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Perverse sheaves with singularities along the curve $y^n = x^m$

R. MacPherson and K. Vilonen

Introduction

The category of perverse sheaves on a stratified complex analytic variety X (or equivalently if X is smooth the category of holonomic, regular singularities $\mathcal{D}$-modules on X) is important. However its structure is not very explicitly understood except when X is a curve. This paper studies the problem locally for surfaces. We prove the following:

THEOREM 1. *The category of perverse sheaves on the complex two-space $\mathbb{C}^2$, constructible with respect to the stratification $\{0\} \subset \{y^n = x^m\} \subset \mathbb{C}^2$, $n \le m$, is equivalent to the category of $(n+2)$-tuples of vector spaces $A, B_1, \ldots, B_n, C$ together with maps*

$$\begin{array}{ccccc} A & \underset{p_k}{\overset{q_k}{\rightleftarrows}} & B_k & \underset{r_k}{\overset{s_k}{\rightleftarrows}} & C \\ & & \downarrow{\scriptstyle \theta_k} & & \\ & & B_{k+m} & & \end{array}$$

which satisfy the following relations (where all the indices are interpreted modulo n):

(i) $a_k = 1 + q_k p_k$ *is invertible for all* k

(ii) θ_k *is invertible for all* k

(iii) $q_{k+m}\theta_k = a_{k+m} \cdots a_{k+1} q_k$ *for all* k

(iv) $\theta_k p_k = p_{k+m} a_{k+m} \cdots a_{k+1}$ *for all* k

(v) $\sum_{k=1}^n r_k p_k = 0$

(vi) $\sum_{k=1}^n q_k s_k = 0$

(vii) $s_j r_k = -\sum_{\substack{l=k+1 \\ l \equiv j (\mathrm{mod}\, n)}}^{k+m} p_l a_{l-1} \cdots a_{k+1} q_k + \delta_{k+m,j}^{(n)} \theta_k - \delta_{kj}^{(n)}$ *where* $\delta^{(n)}$ *is the Kronecker symbol modulo n.*

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Of course the same description of the category of perverse sheaves holds locally near any plane curve singularity with only one Puiseaux pair. Note that m and n do not have to be relatively prime so the curve $\{y^n = x^n\}$ can be reducible. For example, taking $n = m = 3$, the curve is the union of three lines through the origin.

The proof which uses methods of [MV1] and [MV2] is given in section 2. These methods extend in principle to arbitrary plane curve singularities as well as to the global case. However, we have been unable to formulate in general such an explicit combinatorial result as the one presented here.

As an application of the theorem, we classify perverse sheaves with no vanishing cycles at the origin for the case $y^2 = x^3$ (see section 3). In this case the nontrivial irreducible perverse sheaves are parametrized by one complex number.

Results along these lines have been obtained independently by other people. Galligo, Granger, and Maisonobe [GGM1], [GGM2] treat the case of normal crossings (in arbitrary dimension). Granger and Maisonobe [GrM] independently obtained the same result for $y^2 = x^3$ (see section 2, remark 1). Maisonobe [Ma] gives a geometric procedure by which a combinatorial description could in principle be obtained for a general plane curve singularity. Narvaez [N] treats the case $y^2 = x^p$ using the method of Beilinson and Verdier [V]. Gelfand and Khoroshkin also use this method, and construct explicit presentations of the corresponding $\mathcal{D}$-modules for the case of $x^3 = y^3$ in $\mathbb{C}^2$ and for quadric cones in $\mathbb{C}^n$.

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Notation

We will use the notation of [MV2]. In particular all our vector spaces and sheaves are to be considered over a fixed field. If X is complex manifold and $\mathcal{S}$ is a collection of submanifolds of X we denote $\Lambda_S = \overline{T_S^* X}$ for $S \in \mathcal{S}$ and $\tilde{\Lambda}_S = \Lambda_S - \bigcup_{R \neq S} \Lambda_R$. If $\Lambda = \bigcup_{S \in \mathcal{S}} \Lambda_S$ then we denote by $P_\Lambda(X)$ the category of perverse objects on X whose characteristic variety is contained in Λ.

1. Extending across a smooth curve in $\mathbb{C}^2$

Let S be a smooth curve in $\mathbb{C}^2 - \{0\}$ and let $\Lambda = T^*_{\mathbb{C}^2}\mathbb{C}^2 \cup \overline{T_S^*\mathbb{C}^2 - \{0\}} \cup T^*_{\{0\}}\mathbb{C}^2$. Our aim in this paper is to give an explicit combinatorial description of $P_\Lambda(\mathbb{C}^2)$ in the case S is given by the equation $y^n = x^m$. To do so we will apply theorem 5.4 in [MV2] twice. The first application of the theorem in this section will give a description for $P_\Lambda(\mathbb{C}^2 - \{0\})$. The second application of the theorem in the next section extends the result from $P_\Lambda(\mathbb{C}^2 - \{0\})$ to $P_\Lambda(\mathbb{C}^2)$.

