On low dimensional S-cobordisms.

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On low dimensional S-cobordisms

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One of the most beautiful and powerful theorems in differential topology is S. Smale's h-cobordism theorem (see [26]). It states that the smooth simply-connected h-cobordism (W^n, M_0, M_1) with $n \geqslant 6$ is smoothly trivial, i.e. $W^n \approx_{\text{Diff}} M_0 \times I$. There are analogous PL and TOP versions (see [10, [23]) as well as nonsimply-connected generalization: the s-cobordism theorem of Barden–Mazur–Stallings.

A topological version of the 5-dimensional h-cobordism theorem is available using the work of M. Freedman (see [6]) and F. Quinn (see [21]). An analogous 5-dimensional s-cobordism theorem is proved for a class $G :=$ poly-(finite or cyclic) fundamental groups (see [7], [8]). Whether it holds for all fundamental groups is an open question.

Almost nothing was known concerning 4-dimensional s-cobordisms, although it was clear that even some special cases of the 4-dimensional s-cobordism theorem would be of interest. For example, if any s-cobordism between two copies of $S^1 \times S^2$ is trivial then the Stallings unknotting criterion (see [27]) remains true in the outstanding case of S^2 in S^4.

In this paper we show that the 4-dimensional s-cobordism theorem remains valid for 3-manifolds with poly-cyclic fundamental groups. It turns out that this result can not be generalized to all fundamental groups of 3-manifolds which are in G. Namely, a 4-dimensional s-cobordism fails to be a product for $S^1 \times RP^2$. This can be concluded from [13] but for the completeness of this paper we include a slightly modified construction here. This modification will enable us to construct infinitely many examples of nontrivial 4-dimensional (non-orientable) s-cobordisms. We also show that the above construction naturally leads to the existence of the fake RP^4. Finally we give an application of these results to group actions on 4-manifolds. As a by-product of our considerations we answer a question asked by W. C. Hsiang in [9]. We also take the opportunity to correct an inaccuracy which occurs in [13] and which is connected with the computation of the Whitehead group $Wh(G)$ for $G = Z \times Z_2$. Let us also note that S. Cappell and J. Shaneson (see [2]) have produced a family of orientable 4-dimensional s-cobordisms which are not products.

We start with the following (here poly-cyclic = poly-Z):

THEOREM 1. *Let M_0 be a closed, connected prime 3-dimensional manifold with poly-cyclic fundamental group. Let $(W; M_0; M_1)$ be a topological h-cobordism with prime M_1. Then $W \approx_{\text{TOP}} M_0 \times I$.*

Proof. First assume that $\pi_1(M_0) \neq 0$. We can also assume that M_0, M_1 are irreducible; because if M_0, M_1 are not irreducible then they are S^2 bundles over S^1 and will be considered separately. It is not difficult to see that there is a normal subgroup $N \subset \pi_1(M_0)$, $N \neq 1$, Z_2 with the quotient $\pi_1(M_0)/N \approx Z$. By the theorem of Stallings (see [27]) M_0 fibers over S^1. Then it follows from [18] that $M_0 \approx_{\text{TOP}} M_1$ and hence we can write $(W; M_0, M_0)$ for our h-cobordism. An induction with respect to the rank of $\pi_1(M_0)$ together with the Farrell fibering theorem (see [4]) show that the set $\mathcal{S}_{\text{TOP}}(W \times I^8; \partial(W \times I^8))$ of homotopy TOP structures on $W \times I^8$, rel $\partial(W \times I^8)$ is trivial. This is just the Handlebody Lemma C.6 in [10] p. 284. It is known (see [7], [8]) that the topological surgery works in dimension four for manifolds with poly-(finite or cyclic) fundamental groups.

In particular we have the Wall–Sullivan exact sequence:

$$\cdots \to L_5^s(\pi_1(M_0)) \to \mathcal{S}_{\text{TOP}}(W, \partial W) \xrightarrow{\tau} [W, \partial W; G/\text{TOP}, *] \xrightarrow{\theta} L_4^S(\pi_1(M_0)).$$

This implies that the periodicity for homotopy TOP structures (see [10] p. 283),[1] i.e.

$$\mathcal{S}_{\text{TOP}}(X, \partial X) \approx \mathcal{S}_{\text{TOP}}(X \times I^4; \partial(X \times I^4))$$

where $\dim X \geq 5$ remains valid when $\dim X = 4$ and $\pi_1(X)$ is poly-(finite or cyclic).

