Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 62 (1987)

Artikel: A lower bound for ... on manifolds with boundary.

Autor: Croke, Ch. B. / Derdzinski, Andrzej

DOI: https://doi.org/10.5169/seals-47342

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 06.07.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Comment. Math. Helvetici 62 (1987) 106–121

0010-2571/87/010106-16$01.50 + 0.20/0
© 1987 Birkhäuser Verlag, Basel

A lower bound for λ_1 on manifolds with boundary

CHRISTOPHER B. CROKE[1] and ANDRZEJ DERDZIŃSKI[2]

1. Introduction

The aim of this paper is to establish a geometric lower bound for λ_1 on compact Riemannian manifolds (M, g) with boundary, $\lambda_1 = \lambda_1(M, g)$ being the first eigenvalue of the (positive) Laplacian on functions that vanish on ∂M (Dirichlet boundary condition). We also classify the manifolds for which equality in our estimate is attained.

For v in the unit tangent sphere U_x at $x \in \operatorname{Int} M$, let $l(v) \leq \infty$ be the length of the maximal geodesic γ_v in $\operatorname{Int} M$, tangent to v. Denoting by dv_x the canonical measure on U_x and by $\alpha(n-1)$ the volume of the unit $(n-1)$-sphere, we prove

THEOREM. *Every compact Riemannian manifold (M, g) with boundary satisfies*

$$\lambda_1(M, g) \geq E(M, g), \tag{1}$$

where

$$E(M, g) = \frac{n\pi^2}{\alpha(n-1)} \inf_{x \in \operatorname{Int} M} \int_{U_x} l^{-2}(v)\, dv_x \tag{2}$$

and $n = \dim M$. Furthermore, equality holds in (1) if and only if (M, g) is isometric to a Riemannian hemisphere bundle.

By a *Riemannian hemisphere bundle* we mean the total space M of any $O(m)$-bundle of m-discs over a closed manifold B with the metric g on M naturally determined by any metric h on B and any $O(m)$-connection in the

[1] Research supported in part by National Science Foundation grant NCS 79-01780, Mathematical Sciences Research Institute, and the Sloan Foundation.
[2] Research supported in part by National Science Foundation grant 81-20790.

bundle, so that the fibres are isometric to a round hemisphere (see Section 4). Equivalently (cf. Remark 5) we can just require $(M, g) \to (B, h)$ to be a Riemannian submersion, the fibres of which are isometric to a hemisphere and totally geodesic.

The estimate (1) is due to the first author, while the description of the equality case is a joint result.

The assertion of (1), as well as its proof, is a generalization of Theorem 16 of [3], which applies only to the very special case where $l(v) < \infty$ for all v and gives $\lambda_1 \geq n\pi^2[\sup_{x \in \operatorname{Int} M} \sup_{v \in U_x} l^2(v)]^{-1}$ with equality precisely for round hemispheres. That estimate did not even apply to Riemannian hemisphere bundles, which illustrates how much stronger the present result is.

Inequality (1) is proved in Section 2 of this paper. In Section 4 we consider the equality case. The final Section 5 consists of various remarks and corollaries. In particular, we discuss there the case of non-compact manifolds and an application to domains in S^n.

2. Proof of inequality (1)

Let UM (resp., $U(\operatorname{Int} M)$) be the total space of the unit tangent bundle of (M, g) (resp., of its restriction to $\operatorname{Int} M$). We define $U^+ \partial M$ to be the subset of UM consisting of all unit vectors u tangent to M at ∂M which point inward, i.e., satisfy $\langle T, u \rangle > 0$, T being the inner unit normal field along ∂M. In the sequel, the symbols dv and du will stand for the canonical (local product) measures on UM and $U^+ \partial M$, respectively, while Q will denote the set of all $(u, t) \in U^+ \partial M \times \mathbb{R}$ for which $\exp tu$ exists and lies in $\operatorname{Int} M$, i.e., $0 < t < l(u)$, $l(u)$ being the length of the maximal geodesic γ_u in $\operatorname{Int} M$ with $\lim_{t \to 0} \dot{\gamma}_u(t) = u$.

Define the *geodesic flow* $\zeta : Q \to UM$ by $\zeta(u, t) = \dot{\gamma}_u(t)$. One easily sees that ζ is injective and C^∞. The main tools in our proof of (1) are Santaló's formula

$$\zeta^* \, dv = \langle T, u \rangle \, dt \, du$$

(see [8], [1]) and the expression for λ_1 coming from the Poincaré inequality

$$\lambda_1(M, g) = \inf\left(\int_M |\nabla f|^2 \, dx \Big/ \int_M f^2 \, dx\right), \tag{3}$$

where f runs over all non-trivial continuous functions on M which are C^∞ on $\operatorname{Int} M$ and vanish on ∂M (see [7]).

Let $\tilde{U}M$, $\tilde{U}(\operatorname{Int} M)$, $\tilde{U}^+ \partial M$ and $\tilde{Q}$ be the subsets of UM, $U(\operatorname{Int} M)$, $U^+ \partial M$ and Q characterized by $l(v) < \infty$ and $l(u) < \infty$, respectively. The image $\zeta(\tilde{Q})$ has full measure in $\tilde{U}M$, so that Santaló's formula yields, for any integrable function ϕ on $\tilde{U}M$

$$\int_{\tilde{U}M} \phi \, dv = \int_{\tilde{U}^+\partial M} \int_{t=0}^{l(u)} \phi(\zeta(u, t))\langle T, u\rangle \, dt \, du. \tag{4}$$