Let $S_1, \ldots, S_r$ be the components of S. We use the standard Hermitian metric on $\mathbb{C}^2$ to identify $T^*_{S_i}\mathbb{C}^2$ with $T_{S_i}\mathbb{C}^2$ (over the reals) and then use the tubular neighborhood theorem to establish a diffeomorphism between a neighborhood U_i of the zero-section of $T^*_{S_i}\mathbb{C}^2$ and a neighborhood V_i of S_i in $\mathbb{C}^2$. If A is a local system on $\mathbb{C}^2 - S$ we can form a local system $\tilde{A}_i$ on $\tilde{\Lambda}_{S_i}$ by pulling it back via the maps

$$\pi_1(\tilde{\Lambda}_S) \xleftarrow{\cong} \pi_1(U_i - S_i) \xleftarrow{\cong} \pi_1(V_i - S_i) \rightarrow \pi_1(\mathbb{C}^2 - S).$$

Notice that the local system $\tilde{A}_i$ does not depend on the choice of base points. Because $\tilde{\Lambda}_{S_i}$ is a trivial $\mathbb{C}^*$-bundle on S_i we get an isomorphism $\pi_1(\tilde{\Lambda}_{S_i}) \cong \pi_1(\mathbb{C}^*) \times \pi_1(S_i)$ by choosing a section of the bundle. We assume from now on that such a section has been chosen, and we also fix a base point. We denote the image of the canonical generator of $\pi_1(\mathbb{C}^*)$ in $\pi_1(\tilde{\Lambda}_{S_i})$ by γ_i.

If A is a local system on $\mathbb{C}^2 - S$ then $A[2] \in P_\Lambda(\mathbb{C}^2 - S)$. The following lemma follows immediately from the definition in [MV2].

LEMMA 1.1. *We have* $\psi(A[2]) \cong \psi_c(A[2]) \cong \tilde{A}_i$ *on* $\tilde{\Lambda}_{S_i}$ *and the variation map* $\mathrm{var}: \tilde{A}_i \rightarrow \tilde{A}_i$ *is given by* $\mathrm{var}(a) = \gamma_i(a) - a$.

Consider the category Q_Λ consisting of a local system A on $\mathbb{C}^2 - S$ and a $\pi_1(S_i)$-module B_i for every i such that the diagram

$$\begin{tikzcd} \tilde{A}_i \arrow[rr, "\mathrm{var}"] \arrow[dr] & & \tilde{A}_i \\ & B_i \arrow[ur] & \end{tikzcd}$$

commutes as a diagram of $\pi_1(S_i)$-modules.

PROPOSITION 1.2. *The category* $P_\Lambda(\mathbb{C}^2 - \{0\})$ *is equivalent to* Q_Λ.

Proof. Follows directly from theorem 5.4 in [MV2] by observing that for every i there is only one Gabber–Malgrange map I_{γ_i} for the generator γ_i and $I_{\gamma_i} = Id$.

We now specialize to the case when the curve S is given by the equation $y^n = x^m$. Without loss of generality we can assume that $m \geq n$ so that the projection to the x-axis is "good", i.e., $dx \in \tilde{\Lambda}_{\{0\}}$. For $0 < |\epsilon| \ll 1$ define $D_\epsilon = p^{-1}(\epsilon) \cap B$, where B is the unit poly-disc in $\mathbb{C}^2$. For ϵ real we define the following loops

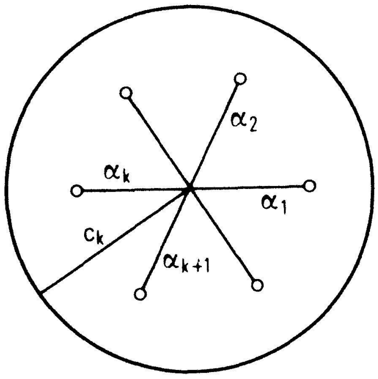

Figure 1. D_ϵ with the loops α_i and path c_k.

$$\alpha_i(t) = \begin{cases} 2t\epsilon^{m/n}\zeta_n^i & 0 \le t \le \dfrac{1-\delta}{2} \\ \epsilon^{m/n}\zeta_n^i\left\{1 - \delta \exp\left[2\pi i\left(t - \dfrac{1-\delta}{2}\right)\Big/\delta\right]\right\} & \dfrac{1-\delta}{2} \le t \le \dfrac{1+\delta}{2} \\ (2-2t)\epsilon^{m/n}\zeta_n^i & \dfrac{1+\delta}{2} \le t \le 1 \end{cases}$$

in $D_\epsilon - D_\epsilon \cap X$ where $\zeta_n = e^{2\pi i/n}$ and δ satisfies $0 < \delta \ll \epsilon$.

It is clear that $\pi_1(D_\epsilon - D_\epsilon \cap S) = \langle \alpha_1, \ldots, \alpha_n \rangle$, the free group generated by $\alpha_1, \ldots, \alpha_n$. It will be convenient sometimes to have the base point at the boundary of the disk D_ϵ. Define a path c_k as follows:

$$c_k = (1-t)\exp\left[2\pi i\left(\frac{k}{m} + \frac{1}{2n}\right)\right] \qquad 0 \le t \le 1$$

We denote $\tilde{\alpha}_i = c_k^{-1}\alpha_i c_k$. We will from here on use the convention that all the indices will be understood as integers modulo n, i.e., $\alpha_k = \alpha_h$ if $k \equiv h \pmod n$.