The existence of the Wall–Sullivan exact sequence is in fact all that is needed in order to obtain this periodicity. Therefore we have

$$\mathcal{S}_{\text{TOP}}(W, \partial W) \approx \mathcal{S}_{\text{TOP}}(W \times I^8; \partial(W \times I^8))$$

and hence $\mathcal{S}_{\text{TOP}}(W, \partial W) = 0$.

The trivial h-cobordism $M_0 \times I$ is an element of $\mathcal{S}_{\text{TOP}}(W, \partial W)$ (any homotopy equivalence $f: M_0 \to M_0$ is homotopic to a homeomorphism (see [18]); therefore by the 5-dimensional s-cobordism (see [7]) we have $W \approx_{\text{TOP}} M_0 \times I$.

Note that we did not meet here the problem of a Whitehead torsion for $(W; M_0, M_1)$. This was because $Wh(\pi_1(M_0)) = 0$ (see [5]). In fact the triviality of

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[1] We refer to [19], [22] for the rigorous proof of this result.

$Wh(\pi_1(M_0))$ was used in the Farrell fibering theorem. Also note that the 12-dimensional Poincaré Conjecture was used in this proof, i.e. we have used the fact that $\mathcal{S}_{TOP}(D^{12}) = 0$. To complete the proof of Theorem 1 we should consider the case when $\pi_1(M_0) = 0$ and the case of S^2 bundles over S^1. If $\pi_1(M_0) = 0$, to omit the Poincaré Conjecture, we assume $M_0 = S^3$.

First we show the following:

PROPOSITION 2. *Let M be a closed, connected, topological manifold homotopy equivalent to $S^1 \times S^3$. Then M is homeomorphic to $S^1 \times S^3$.*

Proof. As it was already mentioned the topological surgery theory is available for 4-manifolds with poly-(finite or cyclic) fundamental groups. In particular we have the following Wall–Sullivan exact sequence:

$$\cdots \to [S^1 \times S^3 \times I, \partial(S^1 \times S^3 \times I); G/\mathrm{TOP}, *] \xrightarrow{\theta_5} L_5^s(Z) \to \mathcal{S}_{TOP}(S^1 \times S^3) \to$$

$$\to [S^1 \times S^3; G/\mathrm{TOP}] \xrightarrow{\theta_4} L_4^s(Z).$$

To prove Proposition 2 it is enough to show that

$$\mathcal{S}_{TOP}(S^1 \times S^3) = 0.$$

Note that because the 6-stage in the Postnikov decomposition of G/TOP is given (see [10]) by

$$K(Z_2; 2) \times K(Z; 4) \times K(Z_2; 6)$$

then

$$[S^1 \times S^3; G/\mathrm{TOP}] \approx H^2(S^1 \times S^3; Z_2) \oplus H^4(S^1 \times S^3; Z) \approx Z.$$

We observe that the map

$$\theta_4 : Z \to L_4^s(Z) \approx L_4^s(1) \approx Z$$

is a bijection.

This follows from:

(1) the description of θ_4, i.e. for $g : S^1 \times S^3 \to G/\mathrm{TOP}$ the surgery obstruction

is given (see [31]) by $\frac{1}{8}\langle g^*L(G/\mathrm{TOP}); [S^1 \times S^3]\rangle$, where L is the Hirzebruch polynomial. (Note that $S^1 \times S^3$ is parallelizable).

(2) the existence (see [6]) of the manifold $|E_8|$.

