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On the volume of a unit vector field on the three-sphere

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A unit vector field on a compact Riemannian manifold M can be pictured as a cross-section, and hence submanifold, of the unit tangent bundle T_1M. We define the *volume of the vector field* to be the volume of this submanifold, measured in the natural Riemannian metric which T_1M inherits from M. It can be expressed by the formula

$$\text{vol } V = \int_M \sqrt{\det (I + (\nabla V)(\nabla V)^t)}\, d \text{ vol}_M,$$

in which we view the covariant derivative ∇V as a linear transformation of the tangent space TM_x to itself.

One hopes that the "visually best organized" unit vector fields on M are rewarded with minimum possible volume. For example, it is clear that on the flat torus, the unit vector fields of minimum volume are precisely those of constant slope. But on the round three-sphere S^3, the story becomes more involved.

Consider the Hopf fibration H of S^3, whose fibres are the unit circles on the complex lines in $R^4 = C^2$. Any fibration congruent to this is also called a Hopf fibration, and a unit vector field V_H tangent to the fibres will be called a *Hopf vector field.* It is natural to regard these as visually the best organized unit vector fields on S^3. We will prove

THEOREM. *The unit vector fields of minimum volume on S^3 are precisely the Hopf vector fields, and no others.*

The proof is by the method of "calibrated geometries" of Federer [F] and Harvey–Lawson [H–L], and is a one-time-deal which fails on the 5-sphere.

To carry out the argument, we will find a smooth closed 3-form μ on the unit tangent bundle T_1S^3, such that

$$\mu(u \wedge v \wedge w) \leq \text{vol}\,(u \wedge v \wedge w), \qquad (*)$$

with equality holding for any properly oriented tangent 3-plane to a Hopf vector field V_H, viewed as a submanifold of T_1S^3.

It will follow immediately that the Hopf vector fields are absolutely volume minimizing in their homology classes in T_1S^3. For if M^3 is a 3-manifold in the same homology class as V_H, then

$$\operatorname{vol} V_H = \int_{V_H} \mu = \int_{M^3} \mu \leq \operatorname{vol} M^3,$$

by equality in $(*)$, Stokes' theorem, and inequality in $(*)$, respectively. If V is another unit vector field on S^3, then it is easy to see that it is in the same homology class as V_H when viewed as a 3-dimensional submanifold of T_1S^3, since the projection $T_1S^3 \to S^3$ is an isomorphism on 3-dimensional homology. Hence $\operatorname{vol} V_H \leq \operatorname{vol} V$, so the Hopf vector fields on S^3 minimize volume.

Call an oriented 3-dimensional submanifold of T_1S^3 a *μ-submanifold* if the equality $\mu(u \wedge v \wedge w) = \operatorname{vol}(u \wedge v \wedge w)$ holds for each of its tangent planes. By examining the 3-planes for which this equality holds, we will find other μ-submanifolds besides the V_H. But they lie in other homology classes in T_1S^3 and hence do not come from vector fields. Furthermore, they all have volumes > $\operatorname{vol} V_H$. Since the μ-submanifolds are the *only* volume minimizing submanifolds in their homology classes, it will follow that the Hopf vector fields are the only volume minimizing unit vector fields on S^3, completing the proof of the theorem.

The family $\{V_H\}$ of Hopf vector fields is invariant under the group of isometries of the unit tangent bundle T_1S^3. Hence if there is any form μ on T_1S^3 which "calibrates" the Hopf vector fields, as above, then we can average it over the group and obtain an isometry-invariant form which does the same. Hence there is no loss in restricting our search for μ to the isometry-invariant forms. The advantage is that such forms are explicitly calculable. It turns out that there is, up to constant multiple, just one isometry-invariant closed 3-form, and it does the job.

The drawback to using the method of calibrated geometries for this problem is that we must prove a little more than we want: the V_H have minimum volume among all 3-manifolds in the same homology class in T_1S^3 whether or not these 3-manifolds come from unit vector fields on S^3. As a result, the *method* will fail on the five-sphere S^5, because there is a 5-manifold in T_1S^5 in the same homology class as $2V_H$, but with less volume. The corresponding isometry invariant closed 5-form μ on T_1S^5 provides a calibrated geometry which distinguishes these submanifolds instead of the Hopf vector fields. And likewise on S^7, S^9, $S^{11}, \ldots$.

Whether the theorem itself remains true on these higher dimensional spheres, we do not know. We can, however, use the method of calibrated geometries to see a little in this direction.

