Comment. Math. Helvetici 61 (1986) 193–202 0010-2571/86/020193-10$01.50 + 0.20/0

Cyclic homology of groups and the Bass conjecture

BENO ECKMANN

0. Introduction

0.1. The cyclic homology $HC_i(\mathbb{Q}G)$ of a group algebra $\mathbb{Q}G$ decomposes into a direct sum indexed by the conjugacy classes $[x]$ in G, as shown by Burghelea [7] (see Section 1.3 below). We will consider certain classes of groups of finite homology dimension over $\mathbb{Q}$, $hd_{\mathbb{Q}}G = n$, and show that for $i \geq n$ the terms in $HC_i(\mathbb{Q}G)$ corresponding to conjugacy classes $[x]$ of elements of infinite order vanish. The groups G with $hd_{\mathbb{Q}}G = n < \infty$ for which this will be done are

(a) Nilpotent groups G,
(b) Torsion-free solvable groups,
(c) Linear groups $G \subset GL_r(F)$ where F is a field of characteristic 0,
(d) Groups of cohomology dimension $cd_{\mathbb{Q}}G \leq 2$ (here $n \leq 2$).

We recall (Serre [10]) that if in (c) F is a number field and G finitely generated, then G is always of finite virtual cohomology dimension, whence $hd_{\mathbb{Q}}G = n < \infty$. The case (b) actually falls under (c), but we prefer to give a simple direct argument, cf. Remark 2.3′ below.

As an immediate consequence of that vanishing result it follows that the character maps from K-theory of $\mathbb{Q}G$ to $HC_*(\mathbb{Q}G)$, see Karoubi [8], have vanishing components in the summands indexed by $[x]$ with x of infinite order—for all the groups listed above.

In particular, the character map $Ch_0^0: K_0(\mathbb{Q}G) \to HC_0(\mathbb{Q}G)$ can easily be seen to be the "Hattori–Stallings rank" r_P of finitely generated projective $\mathbb{Q}G$-modules P (representing elements of $K_0(\mathbb{Q}G)$), see Section 3.2. For the groups above it thus follows that r_P is concentrated on the conjugacy classes $[x]$ of elements x of finite order; hence on [1] if G is torsion-free. This is a contribution towards the strong Bass conjecture [3, p. 156]. Note that the case (c) yields a weaker statement than Bass' result on linear groups [3, p. 156/57]; but our method is entirely different and stems from a result more general in another direction. The result establishing the Bass conjecture over $\mathbb{Q}G$ in the other cases seems to be new.

1. Cyclic homology of groups

1.1. Let G be a group, $\mathbb{Q}G$ its rational group algebra, and $HC_i(\mathbb{Q}G)$, $i \in \mathbb{Z}$, the cyclic homology of $\mathbb{Q}G$ in the sense of Connes; we will call it here in short the cyclic homology of G. It is related to the Hochschild homology $HH_i(\mathbb{Q}G)$ of $\mathbb{Q}G$, with bimodule-coefficients in $\mathbb{Q}G$ by left and right multiplication, through the "Connes–Gysin exact sequence"

$$\cdots \to HH_i(\mathbb{Q}G) \to HC_i(\mathbb{Q}G) \xrightarrow{S} HC_{i-2}(\mathbb{Q}G) \to HH_{i-1}(\mathbb{Q}G) \to \cdots. \tag{1.1}$$

It is a standard fact (see [9]) that Hochschild homology of $\mathbb{Q}G$ with bimodule coefficients can be expressed as homology of G with the same coefficient module turned into a right G-module; in the present case this is $\mathbb{Q}G$ with G-action by conjugation in G. It thus follows that for groups G of finite homology dimension $hd_{\mathbb{Q}}G = n < \infty$ over $\mathbb{Q}$ (i.e., for all $\mathbb{Q}G$-module coefficients) the cyclic homology of G stabilized above n:

$$HC_{n+2k}(\mathbb{Q}G) = HC_n(\mathbb{Q}G),$$
$$HC_{n+2k+1}(\mathbb{Q}G) = HC_{n+1}(\mathbb{Q}G)$$

for $k = 0, 1, 2, \ldots$.