We consider briefly the map θ_5,

$$\theta_5 : [S^1 \times S^3 \times I,\ \partial(S^1 \times S^3 \times I); G/(\mathrm{TOP}, *] \to L_5^s(Z).$$

We have

$$[S^1 \times S^3 \times I,\ \partial(S^1 \times S^3 \times I); G/\mathrm{TOP}, *] \approx [\textstyle\sum (S^1 \times S^3); G/\mathrm{TOP}]$$

$$\approx H^2(\textstyle\sum (S^1 \times S^3); Z_2) \oplus H^4(\textstyle\sum (S^1 \times S^3); Z) \approx Z_2 \oplus Z.$$

The isomorphism $L_5^s(Z) \approx L_4^s(1)$ can be described geometrically as follows: Let $f:(N, \partial N) \to (S^1 \times S^3 \times I; \partial(S^1 \times S^3 \times I))$ be a normal map. Then $\theta_5(f)$ is given by $\theta_4(\bar{f})$, where

$$\bar{f} : (f^{-1}(pt \times S^3 \times I); f^{-1}(\partial(pt \times S^3 \times I))) \to (pt \times S^3 \times I;\ \partial(pt \times S^3 \times I)).$$

(Of course f was made transverse along this submanifold). The obstruction $\theta_4(\bar{f})$ is a signature type obstruction and once more the existence of the manifold $|E_8|$ implies that

$$\theta_5 : Z_2 \oplus Z \to Z$$

when restricted to the Z-summand in $Z_2 \oplus Z$ is a bijection. But the exactness of the Wall–Sullivan sequence implies that $\mathcal{S}_{\mathrm{TOP}}(S^1 \times S^3) = 0$ which completes the proof of Proposition 2.

Remark 3. Note that Proposition 2 answers positively a question asked by W. C. Hsiang (see [9]); also compare Remark 13.

Now let us return to 3-manifolds which are S^2 bundles over S^1. Observe that there are only two such manifolds, namely $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$.

With the help of Proposition 2 we show the following:

PROPOSITION 4. *Any topological h-cobordism of S^3, $S^1 \times S^2$, $S^1 \tilde{\times} S^2$ to itself is homeomorphic to $S^3 \times I$, $S^1 \times S^2 \times I$, $S^1 \tilde{\times} S^2 \times I$.*

Proof. First consider the trivial S^2 bundle $S^1 \times S^2$. Let W be an h-cobordism of $S^1 \times S^2$ to itself. We form a manifold $\bar{W}$ by gluing copies of $S^1 \times D^3$ to the two ends of W. It follows that $\bar{W}$ is homotopy equivalent to $S^1 \times S^3$ and hence homeomorphic to $S^1 \times S^3$. Using a general position argument any two homotopic embeddings of $S^1 \cup S^1$ into $S^1 \times S^3$ are isotopic. (Note, that the topological general position above can be reduced to the the PL one by R. Miller's result (see [16])). But it is not difficult to see that this implies that W is homeomorphic to $S^1 \times S^2 \times I$.

The case of $S^1 \tilde{\times} S^2$ bundle can be treated similarly using a modification of Proposition 2. The case of S^3 follows directly from the analysis of Wall–Sullivan sequence.

As a consequence of Proposition 4 we have the following (comp. [7]):

COROLLARY 5. *A locally flat topological knot* $i: S^2 \to S^4$ *is trivial if and only if* $\pi_1(S^4 - i(S^2)) \approx Z$.

Proof. Let $i: S^2 \to S^4$ be a locally flat topological knot and let X be its complement, i.e. $X = S^4 - \mathring{N}(S^2)$, where $\mathring{N}(S^2)$ is the interior of a tubular neighborhood. The existence of such a neighborhood is guaranteed by [7]. Let K be a closed simple curve in the interior of X which links S^2 once and let $\mathring{N}(K)$ be its open tubular neighborhood. Now if $\pi_1(X) \approx Z$ then $X \simeq S^1$ (see [12]) and $W = X - \mathring{N}(K)$ is an h-cobordism between $\partial(K \times D^3) = S^1 \times S^2$ and $\partial(X) = S^1 \times S^2$. By Proposition 3 $W \approx_{\mathrm{TOP}} S^1 \times S^2 \times I$ and hence $X = S^1 \times D^3$ which means that $i: S^2 \to S^4$ is unknotted.

