\}$. If we identify antipodal points in SE we obtain the projective bundle $\bar{\pi}: \bar{S}E \to B$ of E and 2-sheeted coverings $SE \to \bar{S}E$ resp. $Z \to \bar{Z}$; the latter are characterized by their characteristic classes $u \in H^1\bar{S}E$ resp. $(u \mid \bar{Z}) \in H^1\bar{Z}$.

Cohomology H^* is understood in the Čech sense with coefficients mod 2. Let $H^*B[t]$ be the polynomial ring over $H^*(B)$ in one indeterminate t. Since $H^*\bar{Z}$ is an H^*B-algebras (via $\bar{\pi}^*$) we can substitute $u \mid \bar{Z}$ for the indeterminate t; every polynomial $q(t)$ then defines an element of $H^*\bar{Z}$ which we denote by $q(t) \mid \bar{Z} = q(u \mid \bar{Z})$. We contribute to the Borsuk–Ulam problem by showing that $q(t) \mid \bar{Z} \neq 0$ in many cases – giving lower bounds for the "size" of $\bar{Z}$ and hence of Z. We use Stiefel–Whitney classes $w_jE, w_jE' \in H^jB$ for this purpose, and the Stiefel–Whitney polynomials $w(E; t), w(E'; t) \in H^*B[t]$; $w(E; t) = \sum_{j=0}^{m} (w_jE)t^{m-j}$, where m = fibre-dimension of E. For every polynomial $q(t) \in H^*B[t]$ we prove that

$$q(t) \mid \bar{Z} \neq 0 \quad \textit{or} \quad w(E; t) \text{ divides } q(t)w(E'; t)$$

in the situation described above, and under more general circumstances (theorems 1.3, 1.14, 2.2). For readers who prefer singular to Čech cohomology the same proof shows:

$q(t) \mid V \neq 0$ for all neighborhoods V of $\bar{Z}$ in $\bar{S}E$

or

$w(E; t)$ divides $q(t)w(E'; t)$

in singular cohomology. – Further comments can be found after theorem 1.3, and in §3.

§1. Parametrized Borsuk–Ulam theorems for sphere-bundles with free involution

1.1. We take up and partly repeat the notation from the introduction. Thus $\pi: E \to B$, $\pi': E' \to B'$ are vector-bundles of fibre-dimensions m, n over a paracompact space B, with Stiefel–Whitney classes $w_j(E)$, $w_j(E') \in H^j B$, and *Stiefel-Whitney polynomials* $w(E; t)$, $w(E'; t) \in H^* B[t]$,

$$(1.2) \quad w(t) = w(E; t) = \sum_{j=0}^{m} w_j(E) t^{m-j}.$$

Cohomology H^j resp. H^* is understood to be *Čech cohomology with coefficients* $\mathbf{Z}/2$, and $H^*B[t]$ is the polynomial ring over H^*B in one indeterminate t of degree 1. Thus $w(t)$ has degree m, both with respect to t and as an element of the graded ring $H^*B[t]$.

We consider maps $f: SE \to E'$ such that $\pi' f = \pi \mid SE$ and $f(-x) = -f(x)$, and we put $Z = \{x \in SE \mid f(x) = 0\}$. The antipodal action $\tau: x \mapsto -x$ is fixed point free in SE and in Z so that the projection maps $SE \to \bar{S}E$ resp. $Z \to \bar{Z}$ onto the orbit spaces of τ are 2-sheeted covering maps. Their characteristic classes are denoted by $u \in H^1(\bar{S}E)$ resp. $(u \mid \bar{Z}) \in H^1 \bar{Z}$. We can substitute these classes for the indeterminate t and obtain homomorphisms of H^*B-algebras

$$\sigma: H^*B[t] \to H^*(\bar{S}E) \to H^*\bar{Z}, \qquad t \mapsto u \quad \text{resp.} \quad t \mapsto u \mid \bar{Z}.$$

We find it convenient to write $p(t) \mid \bar{S}E$ resp. $p(t) \mid \bar{Z}$ for the images $p(u)$ resp. $p(u \mid \bar{Z}) \in H^*\bar{Z}$ of $p(t) \in H^*B[t]$.