PROPOSITION 1.3. *We have $\pi_1(\mathbb{C}^2 - \bar{S}) \cong \langle \alpha_1, \ldots, \alpha_n \rangle / R$ where R is the group generated by the relations $\alpha_1 \cdots \alpha_m = \alpha_k \cdots \alpha_{m+k-1}$ for all $2 \le k \le n$.*

Proof. The result is classical but we include the argument because we will be using similar techniques later. For convenience we will use the loops $\tilde{\alpha}_i$ as our basis instead of the loops α_i. We have $\pi_1(\mathbb{C}^2 - \bar{S}) \cong (S^3 - S^3 \cap S) \cong \pi_1(T_1 \cup (T_2 - T_2 \cap S))$ where $T_2 = \{(x, y) \in B \mid |x| = \epsilon\}$, $T_1 = \{(x, y) \in B \mid |y| = 1\}$ are solid tori and $T_1 \cup T_2 \cong S^3$. We now want to use van Kampen to compute $\pi_1(\mathbb{C}^2 - \bar{S})$. We first compute $\pi_1(T_2 - T_2 \cap S)$.

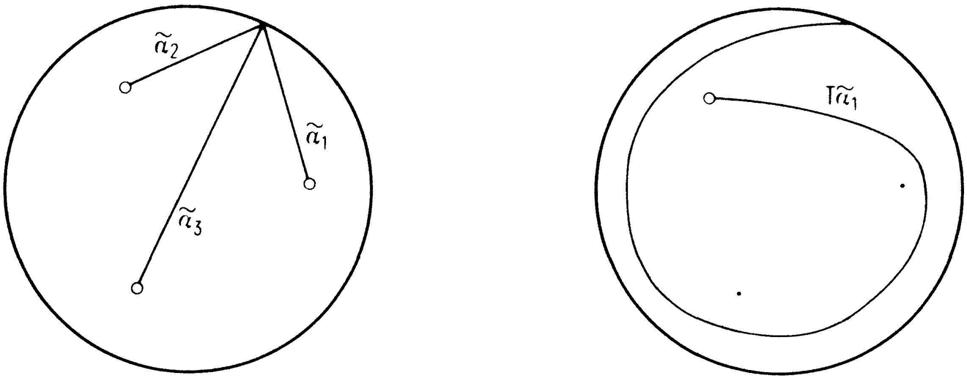

Figure 2. The loops $\tilde{\alpha}_i$ and $T\tilde{\alpha}_1$.

We have a fibration $T_2 - T_2 \cap S \to S^1$ with fibre $D_\epsilon - D_\epsilon \cap S$. We can view $T_2 - T_2 \cap S$ up to homotopy as a CW-complex whose cell decomposition is given by $c_k(0)$, $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$, $c = \{(x, y) \in B \mid y = c_k(0), |x| = \epsilon\}$, the sweep of $c_k(0)$, and the sweeps of all the $\tilde{\alpha}_i$ which we denote by $\tilde{\beta}_i$. This gives

$$\pi_1(T_2 - T_2 \cap S) \cong \langle \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n, c \rangle / \langle \tilde{\beta}_1, \ldots, \tilde{\beta}_n \rangle.$$

It is now easy to check that the monodromy T takes $\tilde{\alpha}_i$ to $\tilde{\alpha}_{i+1}^{-1} \cdots \tilde{\alpha}_{i+m-1}^{-1} \tilde{\alpha}_{i+m} \tilde{\alpha}_{i+m-1} \cdots \tilde{\alpha}_{i+1}$. See figure 2 for the case $y^3 = x^4$.

Therefore each $\tilde{\beta}_i$ gives us a relation

$$c\tilde{\alpha}_{i+1}^{-1} \cdots \tilde{\alpha}_{i+m-1}^{-1} \tilde{\alpha}_{i+m} \tilde{\alpha}_{i+m-1} \cdots \tilde{\alpha}_{i+1} c^{-1} \tilde{\alpha}_i^{-1} = e.$$

Now by using van Kampen's theorem we see that

$$\pi_1(\mathbb{C}^2 - \bar{S}) \cong \pi_1(T_1 \cup (T_2 - T_2 \cap S)) \cong \langle \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \rangle / R$$

because we get a further relation $c = e$.