To prove (1), we can now proceed as follows. For any C^∞ function f on M that vanishes on ∂M,

$$\begin{aligned}\frac{1}{n}\alpha(n-1)\int_M |\nabla f|^2 \, dx &= \int_M \int_{U_x} (df(v))^2 \, dv_x \, dx \\ &= \int_{UM} (df(v))^2 \, dv \geq \int_{\tilde{U}M} (df(v))^2 \, dv \\ &= \int_{\tilde{U}^+\partial M} \int_{t=0}^{l(u)} (df(\zeta(u, t))^2 \langle T, u\rangle \, dt \, du.\end{aligned} \tag{5}$$

For $u \in \tilde{U}^+ \partial M$ and $0 < t < l(u)$, set $f_u(t) = f(\gamma_u(t))$. Thus, f_u has the limits $f_u(0) = f_u(l(u)) = 0$, as $\gamma_u(0)$, $\gamma_u(l(u)) \in \partial M$. The one-dimensional Poincaré inequality gives

$$\begin{aligned}\int_0^{l(u)} (df(\zeta(u, t)))^2 \, dt &= \int_0^{l(u)} (f_u'(t))^2 \, dt \\ &\geq \pi^2 l^{-2}(u) \int_0^{l(u)} (f_u(t))^2 \, dt \\ &= \pi^2 \int_0^{l(u)} l^{-2}(u) f^2(\gamma_u(t)) \, dt.\end{aligned} \tag{6}$$

Therefore, we have from (5) and (4)

$$\alpha(n-1)\int_M |\nabla f|^2 \, dx \geq n\pi^2 \int_{\tilde{U}M} l^{-2}(v) f^2(v) \, dv.$$

Since $l(v) = \infty$ for $v \in UM - \tilde{U}M$, the last integral can equally well be taken over UM, so that

$$\begin{aligned}\alpha(n-1)\int_M |\nabla f|^2 \, dx &\geq n\pi^2 \int_{\operatorname{Int} M} f^2(x) \left\{ \int_{U_x} l^{-2}(v) \, dv_x \right\} dx \\ &\geq n\pi^2 \inf_{x \in \operatorname{Int} M} \left\{ \int_{U_x} l^{-2}(v) \, dv_x \right\} \cdot \int_M f^2 \, dx,\end{aligned}$$

and (1) follows from (3).

COROLLARY 1. *Let (M, g) be a compact Riemannian manifold with $\partial M \neq \varnothing$ for which equality holds in* (1). *Then M admits a C^∞ function f with*

$$f \geq 0, \qquad f^{-1}(0) = \partial M \neq \varnothing, \tag{7}$$

$$\Delta f = mf \tag{8}$$

for some real m, and

$$(\nabla_v H)(v, v) = 0, \qquad H(v, v) \leq 0 \tag{9}$$

for any tangent vector v in $\operatorname{Int} M$, *where*

$$H = f^{-1} \nabla\, df \quad \textit{in } \operatorname{Int} M. \tag{10}$$

Proof. Let f be a λ_1 eigenfunction of Δ, so that (8) holds with $m = \lambda_1$ and we obtain (7) from the maximum principle, changing the sign of f if necessary. Since all inequalities in the preceding argument now become equalities, we conclude from (6) that, for almost all $u \in \tilde{U}^+ \partial M$, $f_u'' = -\pi^2 l^{-2}(u) f_u$. As $f_u'' = (f \circ \gamma_u)'' = (\nabla\, df)(\dot{\gamma}_u, \dot{\gamma}_u)$, this just means that $H(\dot{\gamma}_u, \dot{\gamma}_u) = -\pi^2 l^{-2}(u)$ and $(\nabla_{\dot{\gamma}_u} H)(\dot{\gamma}_u, \dot{\gamma}_u) = \dot{\gamma}_u(H(\dot{\gamma}_u, \dot{\gamma}_u)) = 0$, H being defined by (10). Thus, (9) holds for almost all $v \in \tilde{U}(\operatorname{Int} M)$. The equality in (5) gives $df(v) = 0$ for almost all $v \in UM - \tilde{U}M$. Let $W \subset \operatorname{Int} M$ be the projection of the part of the interior of $UM - \tilde{U}M$, lying over $\operatorname{Int} M$. Thus, f is locally constant in W, which gives $\nabla\, df = 0$ there and hence (9) for all v in the interior of $UM - \tilde{U}M$ over $\operatorname{Int} M$. Consequently, (9) holds for all $v \in U(\operatorname{Int} M)$, which completes the proof.

Remark 1. From the proof of Corollary 1 it is obvious that if (9) and (10) are satisfied by a function f on a Riemannian manifold with boundary, then, for each geodesic γ, the function $t \mapsto f(\gamma(t))$ is *sinusoidal, linear or constant* in the sense that it is given by $p \sin(qt + s)$, $pt + s$ or s with some constants $p \neq 0$, $q \neq 0$ and s, depending on γ.

Remark 2. For compact Riemannian manifolds (M, g) with boundary, the converse to Corollary 1 is also true. We omit the details of this argument, based on repeating the steps in the proof of (1) and on Remark 1, together with the obvious relation $\alpha(n-1)(\Delta f)(x) = -n \int_{U_x} (\nabla\, df)(v, v)\, dv_x$ for $x \in \operatorname{Int} M$.

Remark 3. For a compact Riemannian manifold (M, g) with boundary, the formula

$$h(x) = \int_{U_x} l^{-2}(v)\, dv_x \tag{11}$$

defines a finite continuous function h on Int M. Moreover, (11) also makes sense for $x \in \partial M$ and the resulting extension of h to M always attains its minimum. These facts can be proved by a standard but lengthy argument that will not be presented here. Note that $h = \infty$ on the boundary of a flat disc and the function $v \mapsto l(v) \in [0, \infty]$ may fail to be continuous on $U(\operatorname{Int} M)$ if ∂M is not convex.