1.3. THEOREM. *If $q(t) \in H^*B[t]$ is such that $q(t) \mid \bar{Z} = 0$ then*

$$(1.4) \quad q(t) w(E'; t) = w(E; t) q'(t)$$

for some polynomial $q'(t) \in H^*B[t]$.

1.5. COROLLARY. *If m, n are the fibre-dimensions of E, E' then $q(t) \mid \bar{Z} \neq 0$ for all polynomials $q(t)$ whose degree with respect to t is smaller than $m - n$. In other words, the H^*B-homomorphism*

$$(1.6) \quad \bigoplus_{i=0}^{m-n-1} (H^*B) \cdot t^i \to H^*\bar{Z}, \qquad t^i \to t^i \mid \bar{Z}$$

is monomorphic. In particular, if $m > n$ then the cohomological dimension (in terms of $H^* = \check{H}(-; \mathbf{Z}/2)$) *satisfies*

(1.7) $\text{cohom . dim . }(\bar{Z}) \geq \text{cohom . dim . }(B) + m - n + 1.$

Further consequences, examples and illustrations are discussed in section 3.

Acknowledgements

If B is a single point then 1.3, or 1.5, is the classical Borsuk–Ulam theorem (if $m = n + 1$), resp. a generalization of it by Bourgin [B] and Yang [Y]. The first to study the zeros of $SE \to E'$ in the same spirit was Jaworowski [J] (The general program of parametrizing homotopy theory had been developed before by several authors, especially I. M. James in his papers around 1969). Jaworowski showed that $bt^{m-n-1} \mid \bar{Z} \neq 0$ for every non-zero $b \in H^*B$ assuming $w_j E' = 0$ for all $j > 0$. Nakaoka [N] proved that the assumption $w_j E' = 0$ is redundant. E. Fadell lectured on the problem in Heidelberg in summer 1986; cf. [FH2] for the subject of his lecture, and [FH1], [F] for earlier work. He advocated the use of the map σ and its kernel (= index of Z in his terminology). He did not use characteristic classes but had obtained (with S. Husseini) the result of the corollary 1.5. His lecture prompted the present work.

The proof of theorem 1.3 will apply to more general situations, as follows.

1.8. DEFINITION. A G-sphere-bundle (of fibre-dimension $m - 1$) is a map $\pi : S \to B$ together with a free fibre-wise G-action τ on S such that (i) (π, τ) is G-locally trivial, i.e. B is covered by open sets U such that $\pi^{-1}(U) \approx U \times Y$ as G-spaces over U, $\tau(u, y) = (u, \tau y)$. (ii) The fibre is G-homotopy equivalent to a compact finite-dimensional G-space, and (iii) $H^*Y \cong H^*S^{m-1}$, $S^{m-1} = (m-1)$-sphere. Moreover, all spaces involved are assumed to be paracompact.

For our purposes, $G = \mathbf{Z}/2$ and $H^* = \check{H}(-; \mathbf{Z}/2)$, but other subgroups $G \subset S^1$ and other coefficients are also of interest (cf. 3.8). Examples for 1.8 are, of course, the unit sphere-bundles of vector-bundles with the antipodal action. Or, τ could be any (linear or not) fibre-wise action on a vector bundle, or sphere bundle E, and $S = E - \text{Fix}(\tau)$. One has to make sure that local triviality includes the action in the sense of (i). For *proper* $\pi : S \to B$, local triviality of π alone implies (i) by a result of Edmonds [E], but in general one has to assume (i) as it stands.

1.9. DEFINITION. Since Y, the fibre of $\pi: S \to B$, is a chohomology sphere and $\dim(Y) < \infty$, the orbit space $\bar{Y} = Y/G$, with $G = \mathbf{Z}/2$, is a cohomology projective space

$$(1.10) \quad H^*Y = H^*S^{m-1}, \qquad H^*\bar{Y} = \mathbf{Z}/2[u]/(u^m),$$

where $u \in H^1\bar{Y}$ is the characteristic class of the action. (This follows from the Gysin sequence of the S^0-bundle $Y \to \bar{Y}$ because $H^j\bar{Y} = 0$ for large j.) Since u is also defined on $\bar{S} = S/G$ we can apply the Leray–Hirsch theorem to the fibre-bundle $\bar{\pi}: \bar{S} \to B$ and find that

(1.11) $H^*\bar{S}$ is freely generated, as an H^*B-module, by $1, u, \ldots, u^{m-1}$.