1.2. The conjugation module $\mathbb{Q}G$ obviously decomposes into a direct sum of right $\mathbb{Q}G$-module indexed by the conjugacy classes $[x]$ of G; x is an arbitrary but fixed representative of $[x]$:

$$\mathbb{Q}G = \bigoplus_{[x]} D_x$$

where D_x is the $\mathbb{Q}$-module over the elements $x' \in [x]$ as basis, and with $\mathbb{Q}G$-action by $x' \mapsto y^{-1}x'y$, $y \in G$. If C_x is the centralizer of x in G, D_x is isomorphic to $\mathbb{Q}(G/C_x)$, the right $\mathbb{Q}G$-module generated by the right cosets modulo C_x; the isomorphism is given by $x' \mapsto C_x z$ where $z \in G$ is such that $z^{-1}xz = x'$. Thus

$$\mathbb{Q}G = \bigoplus_{[x]} \mathbb{Q}(G/C_x) = \bigoplus_{[x]} (\mathbb{Q} \otimes_{C_x} \mathbb{Q}G),$$

and finally

$$HH_i(\mathbb{Q}G) = \bigoplus_{[x]} H_i(G; \mathbb{Q} \otimes_{C_x} \mathbb{Q}G) = \bigoplus_{[x]} H_i(C_x; \mathbb{Q}) \tag{1.2}$$

with trivial G-module coefficients $\mathbb{Q}$.

Remark 1.1. $HH_0(\mathbb{Q}G)$ is the $\mathbb{Q}$-module having the conjugacy classes $[x]$ in G as basis. This can also be seen directly from the well-known fact that $HH_0(\mathbb{Q}G)$ is $\mathbb{Q}G/\{\lambda\mu - \mu\lambda\}$, where $\{\lambda\mu - \mu\lambda\}$ denotes the $\mathbb{Q}$-submodule generated by all $\lambda\mu - \mu\lambda$, $\lambda, \mu \in \mathbb{Q}G$; i.e., the $\mathbb{Q}$-submodule generated by all $xy - yx$, $x, y \in G$. We write $\overline{QG}$ for this factor-$\mathbb{Q}$-module of $\mathbb{Q}G$, $T: \mathbb{Q}G \to \overline{\mathbb{Q}G}$ for the canonical map, with $T(\lambda\mu) = T(\mu\lambda)$, $\lambda, \mu \in \mathbb{Q}G$.

1.3. A direct sum decomposition of $HC_i(\mathbb{Q}G)$, with terms indexed by the conjugacy classes $[x]$ in G, has been given by Burghelea [7] using topological (simplicial) constructions:

$$HC_i(\mathbb{Q}G) = \bigoplus_{[x]}{}' \, [H_*(C_x/\langle x\rangle; \mathbb{Q}) \otimes H_*(\mathbb{C}P^\infty; \mathbb{Q})]_i \oplus \bigoplus_{[x]}{}'' \, H_i(C_x/\langle x\rangle; \mathbb{Q}). \tag{1.3}$$

Here $\langle x\rangle$ denotes the cyclic subgroup of G generated by x, and $\oplus'$ is summation over all $[x]$ with finite $\langle x\rangle$, $\oplus''$ over all $[x]$ with infinite $\langle x\rangle$.

The methods of [7] also yield the $\oplus$-decomposition (1.2) of $HH_i(\mathbb{Q}G)$ and shows that the Connes–Gysin sequence (1.1) decomposes into exact sequences of the same type, one for each $[x]$ in G.