3. Some auxiliary results

Given a manifold M with a metric g, usually denoted by $\langle , \rangle$, we use the symbol ∇ for the Riemannian connection of (M, g). Thus, for vector fields u, v, w on M,

$$2\langle \nabla_u v, w \rangle = \langle [u, v], w \rangle - \langle v, [u, w] \rangle - \langle u, [v, w] \rangle \\ + u\langle v, w \rangle + v\langle u, w \rangle - w\langle u, v \rangle, \tag{12}$$

$uf = df(u) = \nabla_u f$ being the u-directional derivative of any function f on M. The Hessian $\nabla\, df$ of f is given by

$$(\nabla\, df)(u, v) = (\nabla\, df)(v, u) = \langle \nabla_u \nabla f, v \rangle = uvf - df(\nabla_u v) \tag{13}$$

for vector fields u, v, where ∇f is the gradient of f. The Laplacian Δ is an operator with *negative* symbol:

$$\Delta f = -\operatorname{Tr}_g \nabla\, df.$$

The formula for the covariant derivative ∇H of a 2-tensor field H is

$$(\nabla_u H)(v, w) = u(H(v, w)) - H(\nabla_u v, w) - H(v, \nabla_u w). \tag{14}$$

The following lemma and remark recall well-known facts that we use later.

LEMMA 1. *Let N be a regular level of a function f on a Riemannian manifold with boundary (empty or not) such that $\nabla\, df = 0$ at each point of N. Then N is totally geodesic and $|\nabla f|$ is constant along each component of N.*

In fact, this is immediate from (13) for vector fields u, v defined near N and tangent to N, together with the obvious equality $d\,|\nabla f|^2 = 2(\nabla\, df)\,(\nabla f, \cdot)$ (cf. (13)).

Remark 4. For a distribution V on a Riemannian manifold we write $X \in V$ when X is a *local* vector field (defined in an open set) which is a section of V. Operations on such fields are to be considered only in the intersection of their definition domains. Thus, V is integrable if and only if $[X, Y] \in V$ whenever $X, Y \in V$; integrable distributions with totally geodesic leaves are characterized by the condition $\nabla_X Y \in V$ for all $X, Y \in V$. Moreover,

(i) If V is integrable, it has totally geodesic leaves if and only if $\nabla_X X \in V$ for all $X \in V$.

(ii) A local vector field u *preserves* V (i.e., its local flow leaves V invariant) if and only if $[u, X] \in V$ whenever $X \in V$.

(iii) If V is integrable, the local fields $\xi \in V^\perp$ preserving V fill $V^\perp$ at each point. In fact, locally, the leaves of V are fibres of a bundle with connection $V^\perp$ over some base B and the sections ξ in question are just the horizontal lifts of vector fields in B.

(iv) Suppose that V is integrable and a local vector field u preserves V. Then u preserves the metric restricted to V (i.e., its local flow sends the leaves isometrically into one another) if and only if $\nabla_u |X|^2 = 0$ for all $X \in V$ with $[u, X] = 0$ (such X fill V at each point).

(v) Let V be integrable. Then its leaves are all totally geodesic if and only if all $\xi \in V^\perp$ that preserve V also preserve the metric restricted to V. This is immediate from (i), (iii), (iv) and (12) with $u = v = X \in V$, $w = \xi \in V^\perp$ and $[\xi, X] = 0$. See also [6].

(vi) If V is integrable, its leaves are, locally, the fibres of a Riemannian submersion if and only if for any $\xi \in V^\perp$ preserving V (a "horizontal lift"), $|\xi|$ is constant along V. Note that, for such ξ and any $X \in V$, $\langle \nabla_X \xi, \xi \rangle = \langle \nabla_\xi X, \xi \rangle = -\langle \nabla_\xi \xi, X \rangle$. Therefore, another equivalent condition is $\nabla_\xi \xi \in V^\perp$ for *all* $\xi \in V^\perp$ (since the latter condition is tensorial, cf. (iii)).

Suppose we are given a principal bundle P with structure group G over a closed manifold B and an isometric action ρ of G on a compact Riemannian manifold (F, g_F) with boundary. It is easy to see that, for any connection ω in P and any metric h on B, there is a unique smooth metric g on the total space M of the bundle with fibre F associated with P, such that

(i) the ω-horizontal spaces in M are g-orthogonal to the fibres,
(ii) the projection $(M, g) \to (B, h)$ is a Riemannian submersion,
(iii) on each fibre g induces the natural metric, isometric to g_F.

DEFINITION 1. A compact Riemannian manifold (M, g) with boundary is called a *Riemannian fibre bundle* if it is obtained by the above construction from some $P, G, \rho, B, F, g_F, \omega$ and h. The manifolds (B, h) and (F, g_F) will be called the (Riemannian) *base* and *fibre* of (M, g), respectively.

REMARK 5. Riemannian fibre bundles can equivalently be defined, up to an isometry, to be the compact Riemannian manifolds (M, g) with boundary admitting a *Riemannian submersion* $(M, g) \to (B, h)$ *with totally geodesic fibres*, where $\partial B = \varnothing$. In fact, by Remark 4(v) the fibres are totally geodesic in (M, g) if and only if the parallel displacements with respect to the g-orthogonal connection in $M \to B$ send them isometrically onto one another.

PROPOSITION 1. *Let* (M, g) *be a Riemannian fibre bundle with* $P, G, \rho, B, F, g_F, \omega$ *and* h *as above, such that* $\partial F \neq \varnothing$.

(i) $\lambda_1(M, g) = \lambda_1(F, g_F)$.