(This is familiar for singular cohomology. In Čech cohomology one can prove it with J. D. Lawson's method [L, §3] because $\bar{Y}$ is essentially compact. If $\bar{Y}$ were not compact but B locally contractible, or locally compact, one could still prove it using [L, §3].) We can express u^m in terms of the basis 1.11, i.e. there are unique elements $w_j \in H^jB$, $j = 1, 2, \ldots, m$, such that

$$(1.12) \quad u^m + w_1 u^{m-1} + \cdots + w_m = 0,$$

and (following Grothendieck) we call these elements the *Stiefel–Whitney classes* of (π, τ) – putting $w_0 = 1$, $w_i = 0$ for $i > m$. As before, we define the Stiefel–Whitney polynomial $w(t) = \sum_{j=0}^m w_j t^{m-j}$, and have that

$$(1.13) \quad H^*B[t]/(w(t)) \cong H^*\bar{S}, \qquad t \mapsto u,$$

as H^*B-algebras.

We can now formulate the cohomological generalisation of theorem 1.3 as follows.

1.14. THEOREM. *Let $\pi: S \to B$ be a G-sphere bundle and let E' be a space with a G-action τ' and a map $\pi': (E' - Z') \to B$, $Z' = \mathrm{Fix}(\tau')$, such that (π', τ') is a G-sphere bundle, $G = \mathbf{Z}/2$. Let $f: S \to E'$ be a G-map which is fibre-preserving ($\pi' f = \pi$) in $S - f^{-1}(Z')$. Put $Z = f^{-1}(Z')$, $\bar{Z} = Z/\tau (\subset S/\tau)$. Now, if $q(t) \in H^*B[t]$ is a polynomial such that $q(t) \mid \bar{Z} = 0$ then*

$$q(t)w'(t) = w(t)q'(t)$$

for some polynomial $q'(t) \in H^*B[t]$. – The unexplained notation is as in theorem

1.3

$$w(t)=\sum_{j=0}^{m} w_j(E)t^{m-j}, \qquad w'(t)=\sum_{j=0}^{n} w_j(E'-Z')t^{n-j}$$

where m, n are the respective cohomological fibre-dimensions. To obtain $(q(t)\mid \bar{Z})\in H^*\bar{Z}$ one substitutes $t\mapsto u\mid \bar{Z}$ in q(t).

As an example of a $\mathbf{Z}/2$-space E' where π' is not defined on all of E' one might take the Thom-space (or the cone) of a vector-bundle.

Proof of theorem 1.14. If $q(t)\mid \bar{z}=0$ then (by continuity of Čech-cohomology) $q(t)$ vanishes in an open neighborhood $V\subset \bar{S}$ of $\bar{Z}$; thus $q(t)\mid V=0$. By exactness of $H^*(\bar{S}, V)\xrightarrow{j^*} H^*\bar{S}\to H^*V$ there is $v\in H^*(\bar{S}, V)$ such that $j^*(v)=q(t)\mid \bar{S}$.

On the other hand, the map $f:(S-Z)\to(E'-Z')$ induces $\bar{f}^*:H^*(\bar{E}'-\bar{Z}')\to H^*(\bar{S}-\bar{Z})$ on orbit spaces, and

(1.15) $\quad w'(t)\mid(\bar{S}-\bar{Z})=w'(u)=w'(\bar{f}^*u')=\bar{f}^*(w'(u'))=0,$

the 1st equation by definition of $p(t)\mid(\bar{S}-\bar{Z})$, the 2nd because $\bar{f}^*$ is an H^*B-homomorphism ($\bar{f}$ is a map over B), and the 4th because $w'(u')=0$ by definition 1.12/13 of w'.