We will consider in Section 2 groups of finite dimension $hd_{\mathbb{Q}}G = n < \infty$ and show that, for certain classes of such groups, one has $\bigoplus''_{[x]} = 0$ in the stable value $HC_n(\mathbb{Q}G)$ and $HC_{n+1}(\mathbb{Q}G)$; in other words, *the stable value is concentrated on the conjugacy classes* $[x]$ *of elements* x *of finite order* – hence for torsion-free groups on the conjugacy class [1]. This will be done for the classes (a)–(d) listed in the introduction. Immediate consequences (Section 3) concern the character maps from K-theory of $\mathbb{Q}G$ to cyclic homology and, in particular, the Hattori–Stallings rank as mentioned in the introduction.

2. Groups of finite dimension

2.1. "Finite dimension" for groups G will refer here, unless otherwise specified, to the homology dimension $hd_{\mathbb{Q}}G$ over $\mathbb{Q}$, i.e., with respect to

$\mathbb{Q}G$-module coefficients. For any subgroup $S \subset G$, in particular for the centralizers C_x, we have $hd_{\mathbb{Q}}S \leq hd_{\mathbb{Q}}G = n < \infty$.

In our context we are thus interested in the homology of factor groups $G/\langle x \rangle$ where $hd_{\mathbb{Q}}G = n < \infty$ and x is a central element; actually in homology with trivial $\mathbb{Q}$-coefficients only, and in its vanishing above n. In other words, we are looking at $thd_{\mathbb{Q}}G/\langle x \rangle$, the *trivial homology dimension over* $\mathbb{Q}$; i.e., defined exactly as $hd_{\mathbb{Q}}$ but referring to trivial $\mathbb{Q}$-module coefficients only. One always has $thd_{\mathbb{Q}} \leq hd_{\mathbb{Q}}$. For that type of dimension we recall the following very simple but useful sum formula (Bieri [6]):

LEMMA 2.1. *Let U be a central subgroup of the group V, and $W = V/U$. If both $thd_{\mathbb{Q}}U$ and $thd_{\mathbb{Q}}W$ are finite then*

$$\mathrm{thd}_{\mathbb{Q}}V = thd_{\mathbb{Q}}U + thd_{\mathbb{Q}}W.$$

A further preliminary remark concerns the case where $\langle x \rangle$ is finite: then the spectral sequence

$$H_i(G/\langle x \rangle; H_j(\langle x \rangle; \mathbb{Q})) \Rightarrow H_{i+j}(G; \mathbb{Q})$$

shows that $thd_{\mathbb{Q}}G/\langle x \rangle = thd_{\mathbb{Q}}G$, hence $\leq n$. In all what follows we therefore restrict attention to central elements x of *infinite* order. In that case the spectral sequence does *not* imply that $thd_{\mathbb{Q}}G/\langle x \rangle$ is finite; however, if it *is* finite then the sum formula yields

$$thd_{\mathbb{Q}}G/\langle x \rangle \leq n - 1.$$

2.2. Nilpotent groups. We recall (Stammbach [11]) that if G is nilpotent then $hd_{\mathbb{Q}}G$ is equal to the Hirsch number hG (the sum of the torsion-free ranks of the factors of any normal series of G with Abelian factors); this holds, more generally, for solvable groups. We thus assume $hG = n < \infty$.

Let $x \in G$ be a central element of infinite order, S a finitely generated subgroup of $G/\langle x \rangle$, and T the preimage of S in G, $T/\langle x \rangle = S$. Since S is finitely generated nilpotent it is polycyclic, and therefore $hd_{\mathbb{Q}}S = hs$ is finite (equal to the number of infinite cyclic factors in a normal series with cyclic factors). The sum formula now yields

$$thd_{\mathbb{Q}}S = thd_{\mathbb{Q}}T - 1 \leq n - 1.$$

$G/\langle x \rangle$ is the direct limit of its finitely generated subgroups S; and since homology commutes with direct limits it follows that $thd_{\mathbb{Q}}G/\langle x \rangle$ is $\leq n - 1$:

