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The non-vanishing of the deviations of a local ring

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Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{k}$. Then (cf. [3], [6]) $\operatorname{Tor}_*^R(\mathbf{k}, \mathbf{k})$ has the structure of a free divided powers algebra on a graded $\mathbf{k}$-vector space $V = \oplus_i V_i$. In particular the Poincaré series for R has the form

$$\sum_{i=0}^{\infty} [\dim \operatorname{Tor}_i^R(\mathbf{k}, \mathbf{k})]t^i = \frac{\prod_{j=1}^{\infty} (1 + t^{2j+1})^{\dim V_{2j+1}}}{\prod_{j=1}^{\infty} (1 - t^{2j})^{\dim V_{2j}}} \tag{1}$$

The integers $e_j = \dim V_j$ are called the *deviations* of R. The equation above shows they are completely determined by the betti numbers $\dim \operatorname{Tor}_i^R(\mathbf{k}, \mathbf{k})$, and conversely. Moreover ([1], [11], [12]) the Yoneda Ext-algebra, $\operatorname{Ext}_R^*(\mathbf{k}, \mathbf{k})$, is naturally the universal enveloping algebra of a graded Lie algebra L_R dual to V, and hence

$$e_i = \dim L_R^i, \quad \text{all } i.$$

Let $\hat{R}$ denote the completion of R with respect to the powers of $\mathfrak{m}$. By the Cohen structure theorem, $\hat{R}$ has the form $\tilde{R}/I$ where $\tilde{R}$ is a regular local noetherian ring (with maximal ideal $\tilde{\mathfrak{m}}$) and $I \subset \tilde{\mathfrak{m}}^2$. We call R a *weak complete intersection* if I is generated by a regular sequence.

Now in [3] Assmus proves the following

THEOREM A (Assmus). *The following conditions are equivalent:*

(i) *R is a weak complete intersection.*

(ii) $e_j = 0$, $j \geqq 3$.

(iii) $e_3 = 0$.

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This raised the question of whether or not any deviation could vanish if R was not a weak complete intersection.

A first step was taken by Gulliksen [7], [8] who showed that infinitely many e_{2k} must be nonzero for non weak complete intersections. Subsequently Avramov and Halperin [5] showed that only finitely many e_i could be zero. Moreover in special cases (eg. Jacobsson [9], Löfwall [10]) it was known that no deviation was zero.

In this paper we completely settle the question with

THEOREM B. *Suppose R is not a weak complete intersection. Then no deviation can vanish:*

$$e_i(R) \geqq 1, \qquad i \geqq 1.$$

The rest of this paper is devoted to the proof of Theorem B, which depends on a variation of an idea (special variables) of André [2], and an adaptation of the minimal models of Avramov [4].

Our first observation is that betti numbers (and hence deviations are unchanged if we complete R, and hence we may assume without loss of generality that $R = \tilde{R}/I$, $\tilde{R}$ regular, $I \subset \tilde{m}^2$, as above. We make this assumption henceforth.

The next step is to build a suitable DGA model for R. This involves the process, introduced by Tate [13], of "adjoining freely commuting variables" which in our case may be either exterior, symmetric or divided power variables. To simplify we shall use "X" to mean an exterior or symmetric variable and "Y" to mean an exterior or divided power variable. More precisely we establish the *Notation convention.* Let $X_1, \ldots, X_i$ (resp. $Y_1, \ldots, Y_j$) denote symbols of degrees p (resp., q). Denote by $\Lambda(X_1, \ldots, X_i)$ the symmetric (resp. exterior) algebra on the free $\mathbb{Z}$-module with basis X_γ if p is even (resp. odd). Denote by $\Gamma(Y_1, \ldots, Y_j)$ the free divided powers (resp. exterior) algebra on the free $\mathbb{Z}$-module with basis Y_γ if q is even (resp. odd).

Then, if A is any graded algebra (commutative in the graded sense) we adopt the notation

$$A[X_1, \ldots, X_i] = A \otimes_{\mathbb{Z}} \Lambda(X_1, \ldots, X_i)$$

and

$$A[Y_1, \ldots, Y_j] = A \otimes_{\mathbb{Z}} \Gamma(Y_1, \ldots Y_j)$$

and we say we have *adjoined variables* X_γ (or Y_γ) to A.

