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Autor(en): **Bedford, Eric / Dadok, Jiri**

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# Bounded domains with prescribed group of automorphisms

ERIC BEDFORD* and JIRI DADOK*

## § 0. Introduction

By an automorphism of a complex manifold $\Omega$ we mean a biholomorphic mapping $f:\Omega\to\Omega$. A classical result of *H.* Cartan (see [9]) states that for a bounded domain $\Omega\subset\mathbb{C}^n$ Aut $(\Omega)$ has the structure of a Lie group. This is also the case if $\Omega$ is a relatively compact domain in a Stein manifold.

Let $\Omega\Subset\tilde{\Omega}$ be a relatively compact domain in a Stein manifold with a $C^2$, strongly pseudoconvex boundary. It is known [11] that if Aut $(\Omega)$ is not compact then $\Omega$ is biholomorphic to an open unit ball in $\mathbb{C}^n$. Thus the automorphism group of such a domain is either $SU(n, 1)$ or a compact Lie group. It is natural to ask whether every compact group can appear as the automorphism group of such $\Omega$. For the case of the trivial group $G=\{\mathrm{id}\}$, there are triply connected domains in $\mathbb{C}$ with smooth boundary but with no nontrivial automorphisms. Finding contractible strongly pseudoconvex domains $\Omega\subset\mathbb{C}^n$, $n\geq 2$ with Aut $(\Omega)=\{\mathrm{id}\}$ is less easy, but it is possible to take $\Omega$ to be a small, smooth perturbation of the ball $B^n$ (see [4]). The next simplest case is $G=T^1$, the circle group. There is no smoothly bounded Riemann surface $M$ with Aut $(M)=T^1$, but an appropriate domain may be constructed in $\mathbb{C}^2$ (Proposition 1.3). In this paper (Theorems 1, 2) we show how to construct a smoothly bounded domain $\Omega$ in $\mathbb{C}^n$ whose group of biholomorphisms is any prescribed compact group $G$. If $G$ is connected our construction (§ 3) is quite explicit:

THEOREM 1. *Let $G$ be a connected compact Lie Group and $G_{\mathbb{C}}$ its complexification. Then there exists a strongly pseudoconvex domain $\Omega\subset G_{\mathbb{C}}$ (or $\Omega\subset G_{\mathbb{C}}\times\mathbb{C}$ in case the center of $G$ is one dimensional) with real analytic boundary so that* Aut $(\Omega)=G$, *acting by left translations.*

The object in constructing $\Omega\subset G_{\mathbb{C}}$ is to keep it invariant under left translations by $G$ while ruling out additional symmetries. If $G$ acts on a complex

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manifold $M \neq G_{\mathbb{C}}$ by biholomorphisms it may happen that no such $\Omega \subset M$ exists (see example 3.0).

The following two theorems were first obtained by Saerens and Zame [12] independently of our Theorem 1, but the proofs we give in § 4 are shorter and more elementary in nature.

THEOREM 2. *Let $G$ be any compact Lie group. Then there is a strongly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$ with real analytic boundary such that* $\mathrm{Aut}(\Omega) = G$.

THEOREM 3. *Let $G$ be a compact Lie group. Then there exists a surface $\Sigma \subset \mathbb{R}^n$ which is an arbitrarily small smooth perturbation of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ whose group of isometries is linear and isomorphic to $G$. Moreover, if for some affine map $T$ of $\mathbb{R}^n$ $T\Sigma = \Sigma$ then $T \in G \subset O(n)$*

*Remark.* The dimension $n$ in the above two theorems may be taken to be $n = k^2$ if $G$ has a faithful action on $\mathbb{R}^k$.

Note that while Theorem 2 applies to disconnected Lie groups it only gives existence of required domains $\Omega$ (typically with $\dim \Omega \gg \dim G$). Its proof cannot be used to actually construct $\Omega$ without prohibitive calculations.