THEOREM 2.2. *Let G be a nilpotent group of finite dimension* $hd_{\mathbb{Q}}G = n$. *Then one has for any central element* $x \in G$ *of infinite order*

$$H_i(G/\langle x\rangle; \mathbb{Q}) = 0 \quad \text{for} \quad i \geq n.$$

2.3. Torsion-free solvable groups. Let G be torsion-free solvable with $hd_{\mathbb{Q}}G = hG = n < \infty$ (or equivalently, solvable with $hd_{\mathbb{Z}}G < \infty$). We consider the Hirsch–Plotkin radical R of G, i.e., the maximal locally nilpotent normal subgroup of G. For any Abelian subgroup S of G the torsion-free rank $hS = hd_{\mathbb{Q}}S$ is $\leq n$. As G is torsion-free (actually a weaker condition would do) we can apply a theorem of Baer–Heineken [2] which tells that

(α) R is nilpotent

(β) G/R is finitely generated

(γ) G/R contains an Abelian subgroup A of finite index.

From (b) and (c) we infer that $hd_{\mathbb{Q}}G/R = hd_{\mathbb{Q}}A$ is finite, say $= m$. If $x \in G$ is central it must lie in R, and if it is of infinite order Theorem 2.2 tells that $thd_{\mathbb{Q}}R/\langle x\rangle = thd_{\mathbb{Q}}R - 1 \leq n - 1$. From the spectral sequence for $G/R \cong G/\langle x\rangle / R/\langle x\rangle$,

$$H_i(G/R; H_j(R/\langle x\rangle; \mathbb{Q})) \Rightarrow H_{i+j}(G/\langle x\rangle; \mathbb{Q})$$

we see that $H_k(G/\langle x\rangle; \mathbb{Q}) = 0$ for $k > m + n - 1$; i.e., $thd_{\mathbb{Q}}G/\langle x\rangle$ is finite and hence $\leqslant n - 1$.

THEOREM 2.3. *Let G be a torsion-free solvable group of finite dimension* $hd_{\mathbb{Q}}G = n < \infty$. *Then one has for any central element* $x \in G$ *of infinite order*

$$H_i(G/\langle x\rangle; \mathbb{Q}) = 0 \quad \text{for} \quad i \geq n.$$

Remark 2.3′. The groups G above admit faithful linear representations over $\mathbb{Q}$ (cf. [13], p. 25) and thus are included in 2.4 below. However, the proof of the linear embedding starts precisely from the structure properties (α), (β), (γ); thus the simple direct argument seems preferable.

2.4. Linear groups. We now consider a linear group $G \subset GL_r(F)$, where F is a field of characteristic 0, with $hd_{\mathbb{Q}}G = n < \infty$. Let Z be the center of G, $\pi: G \to G/Z$ the canonical map. Since Z is closed in the Zariski topology, G/Z is again a linear group over the same field F. We are going to apply the R. Alperin–Shalen criterion [1] to finitely generated subgroups S of G/Z, in order to prove that they are virtually of finite cohomology dimension.

For this let first U be a finitely generated unipotent subgroup of G/Z and put $\pi^{-1}U = V$, $V/Z = U$. Then U being torsion-free finitely generated nilpotent (polycyclic), $hd_{\mathbb{Q}}U = thd_{\mathbb{Q}}U = hU$ (Hirsch number, see 2.2). By Lemma 2.1,

$$thd_{\mathbb{Q}}Z + thd_{\mathbb{Q}}U = thd_{\mathbb{Q}}V.$$

Now $hd_{\mathbb{Q}}Z \leq n$ and $hd_{\mathbb{Q}}V \leq n$; let $m = thd_{\mathbb{Q}}Z$. Thus $thd_{\mathbb{Q}}U \leq n - m$; i.e., we get a uniform bound for all finitely generated unipotent subgroups of G/Z, the Hirsch numbers hU being $\leq n - m$.