We now fix an integer q arbitrarily and construct a sequence $A(0)\subset A(1)\subset \cdots$ of DGA's augmented to R. Indeed we set $A(0)=\tilde{R}$ and let $\pi:A(0)\to R$ be the quotient map. Then, choosing representatives $x_{11},\ldots,x_{1n_1}\in I$ of a $\mathbf{k}$-basis of $I/\tilde{m}\cdot I$ we set

$$A(1)=\tilde{R}[X_{11},\ldots,X_{1n_1}];\qquad dX_{1i}=x_{1i}$$

Extend π (uniquely) to $A(1)$ by setting $\pi(X_{li})=0$. Then $H_0(\pi):H_0(A(1))\xrightarrow{\cong} R$.

Suppose by induction that $A(k)$, $1\leqq k\leqq i-1$, are constructed and satisfy $H_0(A(1))=H_0(A(k))$ and $H_l(A(k))=0$, $1\leqq l<k$. Choose cycles $z_{i1},\ldots,z_{in_i}$ representing a $\mathbf{k}$-basis of $H_{i-1}(A(i-1))/\tilde{m}\cdot H_{i-1}(A(i-1))$ and define $A(i)$ by

$$A(i)=\begin{cases}A(i-1)[X_{i1},\ldots,X_{in_i}];\ dX_{ij}=z_{ij}\ \text{if } i<q.\\ A(i-1)[Y_{i1},\ldots,Y_{in_i}];\ dY_{ij}=z_{ij}\ \text{if } i\geqq q.\end{cases}$$

The differential in $A(i)$ is then determined by the requirement that $A(i)$ be a DGA and (in the second case when i is even) that

$$d(\gamma^s Y_{ij})=z_{ij}\otimes\gamma^{s-1}Y_{ij},\qquad s\geqq 1,$$

γ^s denoting the divided power operations.

Finally, set $A=\varinjlim_i A(i)$ and note that π extends uniquely to $\pi:A\to R$ with $H(\pi):H(A)\xrightarrow{\cong} R$. We say A is a model for R with *switching degree q*.

PROPOSITION 1. *Let A be as above and set $\bar{q}=q$ if q is even and $\bar{q}=q+1$ if q is odd. Let $m=\dim m/m^2=\dim \tilde{m}/\tilde{m}^2$. Then*

(i) *For any $\alpha_0,\ldots,\alpha_m\in H_+(\mathbf{k}\otimes_{A(q-1)}A)$, $\alpha_0\bullet\cdots\bullet\alpha_m=0$.*

(ii) *The integers n_i satisfy*

$$n_i=e_{i+1},\qquad 1\leqq i<2\bar{q},\quad \textit{and}\quad n_{2\bar{q}}\leqq e_{2\bar{q}+1}.$$

Proof. We construct inductively the commutative DGA diagram below in which the vertical arrows induce homology isomorphisms. Indeed let $y_{01},\ldots,y_{0m}\in\tilde{m}$ represent a basis of $\tilde{m}/\tilde{m}^2$ and set: $B(1)=A[y_{01},\ldots,y_{0m}]$; $dY_{0j}=y_{0j}$. Because $\tilde{R}$ is regular the augmentation $\tilde{R}[Y_{01},\ldots,Y_{0j}]\to\mathbf{k}$ induces an isomorph-

$$\begin{array}{ccccccccc} B(0) & \longrightarrow & B(1) & \longrightarrow \cdots & B(i) & \cdots & \longrightarrow & B(q) & \\ \| & & \downarrow \simeq & & \downarrow \simeq & & & \downarrow \simeq & \qquad (2) \\ A & \longrightarrow & \mathbf{k} \otimes_{\tilde{R}} A & \longrightarrow \cdots & \mathbf{k} \otimes_{A(i-1)} A & \ldots & \longrightarrow & \mathbf{k} \otimes_{A(q-1)} A & \end{array}$$

ism of homology. Because $B(1)$ is a free $\tilde{R}[Y_{01}, \ldots, Y_{0m}]$ module the induced morphism

$$B(1) \to \mathbf{k} \otimes_{\tilde{R}[Y_{01}, \ldots, Y_{0m}]} B(1) = \mathbf{k} \otimes_{\tilde{R}} A$$

also induces an isomorphism of homology.