Now we show the failure of a 4-dimensional s-cobordism theorem, namely:

THEOREM 6. *There is a topological s-cobordism W between two copies of* $S^1 \times RP^2$ *which is not trivial. In fact W is stably nonsmoothable, i.e.* $W \times R$ *is nonsmoothable* (*rel boundary*).

$$\begin{array}{ccccccc}
 & \nearrow & G/PL & \xrightarrow{\psi'} & G/\mathrm{TOP} & \overset{\bar{k}}{\searrow} & \\
\mathrm{TOP}/PL & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & K(Z_2, 4) \\
 & \searrow & BPL & \xrightarrow{\psi} & B\mathrm{TOP} & \underset{k}{\nearrow} & \\
 & & \downarrow & & \downarrow & & \\
 & & BG & \xrightarrow{id} & BG & &
\end{array}$$

Proof. Consider the following sequence of fibrations which form the commutative diagram where $k: B\mathrm{TOP} \to K(Z_2, 4)$ is induced by the universal triangulation obstruction (see [10]) and $\bar{k} = k \circ i_2$.

To simplify notation let us put

$$(S^1 \times RP^2 \times I, \partial(S^1 \times RP^2 \times I)) = (Y, \partial Y).$$

From the above diagram we get the induced exact sequence

$$[Y, \partial Y; \mathrm{TOP}/PL, *] \to [Y, \partial Y; G/PL, *] \xrightarrow{\psi'_*} [Y, \partial Y; G/\mathrm{TOP}, *] \xrightarrow{\bar{k}_*} [Y, \partial Y; K(Z_2, 4)]$$

The 4-stage in the Postnikov decomposition of G/TOP is given by $K(Z_2, 2) \times K(Z, 4)$ and hence

$$[Y, \partial Y; G/\mathrm{TOP}, *] \approx H^2(Y, \partial Y; Z_2) \oplus H^4(Y, \partial Y; Z).$$

The 4-stage for G/PL after localization at 2 is given (see [31]) by:

$$K(Z_2, 2) \underset{\delta Sq^2}{\times} K(Z_{(2)}, 4)$$

which implies

$$[Y, \partial Y; G/\mathrm{TOP}, *]/\psi'_*[Y, \partial Y; G/PL, *] \approx \bar{k}_*[Y, \partial Y; G/\mathrm{TOP}, *] \approx$$
$$\approx \mathrm{red}\, H^4(Y, \partial Y; Z) + Sq^2 H^2(Y, \partial Y; Z_2),$$

where $\mathrm{red}: H^4(Y, \partial Y; Z) \to H^4(Y, \partial Y; Z_2)$ is the reduction of coefficients and

$$\mathrm{red}\, H^4(Y, \partial Y; Z) + Sq^2 H^2)Y, \partial Y; Z_2)$$

is a subgroup of $H^4(Y, \partial Y; Z_2)$.

Note that $H^4(Y, \partial Y; Z) \approx Z_2$. For, by the definition:

$$H^4(Y, \partial Y; Z) \approx H^4(S^1 \times RP^2 \times I, \partial(S^1 \times RP^2 \times I); Z)$$

and

$$H^4(S^1 \times RP^2 \times I, \partial(S^1 \times RP^2 \times I); Z) \approx H^4(\Sigma\,(S^1 \times RP^2); Z) \approx H^3(S^1 \times RP^2; Z).$$

The Universal Coefficient Theorem gives the exact sequence:

$$0 \to \mathrm{Ext}\,(H_2(S^1 \times RP^2; Z), Z) \to H^3(S^1 \times RP^2; Z) \to \mathrm{Hom}\,(H_3(S^1 \times RP^2; Z), Z) \to 0.$$

Now because $H_3(S^1 \times RP^2; Z) = 0$ and $H_2(S^1 \times RP^2; Z) \approx Z_2$ we obtain

$$H^3(S^1 \times RP^2; Z) \approx Z_2 \approx H^4(T, \partial Y; Z).$$

Also observe that red $(H^4(Y, \partial Y; Z)) \approx Z_2$.