By exactness as above, there is $z\in H^*(\bar{S}, \bar{S}-\bar{Z})$ such that $j^*z=w'(t)\mid \bar{S}$. Now

$$(v\cup z)\in H^*(\bar{S}, V\cup(\bar{S}-\bar{Z}))=H^*(\bar{S},\bar{S})=0,$$

hence

(1.16) $\quad q(t)w'(t)\mid \bar{S}=(j^*v)\cup(j^*z)=j^*(v\cup z)=0,$

the 2nd equality by naturality of $\cup$ products [D, VII, 8.6]. But $H^*\bar{S}=H^*B[t]/(w(t))$, by (1.13), hence $q(t)w'(t)$ must be a multiple of $w(t)$. ■

§2. A Borsuk–Ulam theorem in the presence of fixed points

2.1. We now generalize theorem 1.3 by allowing non-zero fixed points of the action τ but assuming that f relates the (non-zero) fixed point sets by a cohomology isomorphism. For simplicity, we shall content ourselves with vector bundles and linear actions (and not treat the cohomological version which corresponds to theorem 1.14). The set-up is then as follows: We have vector-bundles E, F, E', F' over B, and we let $G = \mathbf{Z}/2$ act on $E \oplus F$ resp. $E' \oplus F'$ by $\tau(x, y) = (-x, y)$; i.e. antipodal action on E, E', trivial action on F, F'. We consider a G-map $f: S(E \oplus F) \rightarrow E' \oplus F'$ over B and aim for results on the structure (cohomology, dimension) of $Z = f^{-1}(0)$. The unit sphere bundle $S = S(E \oplus F)$ coincides with the fibre-wise join $SE *_B SF$ ($= SE * SF$ for easier printing). Clearly, f maps $SF = \{z \in (SE * SF) \mid \tau z = z\}$ into $F' = \{z \in (E' \oplus F') \mid \tau z = z\}$. *We assume,* in addition, *that* $f(SF) \subset (F' - \{0\})$, *and that* $f \mid SF: SF \rightarrow F' - (0)$ has *odd degree* (in the fibres). In particular, F, F' must have the *same fibre-dimension, say k.* As before, we denote by $\bar{S}(E \oplus F)$, $\bar{Z}$, $\bar{E}$, etc. the corresponding orbit spaces of the G-action, and we use notations $p(t) \mid \bar{Z}$, $w(E; t)$ etc. as in theorem 1.3. We abbreviate

$$S = S(E + F) - SF, \qquad \bar{S} = \bar{S}(E \oplus F) - SF,$$

as these will play the role of S, $\bar{S}$ in theorem 1.14.

2.2. THEOREM. *If* $q(t) \in H^*B[t]$ *is such that* $q(t) \mid \bar{Z} = 0$ *then*

$$q(t)w(E'; t) = w(E; t)q'(t)$$

for some polynomial $q'(t) \in H^*B[t]$ – the same wording as in 1.3 but considerably weaker assumptions! The corollary 1.5 also holds, of course, under the weaker assumptions. It figured in E. Fadell's Heidelberg lecture, for bundles f with a nowhere vanishing section; compare [F, §7].

Proof. We first remark that τ resp. τ' operate freely in $S = S(E + F) - SF$ resp. $(E' - 0) \oplus F' = E' \oplus F' - F'$ so that their characteristic classes u resp. u' are defined in the orbit spaces $\bar{S} = \bar{S}(E \oplus F) - SF$ resp. $\overline{(E' - 0) \oplus F} = E' \oplus F'/\tau' - F'$. We can substitute these classes into polynomials $p(t) \in H^*B[t]$. For instance $q(t) \mid \bar{S}$, or $q(t) \mid \bar{Z}$; the latter because $Z \subset S$ since $f(SF) \subset (F' - 0))$.