Suppose next that the diagram has been constructed up through $B(i)$. Note that

$$\mathbf{k} \otimes_{A(i-1)} A = \mathbf{k}[X_{i1}, \ldots, X_{in_i}][\cdot \cdots\cdots \;\; \cdots]$$

Thus the X_{ij} $(l \leqq j \leqq n_i)$ are cycles representing a basis of $H_i(\mathbf{k} \otimes_{A(i-1)} A)$. Choose cycles $u_{ij} \in B(i)$ mapping to the X_{ij} $(1 \leqq j \leqq n_i)$ and set $B(i+1) = B(i)[Y_{i1}, \ldots, Y_{in_i}]$; $dY_{ij} = u_{ij}$. When i is odd set $d(\gamma^s Y_{ij}) = u_{ij} \cdot \gamma^{s-1} Y_{ij}$.

We have then the sequence of DGA morphisms

$$\begin{aligned} B(i+1) &= B(i)[Y_{i1}, \ldots, Y_{in_i}] \\ &\to (\mathbf{k} \otimes_{A(i-1)} A)[Y_{i1}, \ldots, Y_{in_i}] \qquad (dY_{ij} = X_{ij}) \\ &= \mathbf{k}[X_{i1}, \ldots, X_{in_i}, Y_{i1}, \ldots, Y_{in_i}][\cdot \cdots \cdot] \\ &\to \mathbf{k} \otimes_{\mathbf{k}[X_{i1} \cdots Y_{in_i}]} \mathbf{k}[X_{i1} \cdots Y_{in_i}][\cdot \cdot \cdot] \\ &= \mathbf{k} \otimes_{A(i)} A. \end{aligned}$$

The first arrow induces an isomorphism of homology because $B(i) \to \mathbf{k} \otimes_{A(i-1)} A$ does, and the second induces an isomorphism of homology because $\mathbf{k}[X_{i1}, \ldots, Y_{in_i}] \to \mathbf{k}$ does. This completes the construction of (2).

Again, because each $B(i)$ is a free A-module and because $A \twoheadrightarrow R$ induces an isomorphism $H(A) \xrightarrow{\cong} R$ it follows that for $0 \leqq i \leqq q-1$

$$B(i+1) \to R \otimes_A B(i+1) = R[Y_{01}, \ldots, Y_{in_i}]$$

induces an isomorphism of homology. In particular $R \otimes_A B(1)$ is just the Koszul complex, K^R, and $R \otimes_A B(i+1)$ is obtained from $R \otimes_A B(i)$ by adjoining a minimal number of exterior or divided power variables so as to kill $H_i(R \otimes_A B(i))$. Thus by adjoining such variables in degrees $> q$ to $R \otimes_A B(q)$, we get Tate's acyclic closure C, of R.

Now Gulliksen's theorem [6] asserts that $d(C) \supset \mathfrak{m}C$. Since $H_+(C) = 0$ it follows that $(\ker d)_+ \subset mC$. Since C is a free $R \otimes_A B(q)$ module it follows that

$$(\ker d)_+ \cap (R \otimes_A B(q)) \subset m \otimes_A B(q).$$

The argument of Gulliksen [7; Lemma 1] now shows that the product of $m+1$ homology classes in $H_+(R \otimes_A B(q))$ is zero – in view of the homology isomorphisms above this proves part (i) of the proposition.