If S is any finitely generated subgroup of G/Z, it follows by [1] that its virtual cohomology dimension, and hence $hd_{\mathbb{Q}}S$, is finite. Putting $\pi^{-1}S = T \subset G$, $T/Z = S$, Lemma 2.1 tells that

$$thd_{\mathbb{Q}}Z + thd_{\mathbb{Q}}S = thd_{\mathbb{Q}}T \leq n,$$

and thus $thd_{\mathbb{Q}}S \leq n - m$. The direct limit argument then yields $thd_{\mathbb{Q}}G/Z \leq n - m$.

THEOREM 2.4. *Let G be a linear group of finite dimension $hd_{\mathbb{Q}}G = n$, over a field of characteristic* 0, *Z is center and $thd_{\mathbb{Q}}Z = m$. Then*

$$H_i(G/Z; \mathbb{Q}) = 0 \quad \textit{for} \quad i > n - m.$$

We are looking for a similar result concerning $G/\langle x \rangle$ where x is a central element of infinite order. Since $G/Z = G/\langle x \rangle / Z/\langle x \rangle$ we can apply the spectral sequence (with trivial $\mathbb{Q}$-module coefficients) together with Theorem 2.2 on $Z/\langle x \rangle$. This immediately yields $H_i(G/\langle x \rangle; \mathbb{Q}) = 0$ for $i > (n - m) + (m - 1)$:

THEOREM 2.4'. *Let G be as in Theorem* 2.4. *and x a central element of G of infinite order. Then*

$$H_i(G/\langle x \rangle; \mathbb{Q}) = 0 \quad \textit{for} \quad i \geq n.$$

2.5. Groups of cohomology dimension ≤2. We write as usual $cd_{\mathbb{Q}}G$ for the cohomology dimension of the group G over $\mathbb{Q}$; i.e., with respect to all $\mathbb{Q}G$-module coefficients. The assumption $cd_{\mathbb{Q}}G \leq 2$ includes all groups which are virtually of cohomology dimension 2 (over $\mathbb{Z}$), but is more general; it of course implies $hd_{\mathbb{Q}}G \leq 2$.

The case $cd_{\mathbb{Q}} \leq 1$ is easily dealt with: it means that G contains a free (normal) subgroup F of finite index. If x is a central element of infinite order then $\langle x \rangle \cap F \neq 1$; hence F having non-trivial center must be cyclic $= \langle c \rangle$. Then both

$G/\langle x, F\rangle$ and $\langle x, F\rangle/\langle x\rangle \cong F/F \cap \langle x\rangle$ are finite, and so is $G/\langle x\rangle$. Thus $hd_{\mathbb{Q}}G/\langle x\rangle = 0$. We thus restrict attention to the case $cd_{\mathbb{Q}}G = 2$.

THEOREM 2.5. *Let G be a group with $cd_{\mathbb{Q}}G = 2$, and x a central element of infinite order. Then*

$$H_i(G/\langle x\rangle; \mathbb{Q}) = 0 \quad \textit{for} \quad i \geq 2.$$

This is a consequence of various known facts concerning group (co-) homology, finiteness properties, and structure theorems (cf. Bieri [4], Corollary 8.7, and [5]), We give a short outline of the proof for our somewhat more special situation. It suffices to prove the claim for finitely generated G: If the subgroup $S \subset G/\langle x\rangle$ is finitely generated so is its preimage $T \subset G$, and from the result for $S = T/\langle x\rangle$ the direct limit argument yields the claim for $G/\langle x\rangle$.

As a first step one shows that G is of type $FP_{\mathbb{Q}}$; i.e., that there exists a finitely generated projective resolution over $\mathbb{Q}G$ (of length 2 since $cd_{\mathbb{Q}}G = 2$)

$$0 \to P_2 \to P_1 \to P_0 \to \mathbb{Q} \to 0.$$

To prove this we use R. Strebel's finiteness criterion [11]: G is of type $FP_{\mathbb{Q}}$ if and only if $cd_{\mathbb{Q}}G$ is finite and the canonical map $H^i(G; \oplus\, \mathbb{Q}G) \to \oplus\, H^i(G; \mathbb{Q}G)$ is an isomorphism for all i and all direct sums $\oplus$. In our case the spectral sequence for $G/\langle x\rangle$ yields

$$H^1(G; \mathbb{Q}G) \cong H^0(G/\langle x\rangle; H^1(\langle x\rangle; \mathbb{Q}G))$$

and

$$H^2(G; \mathbb{Q}G) \cong H^1(G/\langle x\rangle; H^1(\langle x\rangle; \mathbb{Q}G))$$