As in the proof of theorem 1.14 we find $v \in H^*(\bar{S}, V)$ such that $j^*(v) = q(t) \mid \bar{S}$, for some open neighborhood V of $\bar{Z}$. And we find $z \in H^*(\bar{S}, \bar{S} - \bar{f}^{-1}F')$ such that $j^*(z) = w'(t) \mid \bar{S}$, where $w'(t) = w(E'; t)$ is the Stiefel–Whitney polynomial of E' (note that $\overline{(E' - 0) \oplus F}$ and $\overline{E' - 0}$ are homotopy equivalent). We could now conclude that $q(t)w'(t) \mid (V \cup (\bar{S} - \bar{f}^{-1}F') = 0$, repeating theorem 1.14. But this is not enough now. What we'll need is $q(t)w'(t) \mid \bar{S} = 0$, and we shall use the degree-odd assumption on $f \mid SF$ to obtain the sharper result.

Recall that

$$S(E \oplus F) = (SE) * (SF), \qquad \bar{S}(E \oplus F) = (\bar{S}E) * (SF),$$

the fibrewise joins. It follows that $S = S(E \oplus F) - SF$, with its natural retraction onto SE, is just the total space of the vector bundle π^*F, where $\pi : SE \to B$ is the projection; thus $S \approx \pi^*F$. Similarly,

$$\bar{S} = (\bar{S}(E \oplus F) - SF) \approx \bar{\pi}^*F, \quad \text{where} \quad \bar{\pi} : \bar{S}E \to B.$$

And $(S, S - SE)$ is the Thom-space of π^*F, and $(\bar{S}, \bar{S} - \bar{S}E)$ is the Thom-space of $\bar{\pi}^*F$. Let $s \in H^k(\bar{S}, \bar{S} - \bar{S}E)$ and $s \in H^k(F, F - 0)$, resp. $s' \in H^k(F', F' - 0)$ be the Thom-classes of $\bar{\pi}^*F$, F, F'. Consider the following diagram

(2.3)
$$\begin{array}{ccccc}
(\bar{S}, \bar{S} - \bar{f}^{-1}\bar{E}') & \xleftarrow{\ \iota\ } & (\bar{S}, \bar{T}SF) & \xrightarrow[\sim]{\ \iota\ } & (\bar{S}, \bar{S} - \bar{S}E) \\
& \overset{\bar{f}}{\searrow} & \big\downarrow{\scriptstyle \bar{f}} & & \\
& & (\bar{E}' \oplus F', \bar{E}' \oplus F' - \bar{E}') & & \\
& & {\scriptstyle \sim}\big\downarrow{\scriptstyle \text{proj}} & & \\
& & (F', F' - 0) & &
\end{array}$$

where TSF is an open tubular neighborhood of SF in S which is small enough so that $f(TSF) \cap E' = \varnothing$ (if B is not compact this will usually require a variable radius $\rho(b)$ *for the fibres* T_b). The symbol $\sim$ indicates maps which are isomorphic in cohomology. The map f (or $\bar{f}$) maps the fibres of SF with odd degree into the fibres of $F' - 0$; similarly for the fibres of $\bar{T}SF$. Therefore it takes Thom-classes into Thom-classes, i.e.

(2.4) $\bar{f}^*(s') = \iota^* s.$

(The reader might find it more convincing to use the spaces

$$(S(E \oplus F), SF) \subset (S(E \oplus F), TSF \cup SF) \xrightarrow[\sim]{\ \iota\ } (S(E \oplus F), S(E \oplus F) - SE)$$

and the map $f : (S(E \oplus F), TSF \cup SF) \to (E' \oplus F', E' \oplus (F' - 0))$ to prove 2.4.) Consider now $(\bar{f}^*s') \in H^*(\bar{S}, \bar{S} - \bar{f}^{-1}\bar{E}')$ and

(2.5) $(v \cup z \cup \bar{f}^*s') \in H^*(\bar{S}, V \cup (\bar{S} - \bar{f}^{-1}F') \cup (\bar{S} - \bar{f}^{-1}\bar{E}')) = H^*(\bar{S}, \bar{S}) = 0.$

We apply j^* resp. i^* to bring v and z into $H^*\bar{S}$, resp. $\bar{f}^*(s')$ and the product into $H^*(\bar{S}, \bar{T}SF)$. Using naturality of $\cup$-products [D, VIII, 8.6] we obtain

$$\begin{aligned} 0 &= (j^*v) \cup (j^*z) \cup (i^*\bar{f}^*s') = (q(t) \mid \bar{S}) \cup (w'(t) \mid \bar{S}) \cup (\bar{f}^*s') \\ &= (q(t)w'(t) \mid \bar{S}) \cup (\iota^*s) = \iota^*((q(t)w'(t) \mid \bar{S}) \cup s). \end{aligned}$$