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**Notation**

Let $G$ be a compact group and $\mathfrak{g}$ its Lie algebra. We choose a faithful imbedding of $G$ into some unitary group $U(n)$. Thus $\mathfrak{g} \subset \mathcal{U}(n)$ is a subalgebra of skew Hermitian matrices. We set $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ and $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ the connected Lie group corresponding to $\mathfrak{g}_{\mathbb{C}}$. If $G$ itself is connected then $G \subset G_{\mathbb{C}}$ as a totally real submanifold of the Stein manifold $G_{\mathbb{C}}$. If $\omega \subset \mathfrak{g}$ is a small neighborhood of $0 \in \mathfrak{g}$ then $\Omega = G \cdot \exp i\omega \approx G \times \omega$ is a tubular neighborhood of $G$ in $G_{\mathbb{C}}$. Here exp is the matrix exponential function. The groups $L(G)$, $R(G) \subset \mathrm{Aut}(G_{\mathbb{C}})$ are the groups of left and right translations by $G$. If $g \in G$, $X$, $Y \in \mathfrak{g}$ we, as usual, define, $\mathrm{Ad}(g)X = gXg^{-1}$ and $\mathrm{ad}(X)(Y) = XY - YX$ (matrix multiplication).

## § 1. Tori

In this section we give examples of domains whose automorphism groups are $T^n$. First we consider a Reinhardt domain $\Omega \subset \mathbb{C}^n$, i.e. $\Omega$ is invariant under $(z_1, \ldots, z_n) \to (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$ for $\theta_1, \ldots, \theta_n \in \mathbb{R}$. It is obvious that $T^n$ is contained in the automorphism group of a Reinhardt domain. The logarithmic image of $\Omega$ is

$$\omega = \operatorname{Log}(\Omega) = \{(\xi_1, \ldots, \xi_n) : (e^{\xi_1}, \ldots, e^{\xi_n}) \in \Omega\}. \tag{1}$$

The automorphism group for certain $\Omega$ is given as follows (see [1]).

THEOREM 1.1. *If $\Omega$ is a Reinhardt domain, and if* Log$(\Omega)$ *is a bounded convex domain in $\mathbb{R}^n$, then* Aut$(\Omega)$ *consists of transformations of the form*

$$(z_1, \ldots, z_n) \to (c_1 z^{m_1}, \ldots, c_n z^{m_n}) \tag{2}$$

*where the matrix $M$ with rows $m_1, \ldots, m_n$ belongs to $GL(n, Z)$.*

It is evident that a mapping of the form (2) will map $\Omega$ to $\Omega$ if and only if $T\xi = M\xi + \log |c|$ is an affine self-mapping of $\omega$.

COROLLARY 1.2. *Let $\omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $\Omega \subset \mathbb{C}^n$ be the Reinhardt domain with* Log$(\Omega) = \omega$. *If $\omega$ has no nontrivial affine self-mappings, then* Aut$(\Omega) = T^n$.

If $n \geq 2$, then it is clear that a "generic" domain $\omega$ in $\mathbb{R}^n$ has no affine self-mappings. This is not true for $n = 1$, since every interval in $\mathbb{R}$ has an (affine) inversion.

Let $D \subset \mathbb{C}$ be a smoothly bounded triply connected domain with Aut$(D) = id$. Let $0 < r_1(z) < r_2(z)$ be continuous functions on $\bar{D}$ and set

$$\Omega = \{(z, w) \in D \times \mathbb{C} : r_1(z) < |w| < r_2(z)\}.$$

PROPOSITION 1.3. *Let $D \subset \mathbb{C}$ be a smoothly bounded triply connected domain with* Aut$(D) = id$. *If we choose $r_1(z)$, $r_2(z)$ such that $r_1(z) r_2(z)$ is not the modulus of an analytic function on $D$, then with $\Omega$ as above,* Aut$(\Omega) = T^1$.

It is clear that we may arrange for $\Omega$ to have real analytic, strongly pseudoconvex boundary.

For the proof we will use invariant 2-forms, as in [2]. We may choose $a_1$,

$a_2 \in \mathbb{C}$ such that

$$T_j = \frac{dz \wedge dw}{(z-a_j)w}, \qquad j = 1, 2$$

are linearly independent cohomology classes. If $[T_j]$ is the set of holomorphic 2-forms cohomologous to $T_j$, then there exists a unique $\omega_{T_j}$ which minimizes the $L^2$-norm $\|\omega\| = |\int_\Omega \omega \wedge \bar{\omega}|^{1/2}$ over $[T_j]$. We may write

$$\omega_{T_j} = \sum_{k=-\infty}^{\infty} f_j^k(z) w^k \, dz \wedge dw.$$

Since $T_j$ is independent of the rotation $(z, w) \to (z, e^{i\theta}w)$, so is $\omega_{T_j}$. Thus

$$\omega_{T_j} = f_j(z) w^{-1} \, dz \wedge dw.$$

If $f \in \operatorname{Aut}(\Omega)$, then

$$\frac{\omega_{f^*T_1}}{\omega_{f^*T_2}} = f^*\left(\frac{\omega_{T_1}}{\omega_{T_2}}\right). \tag{3}$$