(the action of G on $H^1(\langle x\rangle; \mathbb{Q}G)$ is induced by the trivial conjugation of G on $\langle x\rangle$). For the infinite cyclic group $\langle x\rangle$ one has $H^i(\langle x\rangle; \mathbb{Q}\langle x\rangle) = 0$ for $i \neq 1$, and $H^1(\langle x\rangle; \mathbb{Q}\langle x\rangle) = \mathbb{Q}$ with trivial action; and $H^i(\langle x\rangle; \mathbb{Q}G) = 0$ for $i \neq 1$, $H^1(\langle x\rangle; \mathbb{Q}G) = \mathbb{Q} \otimes_{\mathbb{Q}\langle x\rangle} \mathbb{Q}G \cong \mathbb{Q}(G/\langle x\rangle)$.

For $A = \oplus \mathbb{Q}G$ we get $H^1(\langle x\rangle; A) = \oplus H^1(\langle x\rangle; \mathbb{Q}G)$, and since $G/\langle x\rangle$ is finitely generated, we see that G fulfills the Strebel criterion for $i = 1$ and 2; in dimensions $i \neq 1, 2$ this is trivially the case since $H^i(G; \oplus\, \mathbb{Q}G) = 0 = \oplus\, H^i(G; \mathbb{Q}G)$. Thus G is of type $FP_{\mathbb{Q}}$.

As a second step one draws more conclusions from the above formula for $H^2(G; \mathbb{Q}G)$. We note that

$$H^2(G; \mathbb{Q}G) = H^1(G/\langle x\rangle; \mathbb{Q}(G/\langle x\rangle)).$$

As G is of type $FP_{\mathbb{Q}}$ with $cdG = 2$, $H^2(G;\mathbb{Q}G)$ is $\neq 0$ and finitely generated as a right $\mathbb{Q}G$-module, and so is $H^1(G/\langle x\rangle;\mathbb{Q}(G/\langle x\rangle))$ over $\mathbb{Q}(G/\langle x\rangle)$. This implies that $G/\langle x\rangle$ has more than one end and is *accessible*; in other words, $G/\langle x\rangle$ is the fundamental group of a finite graph of groups with finite edge groups, and with vertex groups V satisfying $H^1(V;\mathbb{Q}V) = 0$ (1 or 0 ends). V is finitely generated, and so is its preimage W in G, $W/\langle x\rangle = V$. As before we get $H^2(W;\mathbb{Q}W) = H^1(V;\mathbb{Q}V)$; but now this is $=0$, whence $cd_{\mathbb{Q}}W = 1$. The above formula for $H^1(G;\mathbb{Q}G)$ applied to W and to $W/\langle x\rangle = V$ yields $H^1(W;\mathbb{Q}W) \cong H^0(V;\mathbb{Q}V) \neq 0$. This implies that V is finite.

We thus have proved that $G/\langle x\rangle$ is the fundamental group of a finite graph of finite edge *and* vertex groups. Such a group is well-known to contain a (normal) free subgroup of finite index; from the corresponding spectral sequence we obtain the required result

$$H_i(G/\langle x\rangle:\mathbb{Q}) = 0$$

for $i \geq 2$.