Let $r = gcd(m, n)$. Then S has r components which we denote by $S_1, \ldots, S_r$. We clearly have $\pi_1(\tilde{\Lambda}_{S_i}) \cong \pi_1(\mathbb{C}^*) \times \pi_1(S_i) \cong \mathbb{Z} \times \mathbb{Z}$. We can identify the canonical generator γ_i of $\pi_1(\mathbb{C}^*)$ with α_i by pushing the base point along α_i till it is in the neighborhood V_i of S_i. We trivialize the fibration $\tilde{\Lambda}_{S_i} \to S_i$ by choosing another generator β_i for $\pi_1(\tilde{\Lambda}_{S_i})$. The α_i and β_i generate $\pi_1(\tilde{\Lambda}_{S_i})$. We will now give a more explicit form of proposition 1.2. Denote the action induced by α_i on A by a_i and the action induced by β_i on B by τ_i and the action induced by β_i on A by b_i. By proposition 1.3 b_i is a word on the $a_1, \ldots, a_n$.

PROPOSITION 1.4. *The category* $P_\Lambda(\mathbb{C}^2 - \{0\})$ *is equivalent to the category of* $(r+1)$*-tuples of vector spaces* $(A, B_1, \ldots, B_r)$ *such that* A *has an action by*

$\langle a_1, \ldots, a_n \rangle / R$ *and each* B_i *has an action by* τ_i *together with* $A \underset{p_i}{\overset{q_i}{\leftrightarrows}} B_i$ *such that*

(i) $q_i p_i + 1 = a_i \quad 1 \leq i \leq r$

(ii) $q_i \tau_i = b_1 q_i \quad 1 \leq i \leq r$

(iii) $\tau_i p_i = p_i b_i \quad 1 \leq i \leq r.$

Proof. Follows directly from proposition 1.2 and proposition 1.3.

Remark. The above proposition is valid locally for any curve singularity if we replace $\langle a_1, \ldots, a_n \rangle / R$ with the appropriate π_1.

We will want to make the above proposition more symmetric in that all of the points in D_ϵ will appear. We will also make a specific choice for the paths β_i. Different choices for the β_i will lead to a different combinatorial but of course equivalent description.

We define a path $\tilde{\beta}_k$ as follows

$$\tilde{\beta}_k(t) = \left\{ \epsilon e^{2\pi i t}, \ \epsilon^{m/n} \exp\left[2\pi i \left(\frac{k}{m} + \frac{tn}{m} \right) \right] \right\}.$$

Finally we define β_k by

$$\beta_k(t) = (1 - \delta) \tilde{\beta}_k(t)$$

where δ is very small. The β_k are paths in $V_i - S_i$ and β_k runs from the neighborhood of the kth point to the neighborhood of the $(k + m)$th point in D_ϵ.

We will denote by abuse of notation by B_k the vector space associated to the corresponding local system evaluated at the point corresponding to $\beta_k(0)$. We denote by $\theta_k : B_k \to B_{k+m}$ the transformation gotten by following the local system around along β_k.

PROPOSITION 1.5. *The category* $P_\Lambda(\mathbb{C}^2 - 0)$ *is equivalent to the category of* $(n + 1)$*-tuples of vector spaces* $(A, B_1, \ldots, B_n)$ *together with maps* $A \underset{q_k}{\overset{p_k}{\leftrightarrows}} B_k$ *and* $\theta_k : B_k \to B_{k+m}$ *such that*

(i) $q_k p_k + 1 = a_k$ *is invertible for all* k

(ii) θ_k *is invertible for all* k

(iii) $q_{k+m} \theta_k = a_{k+m} \cdots a_{k+1} q_k.$

(iv) $\theta_k p_k = p_{k+m} a_{k+m} \cdots a_{k+1}.$

Proof. The result follows quite formally from proposition 1.4. We can construct the data of an object in proposition 1.4 out of that of proposition 1.5 by taking $\tau_k = \prod_{i=1}^{n/r-1} \theta_{k+im}$ and $b_k = (a_m \cdots a_1)^{n/r}$ for all k. It follows from (iii) and (iv) of 1.5 that (ii) and (iii) of 1.4 are satisfied and furthermore that the action by $\langle a_1, \ldots, a_n \rangle$ on A satisfies the relations R.

Conversely we can construct the data of proposition 1.5 out of that of proposition 1.4 by choosing $\theta_k = id$ when $k = h + im$ for $1 \leq h \leq r$, $i < n/r - 1$ and $\theta_k = b_h$ when $k = h + nm/r$ for $1 \leq h \leq r$. We then define q_k and p_k via the formulas (iii) and (iv) in proposition 1.5 for the values k not in the range $1 \leq k \leq r$.

The functors defined above establish an equivalence of categories and therefore proposition 1.5 follows from proposition 1.4.

Remark. Although the proof given above is purely formal the geometry of the situation will be used in the next section.