By the arguments above, $\omega_{T_1}/\omega_{T_2} = m(z)$ is a nonconstant meromorphic function, and the left hand side of (3) is another meromorphic function, $\tilde{m}(z)$. Thus writing $f(z, w) = (f_1(z, w), f_2(z, w))$ we have

$$\tilde{m}(z) = m(f_1(z, w))$$

and so $f_1(z, w)$ depends on the variable $z$ alone. We conclude, then, that $f$ induces a mapping of the vertical fibers $\Omega_{z_0} = \{(z, w) \in \Omega : z = z_0)$ of $\Omega$. Thus $f_1$ is an automorphism of $D$, and therefore $f_1(z, w) = z$.

We conclude from this, that $f_2(z, w)$ must be an automorphism of the fiber

$$\Omega_z = \{w \in \mathbb{C} : r_1(z) < |w| < r_2(z)\}.$$

Therefore either

$$f_2(z, w) = e^{i\theta(z)} w$$

or

$$f_2(z, w) = c(z)/w.$$

In the first case, $\theta(z) = \theta_0$ is constant, since it must be real-valued and holomorphic. In the second case, however, $c(z)$ is a holomorphic function and

$$r_1(z) = |c(z)|/r_2(z).$$

which is a contradiction.

## § 2. Simple and connected groups

Given a simple compact group $G$ we construct (in Lemma 2.3) a domain $\Omega \subset G_{\mathbb{C}}$ in the complexification of $G$ and prove (Proposition 2.6) that the connected component of the identity $\mathrm{Aut}(\Omega)_0 = G$. At first we shall assume that $G$ is compact connected and semi-simple. Simplicity of $G$ is necessary only in Proposition 2.6.

The Killing form $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ is negative definite on $\mathfrak{g}$, the Lie algebra of $G$. In this section we shall use $-\kappa$ as the inner product on $\mathfrak{g}$. This inner product on the left invariant vector fields gives a biinvariant metric on $G$.

Let $\mathrm{Aut}(\mathfrak{g})$ be the Lie group of Lie algebra automorphisms of $\mathfrak{g}$, and $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g})$ be the image of $G$ under the adjoint representation i.e. the inner automorphisms of $\mathfrak{g}$. Recall [7] that $\mathrm{Aut}(\mathfrak{g})/\mathrm{Ad}(G)$ is a finite group (of order $\leq 6$ if $\mathfrak{g}$ is simple) and that each $\sigma \in \mathrm{Aut}(\mathfrak{g})$ preserves the Killing form which we will write as $\mathrm{Aut}(\mathfrak{g}) \subset O(\mathfrak{g})$, the orthogonal transformations on $\mathfrak{g}$. If $I \in O(\mathfrak{g})$ is the identity map we readily observe:

LEMMA 2.1. $-I \notin \mathrm{Aut}(\mathfrak{g})$.

We proceed to construct the domain $\Omega \subset G_{\mathbb{C}}$. Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis for $\mathfrak{g}$ and let $x_1, \ldots, x_d$ be coordinates with respect to this basis.

LEMMA 2.2. *On $\mathfrak{g}\backslash\{0\}$ we may find a smooth function $\psi(x)$ with the following properties:*

i) $\psi(\lambda x) = |\lambda|\, \psi(x)$ *for all* $\lambda \in \mathbb{R}$

ii) $\psi \circ g = \psi$, $g \in O(\mathfrak{g})$ *implies* $g = \pm I$.

*Proof.* Let $\psi_0(x) = \sum_{i=1}^{d} x_i^4$. Note that the maxima of $\psi_0$ on the unit sphere $S^{d-1} \subset \mathfrak{g}$ are at $\pm e_i$ $i = 1, 2, \ldots, d$. Set $v = (1, 2, 3, \ldots, d)$ and

$$\psi_\epsilon(x) = \frac{\psi_0(x) + \epsilon\langle x, v\rangle^4}{|x|^3}.$$

For small enough $\epsilon > 0$ there will be local maxima of $\psi_\epsilon$ at $\pm\tilde{e}_i$ with $\tilde{e}_i$ very close to $e_i$, and with

$$\langle \tilde{e}_i, \tilde{e}_j\rangle > 0 \quad \text{for all } i, j. \tag{4}$$

Further, it is clear that for small enough $\epsilon$ we will have

$$\psi_\epsilon(\tilde{e}_i) \neq \psi_\epsilon(\tilde{e}_j), \qquad \text{if } i \neq j. \tag{5}$$

Thus if we fix $\epsilon > 0$ with above properties and assume that $\psi_\epsilon \circ g = \psi_\epsilon$, $g \in O(\mathfrak{g})$ we obtain, using (5), that $g(\tilde{e}_i) = \pm \tilde{e}_i$. Finally, using (4), we must have $g(\tilde{e}_i) = \tilde{e}_i$ or $g(\tilde{e}_i) = -\tilde{e}_i$ for all $i = 1, 2, \ldots, d$.