2.6. From Theorems 2.2, 2.3, 2.4′, 2.5 we immediately obtain the result claimed in 1.3 for cyclic group homology:

COROLLARY 2.6. *Let G be a group with $hd_{\mathbb{Q}}G = n < \infty$ and belonging to one of the classes (a), (b), (c), or (d), $n = 2$ in the case (d). Then for $i \geq n$ the cyclic homology $HC_i(\mathbb{Q}G)$ vanishes on the conjugacy classes of elements of infinite order.*

3. Cyclic homology characters of $\mathbb{Q}G$

3.1. The Connes character Ch_0^l of $\mathbb{Q}G$ (cf. Karoubi [8]) is a homomorphism of $K_0(\mathbb{Q}G)$ to $HC_{2l}(\mathbb{Q}G)$, $l = 0, 1, 2, \ldots$; we will write Ch^l for Ch_0^l since we will not consider here the higher characters Ch_i^l (see however Remark 3.2). The Ch^l are compatible with the map S in the Connes–Gysin sequence (1.1), i.e.,

$$\begin{array}{ccc} & & HC_{2l}(\mathbb{Q}G) \\ & \overset{Ch^l}{\nearrow} & \big\downarrow S \\ K_0(\mathbb{Q}G) & \xrightarrow[Ch^{l-1}]{} & HC_{2l-2}(\mathbb{Q}G) \end{array}$$

is commutative. Corollary 2.6 immediately yields

THEOREM 3.1. *Let G be a group of finite dimension $hd_{\mathbb{Q}}G = n$ belonging to one of the classes (a), (b), (c) or (d). Then the characters Ch^l, $l = 0, 1, 2, \ldots$ all have 0-components in the summands $\bigoplus''_{[x]}$ corresponding to elements x of infinite order. In particular, if G is torsion-free, the Ch^l are concentrated on the [1]-summand, i.e, lie in $[H_*(G; \mathbb{Q}) \otimes H_*(\mathbb{C}P^\infty; \mathbb{Q})]_{2l}$.*

Remark 3.2. A similar result holds, of course, for the higher characters $Ch^l_i : K_i(\mathbb{Q}G) \to HC_{2l+i}(\mathbb{Q}G)$.

3.2. A look at the definitions shows that $Ch^0 : K_0(\mathbb{Q}G) \to HC_0(\mathbb{Q}_G)$ is the same as the Hattori–Stallings rank, as follows.

By (1.2) HC_0 is isomorphic to Hochschild homology HH_0. For any $\mathbb{Q}$-algebra Λ the latter, with Λ as bimodule by left- and right-multiplication for coefficients, $HH_0(\Lambda)$ is well-known to be $\Lambda/\{\lambda\mu - \mu\lambda\}$, where $\{\lambda\mu - \mu\lambda\}$ is the $\mathbb{Q}$-sub-module generated by all $\lambda\mu - \mu\lambda$, $\lambda, \mu \in \Lambda$. In Remark 1.1 we have written $\overline{\mathbb{Q}G}$ for $HH_0(\mathbb{Q}G)$ and $T : \mathbb{Q}G \to \overline{\mathbb{Q}G}$ for the canonical map. Similarly, for the matrix algebra $\mathbb{M}_k(\mathbb{Q}G)$, we have $HH_0(\mathbb{M}_k(\mathbb{Q}G)) = \overline{\mathbb{M}_k(\mathbb{Q}G)}$ with $T : \mathbb{M}_k(\mathbb{Q}G) \to \overline{\mathbb{M}_k(\mathbb{Q}G)}$. The trace of matrices $\mathrm{tr} : \mathbb{M}_k(\mathbb{Q}G) \to \mathbb{Q}G$ induces an isomorphism $\overline{\mathrm{tr}} : \overline{\mathbb{M}_k(\mathbb{Q}G)} \to \overline{\mathbb{Q}G}$, and clearly $T \circ \mathrm{tr} = \overline{\mathrm{tr}} \circ T : \mathbb{M}_k(\mathbb{Q}G) \to \overline{\mathbb{Q}G}$.