2. Extending across the origin

We are now ready to prove our main theorem already stated in the introduction. We define a category Q_Λ in the following manner. Its objects are $(n+2)$-tuples of vector spaces $(A, B_1, \ldots, B_n, C)$ together with the maps $A \underset{p_k}{\overset{q_k}{\leftrightarrows}} B_k \underset{r_k}{\overset{s_k}{\leftrightarrows}} C$ and $\theta_k : B_k \to B_{k+m}$ which satisfy the following conditions where all the indices are to be considered as integers modulo n:

(i) $q_k p_k + 1 = a_k$ is invertible for all k
(ii) θ_k is invertible for all k
(iii) $q_{k+m}\theta_k = a_{k+m} \cdots a_{k+1} q_k$ for all k
(iv) $\theta_k p_k = p_{k+m} a_{k+m} \cdots a_{k+1}$ for all k
(v) $\sum_{k=1}^{n} r_k p_k = 0$
(vi) $\sum_{k=1}^{n} q_k s_k = 0$
(vii) $s_j r_k = -\sum_{\substack{i=k+1 \\ i \equiv j (\mathrm{mod}\, n)}}^{k+m} p_i a_{i-1} \cdots a_{k+1} q_k + \delta_{k+m,j}^{(n)} \theta_k - \delta_{kj}^{(n)}$ where $\delta^{(n)}$ is the Kronecker symbol modulo n,

$$\delta_{ij}^{(n)} = \begin{cases} 0 & i \not\equiv j \bmod n \\ 1 & i \equiv j \bmod n \end{cases}$$

The morphisms of Q_Λ are $(n+2)$-tuples of linear transformations which commute with all the maps p_k, q_k, r_k, s_k, and θ_k.

The theorem in the introduction can now be restated.

THEOREM 2.1. *The category* $P_\Lambda(\mathbb{C}^2)$ *is equivalent to* Q_Λ.

For the proof we want to make another application of theorem 5.4 in [MV2]. We assume now that we are given a perverse sheaf $\mathbf{P}^\cdot \in P_\Lambda(\mathbb{C}^2 - \{0\})$ and the combinatorial data associated to it by proposition 1.5. Because we are extending across a point it suffices to compute $\psi(\mathbf{P}^\cdot)$ and $\psi_c(\mathbf{P}^\cdot)$ for one direction in $\tilde{\Lambda}_{\{0\}}$. Since $m \geq n$ we can take this direction to be dx and therefore we have $\psi(\mathbf{P}^\cdot) = \mathbb{H}^{-1}(D_\epsilon, \mathbf{P}^\cdot)$ and $\psi_c(\mathbf{P}^\cdot) = \mathbb{H}_c^{-1}(D_\epsilon, \mathbf{P}^\cdot)$.

LEMMA 2.2. *Given an object* $\mathbf{P}^\cdot$ *in* $P_\Lambda(\mathbb{C}^2 - 0)$ *we have*

(i) $\psi(\mathbf{P}^\cdot) = \operatorname{Cok}(A \xrightarrow{(p_1,\ldots,p_n)} B_1 \oplus \cdots \oplus B_n)$

(ii) $\psi_c(\mathbf{P}^\cdot) = \operatorname{Ker}(B_1 \oplus \cdots \oplus B_n \xrightarrow{(q_1,\ldots,q_n)} A)$.

In order to apply theorem 5.4 in [MV2] we still need to compute the variation map $\operatorname{var}: \psi(\mathbf{P}^\cdot) \to \psi_c(\mathbf{P}^\cdot)$. If we write an element $(b_1, \ldots, b_n) \in B_1 \oplus \cdots \oplus B_n$ as $\sum_{i=1}^n b_i e_i$ and use the convention that all indices are to be interpreted as integers modulo n then we have

LEMMA 2.3. *The map* $\operatorname{var}: \psi(\mathbf{P}^\cdot) \to \psi_c(\mathbf{P}^\cdot)$ *is given by* $\operatorname{var}(b_k e_k) = -\sum_{i=k+1}^{k+m} p_i a_{i-1} \cdots a_{k+1} q_k b_k e_i + \theta_k(b_k) e_{k+m} - b_k e_k$.

Proof of theorem 2.1. The result follows directly from proposition 1.5, lemma 2.2 and lemma 2.3 by applying theorem 5.4 in [MV2].

Proof of lemma 2.2. Let $K = \bigcup_{k=1}^n c_k([0,1]) \subset D_\epsilon$, where the c_k are the paths defined in section 1. We denote $i: K \hookrightarrow D_\epsilon$ and $j: D_\epsilon - K \hookrightarrow D_\epsilon$. We now have a long exact sequence

$$\cdots \to \mathbb{H}^{-2}(K, \mathbf{P}^\cdot) \xrightarrow{\delta} \mathbb{H}^{-1}(D_\epsilon, j_! j^* \mathbf{P}^\cdot) \to \mathbb{H}^{-1}(D_\epsilon, \mathbf{P}^\cdot) \to \mathbb{H}^{-1}(K, \mathbf{P}^\cdot) \to \cdots$$

One sees immediately that $\mathbb{H}^{-1}(K, \mathbf{P}^\cdot) = 0$ and $\mathbb{H}^{-2}(K, \mathbf{P}^\cdot) \cong A$. We see that $\mathbb{H}^{-1}(D_\epsilon, j_! j^* \mathbf{P}^\cdot) \cong \bigoplus_{k=1}^n B_k$ canonically, and the map $\delta: A \to B_1 \oplus \cdots \oplus_n$ is given by $(p_1, \ldots, p_n)$. This proves part (i) of the lemma.