LEMMA 2.3. *There exists a domain $\omega \subset \mathfrak{g}$ such that*
i) $\omega = -\omega$
ii) $\Omega = G \cdot \exp(i\omega) G_{\mathbb{C}}$ *is strongly pseudoconvex and smoothly bounded.*
iii) *If $\sigma \in \operatorname{Aut}(\mathfrak{g})$ and $\sigma(\omega) = \omega$, then $\sigma = I$.*

*Proof.* Let $\psi(x)$ be as in Lemma 2.2 and set

$$\omega_{\epsilon,\delta} = \left\{ x \in \mathfrak{g} : \sum_{i=1}^{d} x_i^2 + \delta \psi(x) < \epsilon^2 \right\}.$$

From Lemmas 2.1 and 2.2 it follows that for $\epsilon$, $\delta > 0$ properties i) and iii) hold. For $\epsilon \gg \delta > 0$ sufficiently small property ii) holds as well.

For the next lemma we observe that any group automorphism $h$ of $G$ extends to (a holomorphic) automorphism of $G_{\mathbb{C}}$, here denoted also by $h$.

LEMMA 2.4. *Let $\Omega$ be as in Lemma 2.3. Suppose $h$, an automorphism of $G$, and $X \in \omega$ have the property that*

$$R_{\exp iX} \circ h(\Omega) = \Omega.$$

*Then $X = 0$ and $h$ is the identity automorphism.*

*Proof.* The differential $dh \in \operatorname{Aut}(\mathfrak{g}) \subset \operatorname{Aut}(\mathfrak{g}_{\mathbb{C}})$. We thus have

$$h(\exp iY) = \exp(i\, dh(Y)), \qquad Y \in \mathfrak{g}.$$

In $G_{\mathbb{C}}$ consider the curve $\gamma(t) = \exp itX$. Since $dh(\omega) = -dh(\omega)$ it follows that

$$\{t \in \mathbb{R} : \gamma(t) \in \Omega\} = (-a, a), \qquad a > 0$$

is a symmetric interval. Next we observe that if $X \neq 0$ the set

$$\begin{aligned}\{t \in \mathbb{R} : \gamma(t) \in R_{\exp iX} \circ h(\Omega)) &= \{t \in \mathbb{R} : \gamma(t) \exp(-tX) \in h(\Omega)\} \\ &= \{t \in \mathbb{R} : (t-1)X \in dh(\omega)\}\end{aligned}$$

is of the form $(-b+1, b+1)$ and thus not symmetric. This contradiction shows that $X = 0$ and Lemma 2.3 then forces $dh = I$.

COROLLARY 2.5. *Suppose $R_z(\Omega) = \Omega$, $z \in G_{\mathbb{C}}$. Then $z \in Z(G_{\mathbb{C}})$, the center of $G_{\mathbb{C}}$.*

*Proof.* Write $z = g \exp iX$, $g \in G$, $X \in \mathfrak{g}$. Since $L_{g^{-1}}(\Omega) = \Omega$ we have that $R_{\exp iX} \circ h(\Omega) = \Omega$, where $h(x) = g^{-1}xg$ is an inner automorphism of $G$. Lemma 1.4 then implies that $X = 0$ and $gx = xg$ for all $x \in G$ and thus, by extending holomorphically, for all $x \in G_{\mathbb{C}}$. □

PROPOSITION 2.6. *Let $G$ be a simple connected Lie group and $\Omega \subset G_{\mathbb{C}}$ as constructed in Lemma* 2.3. *Then the connected component of the identity is* $\operatorname{Aut}(\Omega) = L(G)$.