Moreover, according to Gulliksen [6; Corollary 1], e_i is the number of variables in C of degree i. But for $1 \leqq i < q$, the number of variables Y_{ij} of degree $i+1$ is n_i, and so

$$n_i = e_{i+1} \qquad 1 \leqq i < q. \tag{3}$$

We complete the proof of part (ii) of the proposition by considering a model A' for R as above, but with switching degree $2\bar{q}+1$. By (3), applied to A', the number of variables of degree j in A' is e_{j+1}, $1 \leqq j \leqq 2\bar{q}$. Thus (ii) will be established once we show that A' has the same number of (resp., at least as many) variables as A in degrees $< 2\bar{q}$ (resp., $2\bar{q}$).

We may, clearly, take $A'(q-1) = A(q-1)$. Suppose by induction that for some $r \geqq q$, A' and A have the same number of variables in degrees $\leqq r-1$, and that there is a DGA morphism $\phi : A'(r-1) \to A(r-1)$ which is an isomorphism in degrees $< 2\bar{q}$.

If $r < 2\bar{q}$ then $H_{r-1}(\phi)$ is necessarily an isomorphism, and so we may choose $A'(r) = A'(r-1)[X_{r1}, \ldots, X_{rn_r}]$ with $H_{r-1}(\phi) : \text{class}\,(dX_{ri}) \mapsto \text{class}\,(dY_{ri})$. Thus A' and A have the same number of variables in degree r. Moreover, because the X_{ri} are either polynomial or exterior variables, we may extend ϕ to a DGA morphism $\phi : A'(r) \to A(r)$ by setting $\phi(X_{ri}) = Y_{ri}$. Since

$$A(r)_{<2\bar{q}} \subset A(r-1) \oplus \oplus_{j=1}^{n_r} A(r-1) \cdot Y_{rj}$$

(and similarly for $A'(r)$) it follows that ϕ is an isomorphism in degrees $< 2\bar{q}$.

Finally, suppose $r = 2\bar{q}$. Since ϕ is an isomorphism in degrees $< 2\bar{q}$ it follows that $H_{r-1}(\phi)$ is surjective. In this case the number of variables of degree r to be adjoined to $A'(r-1)$ is at least as large as the number adjoined to $A(r-1)$. This completes the proof. ■

We now establish Theorem B by supposing $e_3 \neq 0$ and some $e_i = 0$ $(i > 3)$ and deducing a contradiction. Let $s \geqq 3$ be the least integer for which $e_{s+1} = 0$. There are two cases.

Case I. $s = 2k+1$. Let A be a model for R as above with switching degree $2k$.

By Proposition 1 (ii),

$$n_{2k} = e_s > 0 \quad \text{and} \quad n_{2k+1} = e_{s+1} = 0.$$

We construct a deg $-2k$ derivation, θ, of the DGA, (A, d) such that

$$\theta(Y_{2k,1}) = 1 \quad \text{and} \quad \theta(\gamma^q Y_{2k,1}) = \gamma^{q-1} Y_{2k,1}. \tag{4}$$

Indeed, (4), together with the conditions

$$\theta(A_{<2k}) = 0 \quad \text{and} \quad \theta(\gamma^q Y_{2k,i}) = 0, \qquad i > 1,$$

defines a derivation of the DGA, $A(2k)$. Since n_{2k+1} is zero $A(2k) = A(2k+1)$. Suppose θ is extended to some $A(j-1)$, $j-1 \geqq 2k+1$. Then $\theta(dY_{ji})$ is a cycle in A_{j-2k-1}. Because $j > (2k+1)$ and because $H_+(A) = 0$ (since $H(A) = R$) it follows that $\theta(dY_{ji}) = d\Phi_i$, some $\Phi_i \in A_{j-2k}$.

Thus we may extend θ to $A(j)$ by setting

$$\theta(Y_{ji}) = \Phi_i \quad (\text{if } j \text{ is odd})$$

and

$$\theta(\gamma^q Y_{ji}) = \Phi_i \cdot \gamma^{q-1} Y_{ji}, \qquad q \geqq 1, \quad (\text{if } j \text{ is even}).$$