To prove part (ii) we consider the long exact sequence

$$\cdots \to \mathbb{H}_c^{-1}(D_\epsilon, i_* i^! \mathbf{P}^\cdot) \to \mathbb{H}_c^{-1}(D_\epsilon, \mathbf{P}^\cdot)$$
$$\to \mathbb{H}_c^{-1}(D_\epsilon, Rj_* j^* \mathbf{P}^\cdot) \xrightarrow{\delta} \mathbb{H}_c^0(D_\epsilon, i_* i^! \mathbf{P}^\cdot) \to \cdots$$

Again one sees immediately that $\mathbb{H}_c^{-1}(D_\epsilon, i_* i^! \mathbf{P}^\cdot) = 0$ and $\mathbb{H}_c^0(D_\epsilon, i_* i^! \mathbf{P}^\cdot) \cong A$.

A calculation shows that $\mathbb{H}_c^{-1}(D_\epsilon, Rj_*j^*\mathbf{P}^\cdot) \cong \bigoplus_{k=1}^n \mathbb{H}^{-1}(\bar{W}_k, Y_k; \mathbf{P}^\cdot)$ *where* $W_k \subset D_\epsilon$ is the wedge containing $\epsilon^{m/n}\zeta_n^k$ and bounded by the appropriate part of K and ∂D_ϵ. We denote by Y_k the set $\partial D_\epsilon \cap \overline{W}_k$ and by K_k the set $\partial W_k \cap K$. We now choose an isomorphism $\mathbb{H}^{-1}(\overline{W}_k, Y_k; \mathbf{P}^\cdot) \cong \mathbb{H}^{-1}(W_k, K_k; \mathbf{P}^\cdot)$ by moving K_k to Y_k along ∂W_k to the positive direction. As in the first part of the proof $\mathbb{H}^{-1}(W_k, K_k; \mathbf{P}^\cdot) \cong B_k$ canonically and so we have $\mathbb{H}_c^{-1}(D_\epsilon, Rj_*j^*\mathbf{P}^\cdot) \cong \bigoplus_{k=1}^n B_k$. By following through our choices we also see that the map $\delta : B_1 \oplus \cdots \oplus B_n \to A$ is given by $(q_1, \ldots, q_n)$.

To prove lemma 2.3 we introduce the following set $Z = \bigcup_{i=1}^n Z_i$. The sets Z_i are defined as

$$Z_i = \{t\zeta_n^i \mid (1+\delta)\epsilon^{m/n} \leq t \leq 1\} \cup \{\epsilon^{m/n}\zeta_n^i + \delta t \mid |t| \leq 1, t \in \mathbb{C}\}$$

where $0 < \delta \ll \epsilon$. We denote $i : Z \hookrightarrow D_\epsilon$ and $j : D_\epsilon - Z \hookrightarrow D_\epsilon$. A computation shows that $\mathbb{H}^{-1}(D_\epsilon, i_*i^!\mathbf{P}^\cdot) \cong B_1 \oplus \cdots \oplus B_n$ canonically.

Proof of lemma 2.3. To compute the variation map we first choose a monodromy map $\mu : D_\epsilon \to D_\epsilon$ such that $\mu \mid \partial D_\epsilon = id$. We define a function $u : D_\epsilon \to \mathbb{R}$ as follows

$$u(z) = \begin{cases} m & |z| \leq \frac{1}{2} \\ 2m(1 - |z|) & \frac{1}{2} \leq |z| \leq 1. \end{cases}$$

Then we can define

$$\mu(z) = e^{2\pi i u(z)} \cdot z.$$

It now follows from [GM] that the canonical map $B_k \to \psi(\mathbf{P}^\cdot) \to \psi_c(\mathbf{P}^\cdot) \to \bigoplus_{i=1}^n B_i$ is gotten by composing the following maps

$$\begin{tikzcd}
 & \mathbb{H}_{Z_k}^{-1}(D_\epsilon, \mathbf{P}^\cdot) \arrow[dr] & & \\
B_k = \mathbb{H}_{Z_k}^{-1}(D_\epsilon, \mathbf{P}^\cdot) \arrow[ur, "-1"] \arrow[dr, "\mu"'] & & \mathbb{H}_{Z_k \cup \mu Z_k}^{-1}(D_\epsilon, \mathbf{P}^\cdot) & \mathbb{H}_{Z_k \widetilde{\cup} \mu(Z_k)}^{-1}(D_\epsilon, \mathbf{P}^\cdot) \arrow[l, hook'] \arrow[d] \\
 & \mathbb{H}_{\mu Z_k}^{-1}(D_\epsilon, \mathbf{P}^\cdot) \arrow[ur] & & \mathbb{H}_c^{-1}(D_\epsilon, \mathbf{P}^\cdot) \arrow[d] \\
 & & & \bigoplus_{i=1}^n B_i
\end{tikzcd}$$

Here we have denoted by $Z_k \widetilde{\cup} \mu Z_k$ a deformation of $Z_k \cup \mu Z_k$ where we have