*Proof.* Recall that $d = \dim G$. Since $\Omega \subset G_{\mathbb{C}}$ is a small tubular neighborhood of $G$, we have $H_d(\Omega, \mathbb{Z}) = \mathbb{Z}$. By Lemma 2.3 of [3] there exists an orbit of $\operatorname{Aut}(\Omega)$ in $\Omega$ whose dimension is at most $d$. Since $L(G) \subset \operatorname{Aut}(\Omega)$ that orbit must be a finite union of $G$ orbits, and any of these are stable under $\operatorname{Aut}(\Omega)_0$. So suppose $G \cdot x_0$ is $\operatorname{Aut}(\Omega)_0$ stable for some $x_0 \in \Omega$. Restricting the Bergmann metric $ds^2$ to the mainfold $G \cdot x_0 \simeq G$ we see that $\operatorname{Aut}(\Omega)_0$ is naturally a subgroup of the connected component $I_0(G, ds^2)$ of the isometry group. By Theorem 1 of [10] it now follows that any $f \in \operatorname{Aut}(\Omega)_0$ is of the form

$$f(g \cdot x_0) = agb \cdot x_0, \qquad g \in G$$

for some $a, b \in G$. Extending holomorphically to $g \in G_{\mathbb{C}}$ we see that $f = L_a \circ R_{x_0^{-1}bx_0}$, and hence $R_{x_0^{-1}bx_0}(\Omega) = \Omega$. By Corollary 2.5 $b \in Z(G_{\mathbb{C}})$, and so $f = L_{ab} \in L(G)$. □

## § 3. Connected groups and proof of Theorem 1

PROPOSITION 3.1. *Let $G$ be a compact, connected Lie group. Then there exists a piecewise strongly pseudoconvex domain $\Omega \subset G_{\mathbb{C}}$,* (or *$\Omega \subset G_{\mathbb{C}} \times \mathbb{C}$ in case the center of $G$ is one dimensional*) *such that* $G = \operatorname{Aut}(\Omega)$.

One may contemplate constructing such $\omega$ domain $\Omega$ inside other complex manifolds that posses a natural $G$ action. The following example shows that achieving $G = \operatorname{Aut}(\Omega)$ may be impossible.

EXAMPLE 3.0. Let $G = SO(3)$ act on the complex sphere

$$\Sigma = \{z \in \mathbb{C}^3 \mid \textstyle\sum z_i^2 = 1\}$$

Every $G$ orbit on $\Sigma$ intersects the curve

$$\alpha(t) = (\cosh t, i \sinh t, 0)$$

at $\alpha(\pm s)$ for some $s$. Consequently the only $G$ invariant pseudoconvex domains in $\Sigma$ are $\Omega_R = \{z \in \Sigma \mid |z| < R\}$. These domains are also $O(3)$ invariant.

*Proof of* 3.1. Any compact connected Lie group $G$ is of the form

$$G = T^l \times G_1 \times \cdots \times G_k/H \quad \text{where} \quad G_1, \ldots, G_k \text{ are simple,}$$

1-connected and connected, $H \subset Z(T^l \times G_1 \times \cdots \times G_k)$ is finite, and $H \cap T^l = \{e\}$.

In the following we will denote $G_1 \times \cdots \times G_k$ by $G_s$. Let $\Omega^0$ be a domain with $\mathrm{Aut}\,(\Omega^0) = T^l$, as constructed in Section 1. For each simple factor $G_j$ let $\Omega^j$ the domain constructed in Lemma 2.3. Moreover, we may arrange our choice of $\omega_i$'s so that $\Omega_i$ is not biholomorphically equivalent to $\Omega_j$ if $i \neq j$. To see this, we need only to note that if we shrink $\omega_j$, then we obtain a biholomorphically inequivalent $\Omega_j$ (see, for instance, Theorem 3.3 of [2]). Now set

$$D = \Omega_0 \times \Omega^1 \times \cdots \times \Omega^k.$$

We note that $D$ is biholomorphic to a domain in $(T^l \times G_1 \times \cdots \times G_k)_{\mathbb{C}}$ of the form $(T^l \times G_s) \cdot \exp\{i(\omega^0 \times \omega^1 \times \cdots \times \omega^k)\}$. By our choice of $D$ and theorem of *H*. Cartan [9]

$$\mathrm{Aut}\,(D) = T^l \times \mathrm{Aut}\,(\Omega^1) \times \mathrm{Aut}\,(\Omega^2) \times \cdots \times \mathrm{Aut}\,(\Omega^k). \tag{6}$$

Next set $\Omega = D/L(H)$. Again, biholomorphically

$$\Omega = G \cdot \exp\{i(\omega^0 \times \omega^1 \times \cdots \times \omega^k\} \subset G_{\mathbb{C}}.$$