(ii) $E(M, g) = E(F, g_F)$, *the invariant* E *being defined by* (2).

Proof. For $x \in M$ lying in the fibre $F_x = F_y$ of $M \to B$ over $y \in B$, let dx, dy and dz be the Riemannian measures of M, B and F_y, respectively.Suppose f is a λ_1 eigenfunction on (M, g) with $f = 0$ on ∂M. Since $|\nabla f|$ in (M, g) is not less than the same quantity for the restriction of f to F_y with the induced metric, the Poincaré inequality for (F, g_F) gives

$$\int_{F_y} |\nabla f|^2 \, dx \geq \lambda_1(F, g_F) \int_{F_y} f^2 \, dz.$$

Locally, we have $dx = dz\, dy$ and so

$$\begin{aligned} \int_M |\nabla f|^2 \, dx &= \int_B \left(\int_{F_y} |\nabla f|^2 \, dz \right) dy \\ &\geq \lambda_1(F, g_F) \int_B \int_{F_y} f^2 \, dz\, dy = \lambda_1(F, g_F) \int_M f^2 \, dx, \end{aligned} \tag{15}$$

which proves that

$$\lambda_1(M, g) \geq \lambda_1(F, g_F),$$

since

$$\int_M |\nabla f|^2 \, dx = \lambda_1(M, g) \int_M f^2 \, dx.$$

Since $\partial F \neq 0$, the λ_1-eigenspace in (F, g_F) is one-dimensional (cf. [1, Theorem 4.7]). A fixed λ_1-eigenfunction ϕ on (F, g_F) (with $\phi = 0$ on ∂F) is therefore invariant under the action ρ of G. Hence ϕ can be naturally propagated over all fibres of $M \to B$, giving a function f on M invariant under ω-parallel displacements (and so having vertical gradient ∇f in (M, g)) and restricting to a λ_1-eigenfunction on each fibre. With this f we have equality in (15), i.e., by (3), $\lambda_1(M, g) \le \lambda_1(F, g_F)$, which proves (i) in view of the preceding inequality.

Given $x \in \operatorname{Int} M$ and a maximal geodesic $\gamma : I \to \operatorname{Int} M$ of (M, g), defined on an open interval I with $0 \in I$ and $\gamma(0) = x$, consider the C^∞ map $\psi : I \times I \to \operatorname{Int} M$ with

$$\psi(s, t) = \tau_s^t(\gamma(t)), \tag{16}$$

$\tau_s^t : F_{\gamma(t)} \to F_{\gamma(s)}$ being the ω-parallel displacement from t to s along the projection of γ onto B. Now $X = \partial\psi/\partial t$ and $\xi = \partial\psi/\partial s$ are vector fields tangent to M along ψ (i.e., sections of $\psi^* TM \to I \times I$) and X is vertical, while ξ is horizontal and $\gamma(t) = \psi(t, t)$, so that $\dot\gamma(t) = X(t, t) + \xi(t, t)$. Differentiating X, ξ covariantly along ψ, we obtain $\nabla_{X+\xi}(X + \xi) = \nabla_{\dot\gamma}\dot\gamma = 0$ on the diagonal of $I \times I$, while, by Remark 4(i), (vi) and Remark 5, $\nabla_X X$ is vertical, $\nabla_X \xi$, $\nabla_\xi \xi$ are horizontal and so is $\nabla_\xi X$, since $[\xi, X] = 0$ along ψ by definition. Consequently, $\nabla_X X = 0$ on the diagonal of $I \times I$, i.e., for each $s \in I$ the geodesic curvature of the curve $t \mapsto \psi(s, t)$ in the totally geodesic fibre $F_{\gamma(s)}$ vanishes as $t = s$. Since $\tau_{s'}^s : F_{\gamma(s)} \to F_{\gamma(s')}$ is an isometry and $\tau_{s'}^s(\psi(s, t)) = \psi(s', t)$, that geodesic curvature is zero at each $t = s'$, i.e., *the curves* $t \mapsto \psi(s, t)$ *are all geodesics with affine parameter.* Let $v \in T_x M$ be unit, so that

$$v = \cos\theta \cdot X_0 + \sin\theta \cdot \xi_0$$

with $\theta \in \mathbb{R}$ and units vectors X_0, ξ_0 such that X_0 is vertical and ξ_0 is horizontal. We claim that, if v is not horizontal (i.e., $\cos\theta \neq 0$),

$$l(v) = \frac{l(X_0)}{\cos\theta}. \tag{17}$$

In fact, let $\gamma = \gamma_v : I \to \operatorname{Int} M$ be a maximal geodesic with $\gamma_v(0) = x$, $\dot\gamma_v(0) = v$. As we saw, $\gamma_{X_0}(s) = \psi(s/\cos\theta, 0)$, ψ being given by (16) and γ_{X_0} is defined on the *maximal* interval $\cos\theta \cdot I$, since ∂M is invariant under ω-parallel displacements. This proves (17). Using easy spherical integration we now conclude that $h(x)$, defined for F_x with the metric isometric to g_F as in (11), is equal to the corresponding quantity for (M, g) times an appropriate dimensional constant. This implies (ii) and completes the proof.

The following three technical lemmas are the basic local ingredients for the argument of Section 4.