To see it note that the exact sequence

$$0 \to Z \xrightarrow{\times 2} Z \xrightarrow{\text{red}} Z_2 \to 0$$

yields the exact cohomology sequence

$$\cdots \to H^4(Y, \partial Y; Z) \xrightarrow{\times 2} H^4(Y, \partial Y; Z) \xrightarrow{\text{red}} H^4(Y, \partial Y; Z_2) \xrightarrow{\beta} H^5(Y, \partial Y; Z) \to$$

where β is the Bockstein homomorphism. Because $H^5(Y, \partial Y; Z) = 0$, then we infer that

$$\text{red}: H^4(Y, \partial Y; Z) \to H^4(Y, \partial Y; Z_2)$$

is an isomorphism.

Let $g: (Y, \partial Y) \to (G/\text{TOP}, *)$ be a map which represents an element

$$(0, t) \in [Y, \partial Y; G/\text{TOP}, *] \approx H^2(Y, \partial Y; Z_2) \oplus H^4(Y, \partial Y; Z).$$

Let $f: (M, \partial M) \to (Y, \partial Y)$ be a normal map which corresponds to g. Then it is known (see [31]) that the surgery obstruction $\theta(f) := \theta_4(g) \in L_4^s(Z \times Z_2^-) \approx L_4^s(Z_2^-) \oplus L_3^h(Z_2^-) = L_4^s(Z_2^-) \approx Z_2$ which is given by the Kervaire–Arf invariant vanishes. (We refer to [25], assuming that $Wh(Z \times Z_2) = 0$, for the decomposition $L_4^s(Z \times Z_2^-) \approx L_4^s(Z_2^-) \oplus L_3^h(Z_2^-)$.) By the exactness of the Wall–Sullivan sequence

$$\cdots \to S_{\text{TOP}}(Y, \partial Y) \to [Y, \partial Y; G/\text{TOP}, *] \xrightarrow{\theta_4} L_4^s(Z_2^-)$$

we are allowed to take $f: (M, \partial M) \to (Y, \partial Y)$ to be a homotopy equivalence.

The element $(0, \bar{t}) = \bar{k}_*((0, t)) \in Sq^2 H^2(Y, \partial Y; Z_2) + \text{red}\, H^4(Y, \partial Y; Z)$ represents the obstruction to lift

$$g: (Y, \partial Y) \to (G/\text{TOP}, *)$$

through G/PL. But the commutativity of the diagram implies that $(0, \bar{t})$ is just

$$\begin{array}{ccc} G/\mathrm{TOP} & \xrightarrow{\bar{k}} & \\ \downarrow{\scriptstyle t_2} & & K(Z_2, 4) \\ B\mathrm{TOP} & \xrightarrow{k} & \end{array}$$

the difference of Kirby–Siebenmann's invariants

$$k(M) - k(Y).$$

Because $k(Y) = 0$ and $\operatorname{red} H^4(Y, \partial Y; Z) \approx Z_2$ then we obtain $k(M) \neq 0$ (as we can take $(0, \bar{t})$ to be nontrivial).

Now by the result of [11] or [21] $k(M)$ is the obstruction to a relative smoothing of $M \times R$. Therefore there exists an s-cobordism W between two copies of $S^1 \times RP^2$ which is stably nonsmoothable and hence it can not be a product.

What remains to show is the justification for the decomposition $L_4^s(Z \times Z_2^-) \approx L_4^s(Z_2^-) \oplus L_3^h(Z_2^-)$.

Fortunately there is no problem with this because the Whitehead group $Wh(Z \times Z_2)$ is trivial. To see it we first compute $K_1 Z[Z \times Z_2]$. Let us write $Z[Z \times Z_2]$ as $(Z[Z])[Z_2]$. The Milnor square (see [17]) yields the following cartesian square.

$$\begin{array}{ccc} (Z[Z])[Z_2] & \rightarrow & Z[Z] \\ \downarrow & & \downarrow \\ Z[Z] & \rightarrow & Z_2[Z] \end{array}$$

Consider the Mayer–Vietoris sequence associated with this diagram (see [17]).

$$\rightarrow K_2 Z_2[Z] \rightarrow K_1(Z[Z])[Z_2] \rightarrow K_1 Z[Z] \oplus K_1 Z[Z] \rightarrow K_1 Z_2[Z] \rightarrow \cdots.$$

Now (see [29], [20]) $K_2 Z_2[Z] \approx K_2 Z_2 \oplus K_1 Z_2$ and because $K_2 Z_2 = K_1 Z_2 = 0$ (see [17]) then $K_2 Z_2[Z] = 0$.