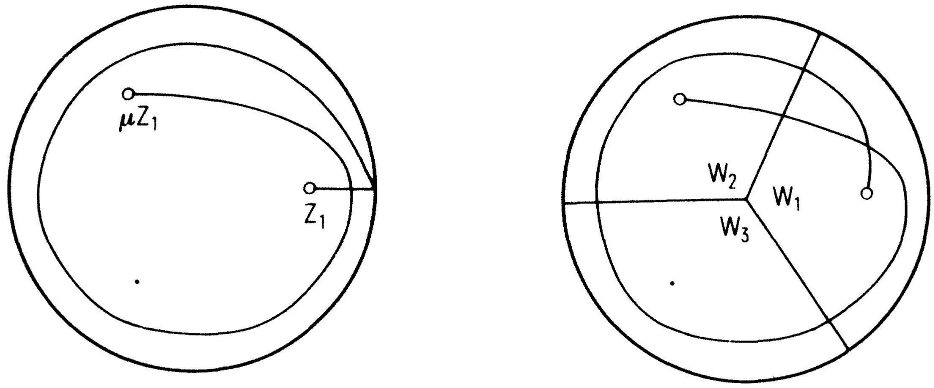

Figure 3. The set $Z_1 \cup \mu Z_1$ and $Z_1 \widetilde{\cup} \mu Z_1$ together with the sets W_i.

pushed the point ζ_n^k, which is the only point of $Z_k \cup \mu Z_k$ lying on the boundary of D_ϵ into the interior of D_ϵ. Because $\mu \mid \partial D_\epsilon = id$ the image of $b_k \in B_k$ in $\mathbb{H}^{-1}_{Z_k \cup \mu Z_k}(D_\epsilon, \mathbf{P}^\cdot)$ lies in $\mathbb{H}^{-1}_{Z_k \widetilde{\cup} \mu Z_k}(D_\epsilon, \mathbf{P}^\cdot)$. Tracing through this diagram and recalling how we defined the map $\mathbb{H}_c^{-1}(D_\epsilon, \mathbf{P}^\cdot) \to \bigoplus_{i=1}^n B_i$ yields the desired formula. This is illustrated in figure 3.

Remark 1. We can see by pure algebra that we could have define Q_Λ by changing three of the conditions as follows

(iii)′ $q_{k+m}\theta_k = a_{k+m-1} \cdots a_{k+1} q_k$ for all k

(iv)′ $\theta_k p_k = p_{k+m} a_{k+m-1} \cdots a_{k+1}$ for all k

(vii)′ $s_j r_k = -\sum_{\substack{i=k+1 \\ i \equiv j (\mathrm{mod}\, n)}}^{k+m-1} p_i a_{i-1} \cdots a_{k+1} q_k + \delta^{(n)}_{k+m,j}\theta_k - \delta^{(n)}_{kj}$.

For $y^2 = x^2$ this gives the description in [GGM1] and for $y^2 = x^3$ that in [GrM].

Remark 2. There are redundancies in our equations defining Q_Λ as can be seen from the formulas in [GrM]. We do not know what the smallest set of equations characterizing Q_Λ is or whether there is a different simpler set of equations characterizing Q_Λ.

Remark 3. It is a very interesting problem to find an analogous description for perverse sheaves in $\mathbb{C}^2$ singular along a curve with more general singularities. In the case of several Puiseaux pairs the methods in this paper work in principle, but the lack of an appropriate generalization of the sets K and Z have kept us from finding formulas as symmetric as the ones presented here.

3. Perverse sheaves on $\mathbb{C}^2$ with no vanishing cycles at the origin

Let S be the curve $y^2 = x^3$ in $\mathbb{C}^2$ and let $\Lambda = T^*_{\mathbb{C}^2}\mathbb{C}^2 \cup \overline{T^*_S\mathbb{C}^2}$. In this section we want to study the category $P_\Lambda(\mathbb{C}^2)$. Let $\mathbb{C}\langle A, B\rangle$ be the free algebra on two generators and let H be $\mathbb{C}\langle A, B\rangle/(ABA + A^2 + A,\ BAB + B^2 + B)$.

PROPOSITION 3.1. *The category $P_\Lambda(\mathbb{C}^2)$ is equivalent to the category of H-modules such that $A+I$ and $B+I$ are invertible transformations. This equivalence of categories is explicitly given by first associating to the H-module a local system on $\mathbb{C}^2-\bar{S}$ by choosing $a_1 = A+I$ and $a_2 = B+I$ and then taking the intersection homology extension to all of $\mathbb{C}^2$.*

Proof. We use theorem 2.1. In order not to have any vanishing cycles at the origin we must have $C=0$. This means that the following system of equations has to be satisfied:

$$\begin{cases} 1+p_1a_2q_1=0 \\ 1+p_2a_1q_2=0 \\ \theta_1=p_2q_1+p_2a_1a_2q_1 \\ \theta_2=p_1q_2+p_1a_2a_1q_2 \end{cases}$$

This system can be simplified to

$$\begin{cases} 1+p_1a_2q_1=0 \\ 1+p_2a_1q_2=0 \\ \theta_1=p_2a_1q_1 \\ \theta_2=p_1a_1q_2 \end{cases} \tag{3.1}$$