LEMMA 2. *Let a symmetric 2-tensor field H on a Riemannian manifold (M, g) satisfy $(\nabla_v H)(v, v) = 0$ for all tangent vectors v and admit a constant eigenvalue c of constant multiplicity. Then the corresponding eigenspace distribution V has the property that $\nabla_u u \in V$ for each local section u of V. Moreover,*

(i) *If V is integrable, its leaves are all totally geodesic.*

(ii) *If $V^\perp$ is integrable, its leaves form, locally, the fibres of a Riemannian submersion.*

Proof. Assume $V \neq TM$ and fix a local section u of V. In a dense open subset of M, the eigenvalues of H form mutually distinct C^∞ functions. For such a function $\phi \neq c$ and a local vector field w with $H(w, \cdot) = \phi\langle w, \cdot\rangle$, we have $\langle w, u\rangle = 0$. Since $H(u, \cdot) = c\langle u, \cdot\rangle$, (14) gives $(\nabla_u H)(u, w) = (c - \phi)\langle\nabla_u u, w\rangle$ and $(\nabla_w H)(u, u) = |u|^2 wc = 0$ as c is constant. Thus, from (9), by polarization, $0 = (\nabla_u H)(u, w) + (\nabla_u H)(w, u) + (\nabla_w H)(u, u) = 2(c - \phi)\langle\nabla_u u, w\rangle$, and so $\nabla_u u \in V$, since the vector fields w as above span $V^\perp$ in a dense set. Our assertion now follows from Remark 4(i), (vi).

LEMMA 3. *Let (M, g) be a Riemannian manifold with boundary admitting a function f with* (7) *and* (9) *for all vectors v in* Int M, *H being defined by* (10). *Then*

(i) *At each point of ∂M, $\nabla\, df = 0$ and $df \neq 0$.*

(ii) *∂M is totally geodesic in (M, g) and $|\nabla f|$ is locally constant along ∂M.*

(iii) *Let M be compact and denote by T the inner unit normal vector field along ∂M. There exist positive constants p, q such that each maximal normal geodesic $t \mapsto \exp tT_y$ issuing from $y \in \partial M$ has length π/q, hits ∂M perpendicularly at $t = \pi/q$ and $f(\exp tT_y) = p \sin(qt)$.*

Proof. Since $f > 0$ in Int M, Remark 1 gives $df \neq 0$ at each point of ∂M. From (9), $f(\nabla^2 df)(v, v, v) = df(v)(\nabla\, df)(v, v)$ for any vector v, wherever $f \neq 0$ and, by continuity, everywhere in M. Consequently, on ∂M the symmetric product $df \odot \nabla\, df$ vanishes, which gives (i), while (ii) now follows from Lemma 1.

Let M be compact. By Remark 1 with (7), for any $y \in \partial M$

$$f(\exp tT_y) = p(y)\sin(q(y)t), \tag{18}$$

where p, q are positive C^∞ functions on ∂M and $0 \leq t \leq \pi/q(y)$. On ∂M, $|\nabla f| = pq$ is locally constant from (ii). However, since ∂M is totally geodesic, any two components of ∂M have a connecting geodesic γ perpendicular to both, and so the constant pq is the same for all components, as it is determined by $f \circ \gamma$ (cf. (18)). For $y \in \partial M$, the geodesic $\gamma(t) = \exp tT_y$ hits ∂M again at $t = t_y = \pi/q(y)$ with $\langle\nabla f, \dot\gamma(t_y)\rangle = -pq = -|\nabla f| \cdot |\dot\gamma(t_y)|$ in view of (18), so that $pq\dot\gamma(t_y) =$

$-(\nabla f)(\gamma(t_y))$ is normal to ∂M. Since all geodesics issuing normally from $y \in \partial M$ intersect ∂M perpendicularly, a standard variational argument shows that their lengths $\pi/q(y)$ are all equal, whether or not ∂M is connected. This implies that p and q are both constant, completing the proof.

LEMMA 4. *Let g be a metric on $M = N \times I$, where N is a manifold without boundary and $I = [0, \pi/2)$ such that the curves $t \mapsto (y, t)$, $y \in N$, are all unit speed geodesics, orthogonal to $\partial M = N \times \{0\}$, and the function $f(y, t) = \sin t$ on (M, g) satisfies* (7), (8) *for some real m and* (9) *for all tangent vectors v in* Int M, *H being given by* (10). *Then*

(i) *m is an integer, $1 \le m \le n = \dim M$.*

(ii) *At each point of* Int M, *H has eigenvalues -1 and 0 of multiplicities m and $n-m$, respectively. The (-1)-eigenspace distribution D of H in* Int M *has a C^∞ extension to ∂M that intersects $T(\partial M)$ along an $(m-1)$-dimensional distribution V on ∂M. If V is integrable, so is D.*

(iii) *At $(y, t) \in M = N \times I$, g is given by $g(T, T) = 1$, $g(X, X) = \cos^2 t \cdot g_0(X, X)$, $g(\xi, \xi) = g_0(\xi, \xi)$, $g(T, X) = g(T, \xi) = g(X, \xi) = 0$, where g_0 is the induced metric on $\partial M = N \times \{0\} \approx N$, T is the vector field with integral curves $t \mapsto (y, t)$ and X* (resp., *ξ*) *is any vector of V* (resp., *of its g_0-complement $V^\perp$ in T_yN*) *at y, regarded as lying in $T_{(y,t)}M$.*