Also note that $K_1 Z[Z] \approx Z_2 \oplus Z$, i.e. it corresponds to the decomposition (see [1]):

$$K_1 Z[Z] \approx K_1 Z \oplus K_0 Z \approx Z_2 \oplus Z.$$

An analogous decomposition of $K_1 Z_2[Z]$ yields

$$K_1 Z_2[Z] \approx K_1 Z_2 \oplus K_0 Z \approx Z$$

Now $K_1(Z[Z])[Z_2]$ injects into $K_1Z[Z] \oplus K_1Z[Z]$ and in fact

$$K_1(Z[Z])[Z_2] \approx Z_2 \oplus Z_2 \oplus Z.$$

To compute the Whitehead group $Wh(Z \times Z_2)$ we should examine the group of units $((Z[Z])[Z_2])^{\times}$ in $(Z[Z])[Z_2]$. Clearly the trivial units $Z_2 \oplus Z_2 \oplus Z$ are in $((Z[Z])[Z_2])^{\times}$. On the other hand it is not difficult to see that these units generate the whole of $K_1(Z[Z])[Z]$. This of course implies that $Wh(Z \times Z_2) = 0$.

Note. The analysis of the Wall–Sullivan exact sequence in fact shows that a topological s-cobordism W between two copies of $S^1 \times RP^2$ is nontrivial if and only if W is stably nonsmoothable.

Remark 7. The false statement that $Wh(Z \times Z_2)$ can be nontrivial was made in [13]. However the result of [13] is not affected by this false claim.

It is worthwhile to observe that the method of the proof of Theorem 6 yields the following result.

PROPOSITION 8. *There is a closed manifold $\mathcal{P}$ which is homotopy equivalent but no homeomorphic to RP^4. In fact the manifold $\mathcal{P}$ is stably nonsmoothable.*

Proof. Consider the Wall–Sullivan exact sequence

$$0 \approx L_5^s(Z_2^-) \to S_{\text{TOP}}(RP^4) \xrightarrow{\tau} [RP^4; G/\text{TOP}] \xrightarrow{\theta_4} L_4^s(Z_2^-).$$

We have

$$[RP^4; G/\text{TOP}] \approx H^2(RP^4; Z_2) \oplus H^4(RP^4; Z) \approx Z_2 \oplus Z_2. \qquad (*)$$

The surgery obstruction map θ_4 is detected in this case by the Kervaire–Arf invariant (see [31]). Therefore for every element $g \in [RP^4; G/\text{TOP}]$ which is of the form $g = (0, t)$ (with respect to the decomposition $(*)$) one obtains

$$\theta_4(g) = \theta_4(0, t) = 0.$$

Also observe that the homomorphism

red: $H^4(RP^4; Z) \to H^4(RP^4; Z_2)$ is nontrivial.

This analogously as in the proof of Theorem 6 gives a homotopy equivalence $h: \mathcal{P} \to RP^4$ with the nontrivial Kirby–Siebenmann invariant $k(\mathcal{P})$.

Remark 9. A different construction of the fake RP^4 was provided by D. Ruberman in [24]. The above construction was given only to illustrate the technique used in the proof of Theorem 6.

Let us also observe that the proper 5-dimensional s-cobordism theorem together with Theorem 6 yields the following:

COROLLARY 10(a). *There exists a topological manifold* $M \approx_{\mathrm{TOP}} R^1 \times S^1 \times RP^2 \times I$ *such that every smoothing of* M *induces an exotic smoothing on its boundary. In particular, there exists an exotic smooth structure on* $R^1 \times S^1 \times RP^2$.

From this one can easily conclude the following compact version of Corollary 10(a).

COROLLARY 10(b). *There exists a topological manifold* $N \approx_{\mathrm{TOP}} S^1 \times S^1 \times RP^2 \times I$ *whose every smoothing induces an exotic smoothing on its boundary. Consequently, there exists an exotic smooth structure on* $S^1 \times S^1 \times RP^2$.