If $f \in \mathrm{Aut}\,(\Omega)_0$ then it is homotopic to the identity and may thus be lifted to $\tilde{f} \in \mathrm{Aut}\,(D)_0$. By (6) and Proposition 2.6 $\tilde{f} = L_{\tilde{g}}$, $\tilde{g} \in T^l \times G^1 \times \cdots \times G^k$ and thus $f = L_g$ for some $g \in G$. Hence $L(G) = \mathrm{Aut}\,(\Omega)_0$ is a normal subgroup of $\mathrm{Aut}\,(\Omega)$. Therefore if $h \in \mathrm{Aut}\,(\Omega)$

$$hL_g h^{-1} = L_{\chi(g)}, \quad \chi \in \mathrm{Aut}\,(G),$$

that is for any $x \in \Omega$

$$h(g \cdot x) = \chi(g) \cdot h(x).$$

Setting $x = e \in G$, $h = R_{h(e)} \circ \chi$ on $G \subset \Omega$ which gives $h = R_{h(e)} \circ \chi$ on $\Omega$ after extending holomorphically to $\chi \in \mathrm{Aut}\,(G_{\mathbb{C}})$. By composing $h$ with a suitable left translation $L_g$, $g \in G$ we may assume that $h(e) = \exp iX$, $X \in \mathfrak{g}$. We write the Lie algebra of $\mathfrak{g}$ as $\mathfrak{g}_0 + \mathfrak{g}_s$ where $\mathfrak{g}_s$ is the Lie algebra of $G_s$ and $\mathfrak{g}_0$ is the center of $\mathfrak{g}$, i.e. the Lie algebra of $T^l$. The differential of $\chi$ must preserve this decomposition, so $d\chi = d\chi_0 \circ d\chi_s = d\chi_s \circ d\chi_0$. Similarly we can write $X = X_0 + X_s$ so $\exp iX_0 \exp iX_s = \exp iX_s \exp iX_0$. We conclude that translation by $X_0$ followed by $d\chi_0$ preserves $\omega^0$ and thus by assumption on $\omega^0$ $X_0 = 0$ and $d\chi_0 = I$. Finally as in the proof of Lemma 2.4 we must have $X_s = 0$ and $d\chi_s(\omega^1 \times \cdots \times \omega^s) = \omega^1 \times \cdots \times \omega^2$. Our choice of these domains forces first $d\chi_s(\omega^j) = \omega^j$ and then $d\chi_s = I$. □

To complete the proof of Theorem 1 we now apply the semicontinuity theorem of Greene and Krantz [6] to smoothen the domain $\Omega$. Let $r(z)$ be a $G$ invariant strongly plurisubharmonic exhaustion function of $\Omega$. For large $\lambda$

$$\Omega_\lambda = \{z \in \Omega : r(z) < \lambda\}$$

is strongly pseudoconvex with smooth real analytic boundary. Evidently $G \subset \mathrm{Aut}(\Omega_\lambda)$. Lemma 3.2 below shows that $\mathrm{Aut}(\Omega_\lambda)$ is a normal family of groups in the sense of Greene and Krantz, and thus by their semicontinuity theorem $\mathrm{Aut}(\Omega_\lambda) \subset G$ for $\lambda$ sufficiently large. The proof of the theorem is now complete.

LEMMA 3.2. *Let $(\lambda_j)$ be a sequence converging to $+\infty$ and let $\varphi_j \in \mathrm{Aut}(\Omega_{\lambda_j})$. Then there exists a subsequence $\{\varphi_{j_k}\}$ converging uniformly on compact sets to an element $\varphi \in \mathrm{Aut}(\Omega)$.*

*Proof.* Since $\Omega$ is bounded we may assume: (by extracting a subsequence) that $\{\varphi_j\}$ converges uniformly on compact subsets to a holomorphic $\psi \in \Omega \to \bar{\Omega}$. By a theorem of H. Cartan [9] either $\psi \in \mathrm{Aut}(\Omega)$ or $\psi(\Omega) \subset \partial\Omega$. We now show, arguing as in [3], that the latter case is impossible. Recall that by construction $\Omega$ is covered by a product of bounded domains. By lifting our maps we may assume that $\Omega$ itself is a product