Proof. Writing $\langle\, ,\rangle$ for g, we have from the generalized Gauss lemma

$$\begin{aligned} &\langle T, T\rangle = 1, \qquad \langle T, \alpha\rangle = 0, \\ &\langle \alpha, \beta\rangle = \langle \alpha, \beta\rangle_t \end{aligned} \tag{19}$$

where $I \ni t \mapsto g_t = \langle\, ,\rangle_t$ is a curve of metrics on N, T is defined in (iii) and α, β denote, from now on, arbitrary vector fields on N viewed at the same time as vector fields on M. Thus

$$\begin{aligned} &\nabla_T T = [T, \alpha] = 0 \\ &\langle T, [\alpha, \beta]\rangle = 0, \end{aligned} \tag{20}$$

so that (12) and (19) yield $\langle \nabla_T \alpha, \beta\rangle = \frac{1}{2}(\partial/\partial t)\langle \alpha, \beta\rangle_t$ and hence

$$\nabla_T \alpha = \nabla_\alpha T = -\tan t \cdot B_t \alpha \tag{21}$$

B_t being the curve of $(1, 1)$ tensor fields on N given by

$$\langle B_t \alpha, \beta\rangle_t = -\tfrac{1}{2}\cot t \frac{\partial}{\partial t}\langle \alpha, \beta\rangle_t. \tag{22}$$

Consequently, from (21) and (19)

$$\langle \nabla_\alpha \beta, T\rangle = \tan t\langle B_t\alpha, \beta\rangle. \tag{23}$$

Since $f(y, t) = \sin t$, we have $Tf = \cos t$, $\alpha f = 0$, i.e., $df(u) = \cos t \cdot \langle u, T\rangle$ for any vector u. Therefore, from (13), (21), (20) and (23), $(\nabla\, df)(T, T) = -\sin t$, $(\nabla\, df)(T, \alpha) = 0$ and $(\nabla\, df)(\alpha, \beta) = -\sin t\langle B_t\alpha, \beta\rangle$. In particular, $\Delta f = (1 + \operatorname{Tr} B_t)f$ and, since $\Delta f = mf$ by hypothesis,

$$\operatorname{Tr} B_t = m - 1. \tag{24}$$

The 2-tensor field $H = f^{-1}\nabla\, df$ on Int M now satisfies

$$\begin{aligned} &H(T, T) = -1, \qquad H(T, \alpha) = 0,\\ &H(\alpha, \beta) = -\langle B_t\alpha, \beta\rangle, \end{aligned} \tag{25}$$

so that $H(T, u) = -\langle T, u\rangle$ for any vector u. Therefore, in view of (14), (23), (21), (19) and (22), $(\nabla_\alpha H)(T, \beta) = \tan t\langle (B_t - B_t^2)\alpha, \beta\rangle$, $(\nabla_T H)(\alpha, \beta) = -\langle (\partial B_t/\partial t)\alpha, \beta\rangle$ (note that (25) holds for *vectors* α, β orthogonal to T, while $T(H(\alpha, \beta)) = -\partial/\partial t\langle B_t\alpha, \beta\rangle_t$). Since, by (9) for all v, $(\nabla_\alpha H)(T, \beta) + (\nabla_T H)(\beta, \alpha) + (\nabla_\beta H)(\alpha, T) = 0$, we now obtain

$$\partial B_t/\partial t = 2\tan t(B_t - B_t^2). \tag{26}$$

Hence, by induction, $\partial B_t^k/\partial t = 2k\tan t(B_t^k - B_t^{k+1})$ and so $\partial/\partial t \operatorname{Tr} B_t^k = 2k\tan t(\operatorname{Tr} B_t^k - \operatorname{Tr} B_t^{k+1})$ for all k. Consequently, by induction we obtain from (24)

$$\operatorname{Tr} B_t^k = m - 1$$

for each k. Since B_t is g_t-self-adjoint, the g_t-norm of $B_t - B_t^2$ now satisfies $|B_t - B_t^2|_t^2 = \operatorname{Tr}(B_t - B_t^2)^2 = 0$. Therefore $B_t^2 = B_t$ and, by (26),

$$\partial B_t/\partial t = 0$$

i.e., $B = B_t$ is independent of t. Thus, at each point of N, B is an orthogonal projection for all metrics g_t and rank $B = m - 1$, which implies (i). Consider the distributions $V = \operatorname{Im} B = \operatorname{Ker}(B - Id)$ and $V^\perp = \operatorname{Ker} B$ in N of dimensions $m - 1$ and $n - m$, respectively. In the sequel, let X (resp., ξ) denote an arbitrary local section of V (resp., $V^\perp$) in N, which we also view as a vector field in M. Thus,

$$BX = X, \qquad B\xi = 0.$$

Hence (22) with $B_t = B$ and (19) imply (iii). On the other hand, (25) gives $H(T, \cdot) = -\langle T, \cdot\rangle$, $H(X, \cdot) = -\langle X, \cdot\rangle$, $H(\xi, \cdot) = 0$, so that (ii) follows from (20) with $\alpha = X$, D being spanned by T and V in the obvious sense. This completes the proof.

Remark 6. In the course of the above proof we have in fact *classified* all manifolds (M, g) satisfying the hypotheses of Lemma 4. Namely, they are in a bijective correspondence with the Riemannian manifolds $(N, g_0)(\partial N = \emptyset)$ endowed with distributions V such that $\mathring{\nabla}_X X \in V$ and $\mathring{\nabla}_\xi \xi \in V^\perp$ for any local sections X of V and ξ of its g_0-complement $V^\perp$ (cf. Lemma 2), $\mathring{\nabla}$ being the Riemannian connection of g_0.

Remark 7. If a manifold (M, g) satisfying the hypotheses of Lemma 4 has bounded curvature, then the distribution D is integrable and has totally geodesic leaves of constant curvature 1. To verify this, it is sufficient to express the curvature of g in terms of the metric g_0 and the distribution V on N, using (iii) and Lemma 2. However, in the case where (M, g) has a Riemannian compactification we shall prove the above assertion by a direct geometric argument (see Section 4).

4. The equality case: Riemannian hemisphere bundles

In this section we classify the compact Riemannian manifolds (M, g) with boundary for which

$$\lambda_1(M, g) = E(M, g), \tag{27}$$

i.e., (1) becomes an equality. It turns out that they form the class defined as follows.