Now we show how to construct infinitely many examples of nontrivial 4-dimensional s-cobordisms.

PROPOSITION 11. *There exist infinitely many examples of nontrivial* 4-*dimensional* s-*cobordisms. In fact all these* s-*cobordisms are stably nonsmoothable.*

Proof. Let W be the h-cobordism between two copies of $S^1 \times RP^2$ as constructed in Theorem 6. Let M_n be the 3-manifold given by $M_n := \#_n S^1 \times S^2$, $n = 1, 2, 3, \ldots$ and let $\bar{W}_n = M_n \times I$ be a trivial h-cobordism. Form a new h-cobordism $(V_n; X_n, X_n)$ by taking the connected sum of W and $\bar{W}_n$ along an arc joining the two ends in W. Of course $X_n = (S^1 \times RP^2) \# (\#_n S^1 \times S^2)$.

We show that the Kirby–Siebenmann invariant $k(V_n)$ is nontrivial. For, consider the classifying map

$$f : V_n = W \# W_n \to (B\mathrm{TOP}, *)$$

for the stable tangent microbundle of V_n. This map factors up to homotopy as follows:

$$V_n = W \# \bar{W}_n \xrightarrow{q} W \vee \bar{W}_n \xrightarrow{f_1 \vee f_2} (B\mathrm{TOP}, *),$$

where q is the projection and f_1 (resp. f_2) classifies the stable microbundle of W (resp. W_n). (Note that the manifold $\bar{W}_n$ is parallelizable). Therefore any lift of f to $(BPL, *)$ is homotopic through lifts to one which is constant on $\bar{W}_n - pt \times I$. But any such lift defines a lift of f, to $(BPL, *)$ because

$$\bar{W}_n - pt \times I \simeq n(S^1 \times S^2 - pt) \times I \simeq n(S^1 \vee S^2)$$

and $\pi_i(\mathrm{TOP}/PL) = 0 \quad i = 0, 1, 2,$ (see [11]). Consequently, one gets for the Kirby–Siebenmann invariants:

$$k(V_n) = 0 \Rightarrow k(W) = 0.$$

This is equivalent to: $k(W) \neq 0 \Rightarrow k(V_n) \neq 0$ and hence the manifold V_n is stably nonsmoothable. Note that there is no problem with the Whitehead torsion for $(V_n; X_n, X_n)$ because (see for example [30])

$$Wh(\pi_1(V_n)) \approx Wh(\pi_1(S^1 \times RP^2) * \pi_1(X_n)) \approx Wh(Z \times Z_2) \oplus Wh(\underbrace{Z * \cdots * Z}_{n\text{-times}}) \approx 0.$$

Remark 12. The s-cobordism constructed in Theorem 6 is the only known example of a nontrivial 4-dimensional s-cobordism where the fundamental group is infinite and where the boundary 3-manifold is prime. The natural attempt (natural with respect to the restriction on fundamental groups) to construct other such examples would be to replace $S^1 \times RP^2$ by some RP^2 bundle over S^1. But there is only the trivial RP^2 bundle over S^1; the reason: every self-homeomorphism $f: RP^2 \to RP^2$ is isotopic to the identity (see [3]).

Remark 13. In [9] W. C. Hsiang has aked whether a manifold which is homotopy equivalent to $S^3 \times S^1$, $S^2 \times S^1 \times S^1 = S^2 \times T^2$, $S^1 \times S^1 \times S^1 \times S^1 = T^4$ is homeomorphic to the corresponding manifold. For T^4 and $S^3 \times S^1$ the answer is positive by [7] and our Proposition 2. To complete the picture we show the following:

PROPOSITION 14. *Every closed connected topological manifold homotopy equivalent to $S^2 \times T^2$ is homeomorphic to $S^2 \times T^2$.*

Sketch of the proof. It follows from the computations of $L_1^s(Z \times Z)$ and $L_0^s(Z \times Z)$ (see [25], [31]) that the set $S_{\mathrm{TOP}}(S^2 \times T^2)$ has two elements. We show that the nontrivial element in $S_{\mathrm{TOP}}(S^2 \times T^2)$ is realized by a self homotopy equivalence $h: S^2 \times T^2 \to S^2 \times T^2$ which is constructed as follows.