But ι^* is isomorphic, hence $(q(t)w'(t) \mid \bar{S}) \cup s = 0$. And $\cup s$ is isomorphic (Thom-isomorphism), hence $q(t)w'(t) \mid \bar{S} = 0$. Further, $\bar{S} = \bar{S}(E \oplus F) - SF$ is homotopy equivalent to $\bar{S}E$, and $H^*(\bar{S}E) = H^*B[t]/w(t)$ as in 1.13. Hence $q(t)w'(t)$ must be a multiple of $w(t)$. ■

§3. Examples and Comments

3.1. A simple example which illustrates well the formula 1.4 is the case of a linear map $\phi: E \to E'$ with *constant rank*. In this case, kernel, image, and cokernel of ϕ are vector bundles K, I, K', and $w(t) = w(E; t) = w(K; t)w(I; t)$, $w'(t) = w(E'; t) = w(I; t)w(K'; t)$, hence

(3.2) $\quad w(K; t)w'(t) = w(t)w(K'; t).$

Also, $Z = (SE \cap \phi^{-1}(0) = SK$, and $w(K; t) \mid \bar{S}K = 0$. Conversely, $q(t) \mid \bar{S}K = 0$ iff $q(t)$ is a multiple of $w(K; t)$, $q(t) = \lambda(t)w(K; t)$. And then $q(t)w'(t) = w(t)q'(t)$ holds with $q'(t) = \lambda(t)w(K'; t)$. – Another (more symmetric) way of putting it is that, *for polynomials q, q' such that $q(t)w'(t) = w(t)q(t)$ we have*

(3.3) $\quad q(t) \mid \bar{S}K = 0 \Leftrightarrow q'(t) \mid \bar{S}K' = 0$

where K' is the kernel of the transposed (linear) map $\phi': E' \to E$ (with respect to some metrics). I don't know whether (3.3) also holds for linear maps of variable rank.

3.4. The set of polynomials $q(t) \in H^*B[t]$ such that $q(t) \mid \bar{Z} = 0$ is an ideal in $H^*B[t]$ which Fadell–Husseini call the *index* of the G-space Z ($G = \mathbf{Z}/2$). The set of polynomials $q(t)$ which satisfy $q(t)w'(t) = w(t)q'(t)$ for some $q'(t) \in H^*B[t]$ is also an ideal which is denoted by $[w(t): w'(t)]$. Our theorems state

(3.5) $\quad \operatorname{index}(Z) \subset [w(t): w'(t)].$

A convenient description of $[w(t): w'(t)]$ is in the ring $H^*B[t, t^{-1}]$ of finite

Laurent series (obtained from $H^*B[t]$ by inverting t). If $\dim B < \infty$, or E' is a bundle of finite type then $(w'(t))^{-1} \in H^*B[t, t^{-1}]$ and

$$(3.6) \quad [w(t) : w'(t)] = (w(t)w'(t)^{-1}H^*B[t]) \cap H^*B[t],$$

an intersection of two free $H^*B[t]$-modules (in $H^*B[t, t^{-1}]$) of rank one.

Note that the right side of 3.5 does not depend on the map f; the ideal $J = [w(t) : w'(t)]$ contains the index $(Z(f))$ for *all* odd maps $SE \to E'$. Moreover, J is unchanged (is "stable") if we replace E, E' by $E \oplus F$, $E' \oplus F$ where F is any vector bundle over B, hence J contains $\text{index}(Z(\phi))$ for all odd maps $\phi : S(E \oplus F) \to E' \oplus F$, all F. Is it minimal with this property?