DEFINITION 2. By a *Riemannian hemisphere bundle* we mean a compact Riemannian manifold (M, g) with boundary which is a Riemannian fibre bundle in the sense of Definition 1 having as the Riemannian fibre a round hemisphere of some dimension $m \geq 1$ and radius $r > 0$ with $G = O(m)$ acting in the natural way.

For a round hemisphere of dimension m and radius r, $\lambda_1 = m/r^2$ and $l(v) = \pi r$ for each unit tangent vector v, which implies (27). Therefore, by Proposition 1, (27) also holds for all Riemannian hemisphere bundles.

We will now show that every compact Riemannian manifold (M, g) with boundary, satisfying (27), is isometric to a Riemannian hemisphere bundle.

In view of Corollary 1, M admits a function f with (7), (8) for some real m and (9) for any tangent vector v in $\operatorname{Int} M$, H being given by (10). By Lemma 3(iii), we can rescale g and f so that $f(\exp tT_y) = \sin t$ for each $y \in \partial M$ and $t \in [0, \pi]$, T being the inner unit normal vector field along ∂M. Connecting any $x \in M$ to ∂M by a shortest geodesic, we see that

$$f(x) = \sin [\operatorname{dist}(x, \partial M)], \tag{28}$$

which easily implies that all geodesics $[0, \pi/2) \ni t \mapsto \exp tT_y$, $y \in \partial M$, are minimizing and mutually disjoint. Using a standard argument ([5, p. 135–136]) we now conclude that the normal exponential map

$$\partial M \times \left[0, \frac{\pi}{2}\right) \to M - f^{-1}(1) \tag{29}$$

is a diffeomorphism. Thus, applying Lemma 4 to $M - f^{-1}(1)$ instead of M we see, by continuity, that assertions (i) and (ii) of Lemma 4 hold *everywhere* in $\operatorname{Int} M$.

Let $B = f^{-1}(1)$. For a geodesic γ connecting y, $z \in B$ in M we have $0 \le f \circ \gamma \le 1 = f(y) = f(z)$ and $f \circ \gamma$ is sinusoidal or constant (Remark 1), so, obviously, $f \circ \gamma = 1$. Thus, γ lies entirely in B and so does its extension beyond y and z if $y \ne z$, i.e., B is a totally convex compact submanifold without boundary. Moreover, for $y \in B$, T_yB is the null-space of H_y and so $\dim B = n - m$. To see this, note that we just showed that T_yB consists of all vectors u at y tangent to geodesics γ with $f \circ \gamma = 1$, which are precisely the vectors u with $(\nabla\, df)(u, u) = 0$, since $f = 1$ and $df = 0$ on B and f is sinuisoidal or constant along each geodesic. As H is semidefinite, these vectors form the null-space of H_y.

Let $\gamma: [0, t_0) \to \operatorname{Int} M$ be a maximal geodesic normal to B at $y = \gamma(0)$ with $|\dot\gamma| = 1$. Thus, $\dot\gamma(0)$ is in the (-1)-eigenspace of H_y. The sinusoidal function $\phi = f \circ \gamma$ (Remark 1) has $\phi(0) = 1$, $\phi \le 1$, $\phi'(0) = 0$ and $\phi''(0) = -1$, i.e., $t_0 = \pi/2$ and $\phi(t) = \cos t$. Connecting any $x \in M$ to B by a distance minimizing segment of such a geodesic γ we see that

$$f(x) = \cos [\operatorname{dist}(x, B)], \tag{30}$$

so that normal geodesics issuing from B are all minimizing and mutually disjoint. Consequently, the normal exponential map from the bundle of radius $\pi/2$ closed normal discs over B is a diffeomorphism onto M, which allows us to identify M with the total space of this bundle. The projection $M \to B$ (restricting to the $(m-1)$-sphere bundle $\partial M \to B$) sends $\exp tT_y$ ($y \in \partial M$, $0 \le t \le \pi/2$) onto $\exp(\pi/2)T_y$. In fact, (28) and (30) give $\operatorname{dist}(x, \partial M) + \operatorname{dist}(x, B) = \pi/2 = \operatorname{dist}(\partial M, B)$ for all $x \in M$, so that each geodesic normal to B hits ∂M perpendicularly, and vice versa.

Let $s \mapsto y(s)$ be a curve in ∂M with $\dot{y}(s)$ tangent to the distribution V at each point (cf. Lemma 4(ii)) and set $y(s, t) = \exp tT_{y(s)}$ for $0 \leq t \leq \pi/2$. Using Lemma 4(iii) with our identification (29) we see that the vector $(\partial/\partial s)y(s, t)$ has length $\cos t\,|\dot{y}(s)|$, which vanishes for $t = \pi/2$ and any s. Therefore, $y(s, \pi/2) = \exp(\pi/2)T_{y(s)}$ is constant, i.e., $y(s)$ lies entirely in a fibre of $\partial M \to B$. For dimensional reasons, this implies that V is integrable and hence so is D by Lemma 4(ii), while the leaves of V are just the $(m-1)$-sphere fibres of $\partial M \to B$. The leaves of D are totally geodesic in view of Lemma 2(i). The normal geodesics issuing from B are all tangent to D, since this is the case at their origins in B (cf. the above description of T_yB). Consequently, the totally geodesic leaves of D are just the m-disc fibres of the bundle projection $M \to B$, which is a Riemannian submersion by Lemma 2(ii). In view of Lemma 4(iii), each of these fibres is isometric to a unit hemisphere. for otherwise it would have a singularity at B. Now Remark 5 implies that (M, g) is isometric to a Riemannian hemisphere bundle.