Now Ch^0 is defined, on a finitely generated projective $\mathbb{Q}G$-module P representing an element of $K_0(\mathbb{Q}G)$, as follows: Let p be an idempotent matrix $\in \mathbb{M}_k(\mathbb{Q}G)$, for suitable k, describing P as a direct summand of a free $\mathbb{Q}G$-module M, and put $Ch^0 p = \overline{\mathrm{tr}} \circ T(p) \in \overline{\mathbb{Q}G}$, i.e., $= T \circ \mathrm{tr}\,(p)$. This is precisely the definition of the Hattori–Stallings rank $r_P \in \overline{\mathbb{Q}G}$, independent of choices and of bases in M. We recall that $\overline{\mathbb{Q}G}$ is the $\mathbb{Q}$-module having the conjugacy classes $[x]$ as basis.

THEOREM 3.3. *For the groups G of finite dimension $hd_{\mathbb{Q}}G$ belonging to one of the classes* (a), (b), (c), (d), *the Hattori–Stallings rank r_P of a finitely generated projective $\mathbb{Q}G$-module P vanishes on the conjugacy classes of elements of infinite order.*

Remark 3.4. The vanishing of the character map $Ch^l : K_0(\mathbb{Q}G) \to HC_{2l}(\mathbb{Q}G)$ on the conjugacy classes of elements of infinite order, in particular of the Hattori–Stallings rank Ch^0, would of course follow from properties much weaker than those established in Section 2 for certain classes of groups. Indeed it suffices that, for a group G under consideration, some iteration $S^k : HC_{2l+2k}(\mathbb{Q}G) \to HC_{2l}(\mathbb{Q}G)$, $1 \leq k \leq \infty$, of S in (1.1) is zero on the conjugacy classes of elements of infinite order; $k = \infty$ refers to the inverse limit. It has been conjectured, for example, that this is the case for $k = \infty$ and for all groups having a finite Eilenberg–MacLane complex, cf. [7].

Note Added in Proof. The proof of Theorem 2.3, without assuming G to be torsion-free (and hence also of Theorem 2.2), becomes much simpler if one uses the fact that the Hirsch number of a factor group of G is less or equal to that of G, combined with Stammbach's theorem [11].

REFERENCES

[1] R. ALPERIN and P. SHALEN, *Linear groups of finite cohomological dimension,* Bull. Am. Math. Soc. (New Series) *4*, 339–341; Inventiones Math. *66*, 89–98 (1982).

[2] R. BAER and H. HEINEKEN, *Radical groups of finite Abelian subgroup rank,* Illinois J. of Math. *16*, 533–586 (1972).

[3] H. BASS, *Euler characteristics and characters of discrete groups,* Inventiones Math. *35*, 155–196 (1976).

[4] R. BIERI, *Homological dimension of discrete groups,* Queen Mary College Lecture Notes (1976).

[5] R. BIERI, *On groups of cohomology dimension 2,* in Topology and Algebra (Enseignement Math. 1978), 55–62.

[6] R. BIERI, *A connection between the integral homology and the centre of a rational linear group,* Math. Z. *170*, 263–266 (1980).

[7] D. BURGHELEA, *The cyclic homology of the group rings,* Comment. Math. Helv. *60*, 354–365 (1985).

[8] M. KAROUBI, *Homologie cyclique et K-théorie algébrique,* C.R. Acad. Sc. Paris, *297* S.I., 447–450 (1983).

[9] S. MACLANE, Homology (Springer Verlag Berlin 1975), Chapt. X.

[10] J.-P. SERRE, *Cohomologie des groupes discrets,* Annals of Math. Studies 70 (Princeton University Press 1971).

[11] U. STAMMBACH, *On the weak homological dimension of the group algebra of solvable groups,* J. London Math. Soc. (2) *2*, 567–570 (1970).

[12] R. STREBEL: *A homological finiteness criterion,* Math. Z. *151*, 263–275 (1976).

[13] B. A. F. WEHRFRITZ, *Infinite linear groups* (Springer Verlag Berlin 1973).

Mathematics
ETH-Zentrum
CH-8092 Zürich/Switzerland

Received November 28, 1985