$$\Omega = \Omega^0 \times \Omega^1 \times \cdots \times \Omega^k.$$

Suppose $\psi(\Omega) \cap \partial\Omega^0 \times \Omega^1 \times \cdots \times \Omega^k \neq 0$. Then, since $\partial\Omega^0$ is strongly pseudoconvex, $\psi(\Omega) \subset \{p_0\} \times \Omega^1 \times \cdots \times \Omega^k$ for some $p_0 \in \partial\Omega^0$. Now let $U$ be a contractible neighborhood of $p_0$ in $\Omega^0$, and let $T$ be a compactly supported cycle representing a nontrivial class in $H_q(\Omega)$, where $q = \dim_{\mathbb{C}} \Omega$. For large $j$ we have $\varphi(T) \subset U \times \Omega^1 \times \cdots \times \Omega^k$ which is homologically trivial in dimension $q$. On the other hand $\varphi_j$ is a diffeomorphism and hence $\psi_j(T)$ cannot be a boundary in $\Omega$ for $j$ large.

## § 4. Existence proofs

In this section we prove Theorems 2 and 3.

PROPOSITION 4.1. *Let $G$ be a compact Lie group. Then there exists an* *orthogonal action of $G$ on $\mathbb{R}^n$ with the following properties*

(i) *If $H \subset O(n)$ is a subgroup such that $Hx = Gx$ for all $x \in \mathbb{R}^n$, then $H = G$.*

(ii) *There exists a set $F \subset \mathbb{R}^n$ consisting of finitely many $G$-orbits such that if $g \in O(n)$ and $gF = F$, then $g \in G$.*

*Proof.* Let $G$ be faithfully imbedded in $O(k)$, and let $G$ act diagonally on

$$\mathbb{R}^n = \mathbb{R}^k \oplus \cdots \oplus \mathbb{R}^k \quad (k \text{ times}). \tag{$*$}$$

First we show that (i) is satisfied for this action. By assumption, the decomposition of $\mathbb{R}^n$ in $(*)$ is also $H$-invariant. For $v_1, \ldots, v_k \in \mathbb{R}^k$, we write $v = (v_1, \ldots, v_k) \in \mathbb{R}^n$ by the decomposition $(*)$. For $h \in H$, and $u = (v, \ldots, v) \in \mathbb{R}^n$, there exists by assumption $g \in G$ such that

$$hu = gu = (w, \ldots, w).$$

Thus we conclude that $H$ acts diagonally on the decomposition $(*)$. Finally, if $\{e_1, \ldots, e_k\}$ is a basis of $\mathbb{R}^k$, we set $v = (e_1, \ldots, e_k)$ to see that $hv = gv$ implies that $h = g$.

For part (ii), we construct a sequence of sets $F_j$, $j = 1, 2, 3, \ldots,$ with the following properties:

1. $F_j$ is the union of $j$ $G$-orbits.
2. If $H_j = \{g \in O(n) : gF_j = F_j\}$, and if $H_j \neq G$, then $H_j \supsetneqq H_{j+1}$.

Since the $H_j$ are compact and each contains $G$, we must have $H_l = G$ form some $l$. Indeed, at each step either the dimension or the number of components must decrease.

Now fix $\epsilon > 0$, pick $x_1$ of length $1 + \epsilon$, and set $F_1 = Gx_1$. We proceed inductively, under the assumption that $H_j \neq G$. By part (i) there exists a point $x_{j+1}$ such that $H_j x_{j+1} \neq Gx_{j+1}$. We then set

$$F_{j+1} = F_j \cup Gx_{j+1}.$$

Since we may take $\|x_{j+1}\| > \|x\|$ for all $x \in F_j$, we see that $H_{j+1}F_j \subset F_j$, and thus $H_{j+1} \subsetneqq H_j$.

PROPOSITION 4.2. *Let $G$ be a compact Lie group. There exists an orthogonal action of $G$ on $\mathbb{R}^n$ and a $G$ invariant domain $\omega \subset \mathbb{R}^n$ which is a small, smooth perturbation of the unit ball with the property that $g\omega = \omega$ and $g$ affine implies $g \in G$.*

*Proof.* Let $Gx_1 \cup \cdots \cup Gx_l$ denote the set obtained in (ii) of Proposition 1. For any $\delta > 0$, we may assume that $1 + \delta > |x_1| > |x_2| > \cdots > |x_l| > 1$. From $S^{n-1}$ we remove a small tubular neighborhood $V_j$ of $|x_j|^{-1} \cdot Gx_j$ such that the area of $V_j$ is small and such that $\bar{V}_i \cap \bar{V}_j = \varnothing$ for $i \neq j$.