Let $D^4 \subset S^2 \times T^2$ be a small disk. Shrink ∂D^4 to a point to obtain a map $c: S^2 \times T^2 \to S^2 \times T^2 \vee S^4$. Let $\eta^2: S^4 \to S^2$ be an essential map and let $x: S^2 \to S^2 \times T^2$ be a map given by

$$x: S^2 = S^2 \times * \hookrightarrow S^2 \times T^2$$

The homotopy equivalence h is given as the composition

$$h := S^2 \times T^2 \xrightarrow{c} S^2 \times T^2 \vee S^4 \xrightarrow{id \vee \eta^2} S^2 \times T^2 \vee S^2 \xrightarrow{(id,x)} S^2 \times T^2.$$

To show that h is indeed a nontrivial element in $\mathcal{S}_{TOP}(S^2 \times T^2)$ it is enough to show that h is not homotopic to a homeomorphism. This can be done using the characteristic variety theorem (comp. [31] p. 237). Namely the nontriviality of h is detected by the characteristic variety $* \times T^2$ in $S^2 \times T^2$. For, one can assume that $h^{-1}(* \times T^2) = W \cup T^2$ with W framed in D^4. The splitting invariant of h is the Arf invariant of W. Now W is the preimage under $\eta^2: S^4 \to S^2$ of one point and hence the Arf invariant of W is equal to 1. Therefore h is not homotopic to a homeomorphism. Now because $\mathcal{S}_{TOP}(S^2 \times T^2)$ has only two elements and the nontrivial one is represented by $h: S^2 \times T^2 \to S^2 \times T^2$ then every closed manifold homotopy equivalent to $S^2 \times T^2$ must be homeomorphic to $S^2 \times T^2$. This completes the proof of our claim.

Now we give an application of our considerations to group actions on 4-manifolds.

It was observed by W. Meeks III and S. Yau in [14] that the Smith Conjecture is a special case of the following more general question.

Let F be a compact surface and let the group Z_k act on $M = F \times I$. Suppose that this action preserves the both ends $F \times \{0\}$ and $F \times \{1\}$. The question is:

Does this action preserve the product structure on M?

Let us recall that an action $\psi: Z_k \times F \times I \to F \times I$ on M preserves the product structure if ψ is conjugate to an action of γ, $\gamma: Z_k: F \times I \to F \times I$ such that

$$\gamma(g, (x, t)) = (\bar{\gamma}(g, t), t)$$

where $\bar{\gamma} = \gamma|_{F \times \{0\}}$ and $(x, t) \in F \times I$.

When $F = D^2$, this question is just the Smith Conjecture. Therefore the above question can be considered as a generalized Smith Conjecture. In [14] W. Meeks III and S. Yau has shown that this generalized Smith Conjecture remains true for some specific surfaces and group actions.

The positive solution of the general case was provided by W. Meeks III and P. Scott in [15].

Here we show that if the surface F is replaced by a 3-manifold then an analogous problem has a negative answer. It is worthwhile to note that our example is in some sense closely related to the original question of M. Meeks III and S. Yau. Namely, though our manifold is 3-dimensional it is given by $S^1 \times S^2$ with the trivial Z_2-action on the S^1 factor.

To be more precise we have the following.

COROLLARY 15. *There is a free involution on* $M = S^1 \times S^2 \times I$ *which preserves the ends and which does not preserve the product structure on* M.

Proof. Let W be an h-cobordism between two copies of $S^1 \times RP^2$ constructed in Theorem 5. The natural 2-fold covering induces a free action of Z_2 on a manifold $\tilde{W}$ which is a trivial h-cobordism $S^1 \times S^2 \times I$ by Proposition 3. Of course this action can not preserve the product structure.

Remark 16. Examples of end preserving exotic free actions of generalized quaternion groups on $S^3 \times I$ are contained in [2]. For a non-free actions of a finite cyclic groups Z_k analogous examples can be obtained by double puncturing the well known Giffen's examples of Z_k actions on S^4 with knotted fixed set (see Amer. J. Math. *88* (1966), 187–198).

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