Finally, observe that θ factors to give a derivation $\bar{\theta}$ of the DGA $\mathbf{k} \otimes_{A(2k-1)} A$. In this quotient DGA, the elements $\gamma^q Y_{2k,1}$ are cycles. It follows from (4) that

$$\bar{\theta}^q(\gamma^q Y_{2k,1}) = 1$$

and hence $\gamma^q Y_{2k+1}$ represents a non-zero homology class for each q. But if $\operatorname{char} \mathbf{k} = p$ or 0 then in $\mathbf{k}[Y_{2k,1}]$, $\gamma^{(1+p+p^2+\cdots+p^m)}(Y_{2k,1})$ is a scalar multiple of $Y_{2k,1} \cdot \gamma^p(Y_{2k,1}) \cdot \cdots \cdot \gamma^{p^m}(Y_{2k,1})$. Thus this latter element also represents a non-zero homology class, which contradicts part (i) of Proposition 1.

Case II. $s = 2k+2$ $(k \geqq 1)$. Again let A be a model for R as above with switching degree $2k$. By Proposition 1 (ii),

$$n_{2k} = e_{s-1} > 0 \quad \text{and} \quad n_{2k+2} \leqq e_{s+1} = 0. \tag{5}$$

Let $y_1, \ldots, y_m \in m$ represent a $\mathbf{k}$-basis for $\tilde{m}/\tilde{m}^2$ and consider the differential

A-module (free as an A-module)

$$M = A \oplus A \cdot Y_1 \oplus \cdots \oplus A \cdot Y_m; \qquad dY_i = y_i.$$

Now the quotient module M/A is isomorphic (as differential modules) with a direct sum of copies of A shifted in degree by 1. Since $H(A) = R$ is concentrated in degree zero, $H(M/A)$ is concentrated in degree 1. From the short exact sequence $A \rightarrowtail M \twoheadrightarrow M/A$ and that fact that $y_1, \ldots, y_m \in \operatorname{Im} d$ we deduce that $H(M) = H_0(M) \oplus H_1(M)$, and that $H_0(M) = \mathbf{k}$.

Using these facts we construct a derivation θ of degree $-2k$ from the DGA, A, to the differential A-module M. (M is a right A-module via $m \cdot a = (-1)^{\deg m \deg a} a \cdot m$.) Indeed we set $\theta(A_{2k-1}) = 0$ and

$$\theta(\gamma^q Y_{2k,i}) = \begin{cases} \gamma^{q-1} Y_{2k,i} & i = 1 \\ 0 & i > 1. \end{cases} \tag{6}$$

This defines θ in $A(2k)$.

Next, for each $Y_{2k+1,i}$ we have $\theta(dY_{2k+1,i}) \in \tilde{m}$ and hence $\theta(dY_{2k+1,i}) = d\Phi_i$ for some $\Phi_i \in \oplus_j \tilde{R} \cdot Y_j$. Extend θ to $A(2k+1)$ by setting $\theta(Y_{2k+1,i}) = \Phi_i$.

Now because of (5), $A(2k+1) = A(2k+2)$. Assume we have extended θ to $A(j-1)$, some $j - 1 \geqq 2k + 2$. Then $\theta(dY_{ji}) \in M_{j-2k-1}$ and $j - 2k - 1 \geqq 2$. Our calculations above ($H(M) = H_0(M) \oplus H_1(M)$) thus imply that $\theta(dY_{ji}) = d\Psi_i$ and we can extend θ to $A(j)$ by setting

$$\theta(Y_{ji}) = \Psi_i \quad \text{if } j \text{ is odd,}$$

and

$$\theta(\gamma^q Y_{ji}) = \gamma^{q-1}(Y_{ji}) \cdot \Psi_i, \qquad q \geqq 1, \quad \text{if } j \text{ is even.}$$

Finally, we extend the projection $A \to \mathbf{k} \otimes_{A(2k-1)} A$ to a map $\rho : M \to \mathbf{k} \otimes_{A(2k-1)} A$ of differential A-modules by setting $\rho(A \cdot Y_i) = 0$. The derivation $\rho \circ \theta$ factors to yield a derivation $\bar{\theta}$ of the DGA $\mathbf{k} \otimes_{A(2k-1)} A$ and we obtain a contradiction exactly as in case I.

This completes the proof. ■

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