5. Remarks and corollaries

A natural question suggested by the theorem of Section 1 is whether $\lambda_1(M, g)$ being close to $E(M, g)$ implies that (M, g) is close, in an appropriate sense, to a Riemannian hemisphere bundle. That this is not the case, for a reasonable notion of closeness, can be seen as follows. Given any closed Riemannian manifold (N, g), consider the manifolds $(M_\varepsilon, g_\varepsilon) = (N \times [0, (\pi/2) - \varepsilon], \cos^2 t \cdot g + dt^2)$, where $0 < \varepsilon < \pi/2$. We then have $\lambda_1(M_\varepsilon, g_\varepsilon) \geq E(M_\varepsilon, g_\varepsilon) \geq n = \dim N + 1$ and $\lambda_1(M_\varepsilon, g_\varepsilon) \to n$ as $\varepsilon \to 0$, which one can verify by using the function $\sin t \cdot \phi_\varepsilon(t)$ with an appropriate cut-off function ϕ_ε. More complicated examples of this type are provided by Riemannian fibre bundles with fibre $(M_\varepsilon, g_\varepsilon)$ as above.

The invariant $E(M, g)$ is still well-defined for *non-compact complete* Riemannian manifolds with boundary. In this case, our proof of (1) remains valid if we define $\lambda_1(M, g)$ by (3) restricted to functions f vanishing on ∂M for which $|\nabla f|$ is square integrable. Consequently, we have

COROLLARY 2. *Every complete Riemannian manifold (M, g) with boundary satisfies $\lambda_1(M, g) \geq E(M, g)$.*

The last estimate is sharp for infinite strips in the plane as well as for Riemannian hemisphere bundles with complete open base manifolds (B, h) satisfying $\lambda_1(B, h) = 0$ in the sense explained above. The simplest examples of such (B, h) are the Euclidean spaces.

For a complete n-dimensional Riemannian manifold (M, g) with boundary and $x \in \operatorname{Int} M$, set (cf. [4])

$$EV_x(M, g) = \frac{1}{n} \int_{U_x} l_+^n(v)\, dv_x,$$

$l_+(v) \leq \infty$ being the maximum length of the geodesic segment in $\operatorname{Int} M$ having v as its tangent vector at the origin, so that

$$l(v) = l_+(v) + l_+(-v). \tag{31}$$

If (M, g) is a Euclidean domain, $EV_x(M, g)$ is just the volume of the region "visible from x". In general, it is the Euclidean volume in T_xM of the set where the exponential map is defined. Note that $EV_x(M, g)$ may be infinite even if M is compact. As a consequence of (1) we obtain the following estimate (which is never sharp).

COROLLARY 3. *For any complete n-dimensional Riemannian manifold (M, g) with boundary,*

$$\lambda_1(M, g) \geq \frac{\pi^2}{4} [\alpha(n-1)]^{2/n} n^{(n-2)/n} \left[\sup_{x \in \operatorname{Int} M} EV_x(M, g) \right]^{-2/n}.$$

Proof. Fix $x \in \operatorname{Int} M$. In view of (31), the Schwarz inequality yields

$$[\alpha(n-1)]^2 = \left(\int_{U_x} dv_x \right)^2 \leq h(x) \int_{U_x} l^2(v)\, dv_x \leq 4h(x) \int_{U_x} l_+^2(v)\, dv_x,$$

h being given by (11). By the Hölder inequality, we have

$$\int_{U_x} l_+^2(v)\, dv_x \leq [\alpha(n-1)]^{(n-2)/n} [nEV_x(M, g)]^{2/n}$$

so that

$$[\alpha(n-1)]^2 \leq 4[\alpha(n-1)]^{(n-2)/n} h(x) \left[n \sup_{x \in \operatorname{Int} M} EV_x(M, g) \right]^{2/n}$$

and our assertion is immediate from (1).

For a domain Ω in the unit n-sphere we have $l(v) = \infty$ or $l(v) \leq 2\pi$, so that $4\pi^2 h(x) \geq \mu(x)$, where h is given by (11) and $\mu(x)$ is the measure of $\{v \in U_x : l(v) < \infty\}$. Therefore, by (1),

$$\lambda_1(\Omega) \geq \frac{n}{4\alpha(n-1)} \min_{x \in \operatorname{Int} \Omega} \mu(x).$$

In particular, $\lambda_1(\Omega) \geq \pi/4$ if Int Ω contains no great circle.

REFERENCES

[1] T. AUBIN, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*. Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[2] M. BERGER, *Lectures on Geodesics in Riemannian Geometry*, Tata Institute, Bombay, 1965.

[3] C. CROKE, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) *13* (1980) 419–535.

[4] C. CROKE, *Curvature free volume estimates*. Invert. Math. *76* (1984) 515–521.

[5] D. GROMOLL, W. KLINGENBERG and W. MEYER, Riemannsche Geometrie im Grossen, Lecture Notes in Math. No. 55, Springer-Verlag, 1968.

[6] R. HERMANN, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle*, Proc. Amer. Math. Soc. *11* (1960) 236–242.

[7] P. LI, *Poincaré inequalities on Riemannian manifolds*. Seminar on Differential Geometry (edited by Shing-Tung Yau), *Annals of Math. Studies* No. 102, Princeton University Press, 1982, pp. 73–83.

[8] L. A. SANTALÓ, *Integral Geometry and Geometric Probability* (Encyclopedia of Mathematics and Its Applications), Addison-Wesley, 1976.

Dept. of Mathematics
University of Pennsylvania
Philadelphia, PA 19104, U.S.A.

Dept. of Mathematics
Ohio State University
Columbus, OH 43210, U.S.A.

Received March 13, 1986