It also follows from the first two equations that $p_2a_1q_1=p_2a_2q_1$ and $p_1a_1q_2=p_1a_2q_2$. A computation shows that $\theta_2\theta_1=-(1+p_1q_1)^3$ and $\theta_1\theta_2=-(1+p_2q_2)^3$. If we assume that the equations (3.1) are satisfied and that a_1 and a_2 are invertible then it follows easily that $1+p_1q_1$ and $1+p_2q_2$ have to be invertible which in turn implies that θ_1 and θ_2 have to be invertible. Now the 3rd and 4th equations imply that p_1 and p_2 have to be surjections and q_1 and q_2 injections. A computation also shows that the equations (iii) and (iv) in the definition of Q_Λ follow from the the system (3.1).

Because p_1 and p_2 are surjections and q_1 and q_2 are injections the first two equations of (3.1) are equivalent to the equations

$$\begin{cases} q_1p_1+q_1p_1a_2q_1p_1=0 \\ q_2p_2+q_2p_2a_1q_2p_2=0. \end{cases}$$

If we now denote $A=a_1-1$ and $B=a_2-1$ then these equations take the form

$$\begin{cases} ABA + A^2 + A = 0 \\ BAB + B^2 + B = 0. \end{cases}$$

Therefore if we have an element in $P_\Lambda(\mathbb{C}^2)$ the above equations have to be satisfied and conversely if they are satisfied then it follows from what was said above that we get an element in $P_\Lambda(\mathbb{C}^2)$ if we define θ_1 and θ_2 by (3.1).

Consider the following family V_γ of representations of H on $\mathbb{C}^2$ given by

$$A = \begin{pmatrix} 0 & 0 \\ 1 & \lambda \end{pmatrix} \qquad B = \begin{pmatrix} \lambda & -(\lambda+1) \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad \lambda \in \mathbb{C}$$

is a parameter. One checks easily that V_λ is irreducible for $\lambda \neq \zeta_3, \zeta_3^2$ (ζ_3 is a primitive 3rd root of unity). For $\lambda = \zeta_3^i$, $i = 1, 2$, the representation V_λ has the trivial one-dimensional representation as a subrepresentation and the quotient representation is given by $A = B = \zeta_3^i$. Let $\tilde{V}_\lambda$ denote the family of irreducible representations of H where we have quotiented out the trivial representations at $\lambda = \zeta_3$ and $\lambda = \zeta_3^2$ in the family V_Λ and added it separately to the family.

PROPOSITION 3.2. *The irreducible objects in $P_\Lambda(\mathbb{C}^2)$ correspond to the irreducible representations $\tilde{V}_\lambda$.*

Proof. We have to show that every irreducible representation of H occurs in $\tilde{V}_\lambda$. Let now V be an arbitrary irreducible representations of H. Observe that A and B satisfy the relations $A^2B = AB^2$ and $B^2A = BA^2$ as an easy calculation shows.

We consider first the case that either A or B has a non-zero eigenvalue. We can assume that there exists a $v \in V$ such that $Bv = \lambda v$, $\lambda \neq 0$. Now we have two cases.

(1) Assume that $Av = \mu v$. Then the relations for H imply that the following equations must be satisfied

$$\begin{cases} \mu\lambda\mu + \mu^2 + \mu = 0 \\ \lambda\mu\lambda + \lambda^2 + \lambda = 0. \end{cases}$$

If $\mu = 0$ then $\lambda = 0$ or $\lambda = -1$ which is impossible. Therefore $\lambda \neq 0$ and a calculation shows that $\lambda = \mu$ and $\lambda^2 + \lambda + 1 = 0$. This gives us a one-dimensional representation occurring in $\tilde{V}_\lambda$.

(2) We now assume that $Av = w$ is not a multiple of v. We now consider the

subspace W of V spanned by v and w. We claim that W is invariant by H. We have

$$Bw = BAv = \frac{1}{\lambda} BABv = -\frac{1}{\lambda}(B^2 + B)v = -(\lambda + 1)v$$

and

$$Aw = AAv = \frac{1}{\lambda} A^2Bv = \frac{1}{\lambda} AB^2v = \lambda Av = \lambda w$$

so $V = W$ occurs in $\tilde{V}_\lambda$.

We now assume that neither A nor B has non-trivial eigenvalues. We could have $A = B = 0$ and get the trivial representation but otherwise one of them has to have a non-trivial Jordan block. Therefore we can assume that there is an element $v \in V$ such that $Bv \neq 0$ but $B^2 = v = 0$. Consider the subspace W of V generated by Bv and ABv. We have

$$B(ABv) = -B^2v - Bv = -Bv$$

and

$$A(ABv) = AB^2v = 0$$

which gives us a representation in $\tilde{V}_\lambda$.

Remark. It would be interesting to have results analogous to those of this section for $y^n = x^m$. If n does not divide m there will be a similarly constructed algebra such that proposition 3.1 holds. We do not know a simple presentation of this algebra and we do not know an analogue of proposition 3.2.

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