Now we may make a small smooth perturbation of $S^{n-1}$ of the form

$$\Sigma = \{r(x)x : x \in S^{n-1}\}$$

where $r(x)$ is a smooth function on $S^{n-1}$ with $r \geq 1$, and $r(x) = 1$ for $x \notin \bigcup_{j=1}^{l} V_j$. Let us write

$$\omega = \{x \in \mathbb{R}^n : |x| < r(x/|x|)\}.$$

Before we specify $r(x)$ more precisely, let us note that if $h$ is an affine transformation of $\mathbb{R}^n$ with $h(\Sigma) = \Sigma$, then $h \in O(n)$. To see this, write

$$\omega_1 = \{x \in \mathbb{R}^n : |x| < 1, x/|x| \notin V_1 \cup \cdots \cup V_l\}.$$

Thus $\omega_1$ is a conical subset of $\omega$ generated by the complement of $V_1 \cup \cdots \cup V_l$. Since $h(\omega) = \omega$, $h$ must preserve volume. And since the volume of $\omega - \omega_1$ is small, $h(\omega_1) \cap \omega_1$ contains an open set. It follows, then, that $|h(x)| = 1$ for $x$ in an open subset of $S^{n-1}$. We conclude, then, that $h \in O(n)$.

Let $\chi \in C^\infty(\mathbb{R})$ be monotone decreasing with $\chi(0) = 1$, $\chi'(0) < 0$ and $\chi = 0$ on $[1, \infty)$. We define

$$r(x) = 1 + (|x_j| - 1)\chi(M \operatorname{dist}^2(x, |x_j|^{-1}Gx_j))$$

for $x \in V_j$ and $r = 1$ elsewhere on $S^{n-1}$. For $M$ sufficiently large, $r$ is smooth. Choosing $\delta > 0$ sufficiently small, we have $r$ close to 1.

Now if $h \in O(n)$ and $h\Sigma = \Sigma$, then $h$ must map $Gx_j$ to a portion of $\Sigma$ with distance $|x_j|$ to the origin. At the same time, $h$ must map $Gx_j$ to a portion of $\Sigma$ where the distance to the origin takes a local maximum. Thus $h(Gx_j) \subset Gx_j$. We conclude from Proposition 1, then, that $h \in G$.

*Proof of Theorem* 3. We let $\omega$ be the domain obtained in Proposition 2, and let $\Sigma = \partial\omega$. If $\omega$ is sufficiently close to the unit ball, then $\Sigma$ is positively curved. Thus $\Sigma$ is rigid, and any isometry $g$ of $\Sigma$ extends to an isometry of $\mathbb{R}^n$ (cf. [8]). It follows that $g \in G$, and thus $G$ is the group of isometries of $\Sigma$.

*Proof of Theorem* 2. Let $\omega \subset \mathbb{R}^n$ be the domain from Theorem 1, and let

$$\Omega = (\omega + i\mathbb{R}^n) - V,$$

where

$$V = \{z_1^2 + \cdots + z_{n+1}^2 = \tfrac{1}{2}\}.$$

We claim that $\operatorname{Aut}(\Omega) = G$. Since $G \subset O(n)$, it follows that $\operatorname{Aut}(\Omega) \supset G$. On the other hand, $\omega$ is contained in a proper cone, and thus is biholomorphic to a bounded domain. Thus any $f \in \operatorname{Aut}(\Omega)$ extends to a holomorphic mapping $f \in \operatorname{Aut}(\omega + i\mathbb{R}^n)$. By the Corollary to Theorem 1 of [5] or by [13] $f(z)$ is of the form

$$f(z) = Az + b + ic$$

where $A \in GL(n, \mathbb{R})$, and $b$, $c \in \mathbb{R}^n$. Since $Az + b$ maps $\omega$ to itself, it follows from Proposition 2 that $b = 0$ and $A$ represents an orthogonal transformation in $G$. Thus $A$ maps $V$ to itself, but it is evident that $V \neq V + ic$ if $c \neq 0$. We conclude, then, that $f \in G$.

To complete the proof of Theorem 2, we now smoothen the domain $\Omega$, as in the proof of Theorem 1. The only difference is that in the normal families argument we now use the fact that $\Omega$ cannot be retracted to $V$, since $H_n(V \cap \Omega, \mathbb{Z}) \neq 0$ but $\Omega$ is contractible. We can then apply the Semicontinuity theorem of Greene and Krantz [6] to smoothen $\Omega$.

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*Indiana University*
*Bloomington, IN47401*

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