3.7. If $m = \dim E \leq \dim E' = n$ then the corollary 1.5 is void whereas the theorem 1.3 still provides non-trivial information about $Z = f^{-1}(0)$, in many cases. In particular, it provides some standard obstructions to immersions. Immersions correspond to the case $Z = \emptyset$, or $f : SE \to (E' - 0)$. If $Z = \emptyset$ (also for non-linear f and in the situation of theorem 1.14) then $1 \mid \bar{Z} = 0$, hence $w' = wq'$, or $w'(t)w(t)^{-1} \in H^*B[t]$. In other words, $[w'(t)w(t)^{-1}] \in H^*B[t, t^{-1}]/H^*B[t]$ is an obstruction for G-maps $SE \to (E' - 0)$ (and every $q(t)$ for which $q(t)w'(t)w(t)^{-1} \notin H^*B[t]$ provides a measure of how much every attempt to immerse will fail). – For $m = n$, G-maps $SE \to (E' - 0)$ can only exist if $w(t) = w'(t)$, and all such maps have odd degree in the fibres.

Also for $m > n$ the theorem gives more information than the corollary. For instance, if E is the Hopf-bundle ($m = 2$) over $B = S^2 = P_1\mathbf{C}$ and E' is the trivial line-bundle ($n = 1$) then the theorem implies $t \mid \bar{Z} \neq 0$ (because $w_2E \neq 0$) which isn't contained in the corollary. On the other hand, the theorem does not give any better dimensional estimates on $\bar{Z}$ than the corollary. I.e. *every monomial* $q(t) = bt^l$ *with* $b \in H^rB$ *and* $r + l \geq (\dim B) + m - n$ *makes* $q(t)w'(t)$ *divisible by* $w(t)$; thus $q(t)w'(t) = w(t)q'(t)$ for some $q'(t)$. To prove this, one can assume $r + l = \dim B + m - n$. Write $k = \dim B - r$, $l = k + m - n$. Then

$$q(t)w'(t) = \sum_{i=0}^{k} (bw_i')t^{k+m-i}$$

(because $bw_i' = 0$ for $i > k$), and one easily solves $q(t)w'(t) = w(t)q'(t)$ for $q'(t) = \sum_{j=0}^{k} b_j t^j$.

3.8. Instead of real vector bundles and fibre-preserving free $\mathbf{Z}/2$-actions we can consider complex vector bundles and fibre-preserving free S^1-actions, $S^1 = \{z \in \mathbf{C} \mid \|z\| = 1\}$, replacing $\check{H}(-; \mathbf{Z}/2)$ by integral Čech cohomology $H^* = \check{H}$

$(-;\mathbf{Z})$. Stiefel–Whitney classes are replaced by Chern-classes $c_j \in H^{2j}B$ whose definition à la Grothendieck applies to G-sphere bundles with $G = S^1$, as in 1.9–1.13. The main notion is the Chern polynomial $c(\xi)$ which is entirely analogous to the Stiefel–Whitney polynomial except for some signs,

$$(3.9) \quad c(E;\xi) = \sum_{j=0}^{m} c_j(E)(-\xi)^{m-j} \in H^{\text{even}}B[\xi]$$

whose indeterminate ξ now has degree 2.

Its crucial property is that $c(E;u) \in H^*\bar{S}E$ vanishes and

$$(3.10) \quad H^*BU[\xi]/c(E;\xi) \cong H^*\bar{S}E, \qquad \xi \mapsto u$$

where $u \in H^2\bar{S}E$ now is the characteristic class of the S^1-action, $\bar{S}E = SE/S^1$.

An interesting point is that we can replace $H^* = \check{H}(-;\mathbf{Z})$ by other cohomology theories, in particular by complex cobordism Ω_U^*, and use the corresponding finer (universal) Chern classes $\omega_j \in \Omega_U^{2j}$. This should provide more information on $Z = f^{-1}(0)$ than $c_j \in H^{2j}$. Or we can use K-theoretic Chern-classes in $K_{\mathbf{C}}B$ which are easier to handle than ω_j.

3.11. Still another possibility is to consider cyclic subgroups $G \subset S^1$ of order $k > 2$ and study G-sphere bundles and their "Chern–Grothendieck" classes $g_i \in H^i(B;\mathbf{Z}/k)$. Projective spaces (and bundles) will then be replaced by all kind of lens-spaces (and -bundles). It looks interesting but I haven't seriously worked on it.

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