If V is a parallel vector field on a compact Riemannian manifold M, then vol V = vol M. It is natural to ask: if M admits no parallel vector fields, is vol V bounded away from vol M for any unit vector field V on M? We will observe that

$$\text{vol } V \geq 2 \text{ vol sphere}$$

for any unit vector field on a unit sphere. By contrast,

$$\text{vol } V_H = 2^n \text{ vol } S^{2n+1},$$

so that starting on S^5 the above inequality is much weaker than the expected one. Nevertheless, this inequality reports that all unit vector fields on an odd-dimensional round sphere fail to be parallel by at least a certain amount.

When trying to show that nicely organized submanifolds minimize volume in their homology classes, it is good to keep in mind the following simple example, which shows that higher dimensions can frustrate the attempt.

The diagonal in $S^1 \times S^1$ has length equal to $\sqrt{2}$ times that of S^1, and certainly minimizes length in its homology class. The diagional in $S^2 \times S^2$ has linear dimensions multiplied by $\sqrt{2}$, and hence

$$\text{area diag } (S^2 \times S^2) = 2 \text{ area } S^2.$$

The diagonal still minimizes area in its homology class, but now the union

$$S^2 \times \text{point} \cup \text{point} \times S^2,$$

which lies in the same homology class, has the same area. Moving up one more dimension, we get

$$\text{vol diag } (S^3 \times S^3) = 2\sqrt{2} \text{ vol } S^3,$$

and now it is

$$S^3 \times \text{point} \cup \text{point} \times S^3,$$

and no longer the diagonal, which minimizes volume in its homology class.

Exactly this phenomenon is at work in the following circumstance. Define the *volume of a map* $f: M \to N$ between Riemannian manifolds to be the volume of its graph, considered as a submanifold of $M \times N$. It follows from the work of Walter Wei [W] that the Hopf map $h: S^3 \to S^2$ does *not* have minimum volume in its homotopy class. Indeed, we will derive in section 4 a general inequality for the volume of a fibre bundle map over a surface, and use it to display a large family of mutually homotopic maps from $S^3 \to S^2$, amongst which the Hopf map has *maximum* volume.

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1. Finding the 3-form μ

We divide the proof of the main theorem into two parts. In this section we find the closed invariant 3-form μ on the unit tangent bundle T_1S^3. In the next we will complete the argument by finding the 3-dimensional submanifolds calibrated by μ and noting that the Hopf vector fields, and no others, have minimum volume among them.

We begin by summarizing the geometry of the situation. The points of the unit tangent bundle T_1S^3 may be regarded as pairs (x, y) of orthogonal unit vectors from S^3. The same is true for the Stiefel manifold V_2R^4 of orthonormal two-frames (x, y) in R^4. Hence as sets, these five-dimensional manifolds are identical. As topological spaces, they are homeomorphic to $S^3 \times S^2$, since the 3-sphere is parallelizable.

The natural Riemannian metric on T_1S^3, defined in terms of covariant derivatives of vector fields, is the same as the one it inherits as the homogeneous space $SO(4)/SO(2)$. The natural Riemannian metric on V_2R^4 is the one it inherits as a subspace of R^8. These two metrics are not identical; we will compare them in a moment. But neither one is the product metric on $S^3 \times S^2$.

The *geodesic flow* g_t on the unit tangent bundle T_1S^3 is defined by

$$g_t(x, y) = (x \cos t + y \sin t,\ -x \sin t + y \cos t).$$

It is an $SO(2)$ action by isometries in either metric.

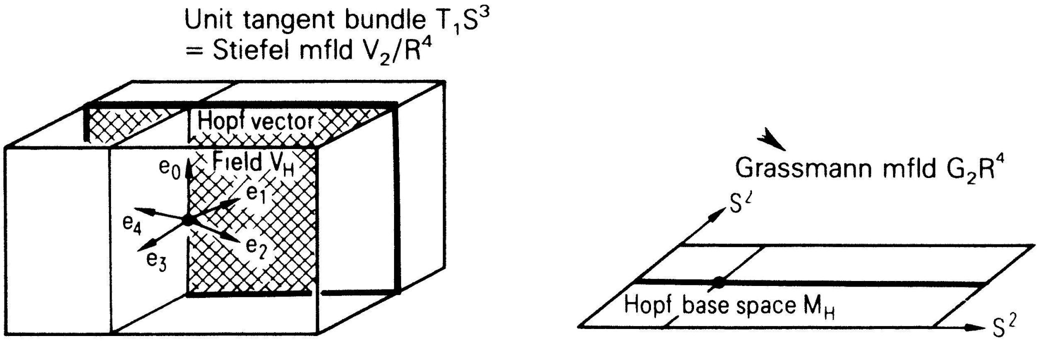
One passes from the natural Riemannian metric on T_1S^3 to that on V_2R^4 by multiplying lengths in the direction of the geodesic flow by $\sqrt{2}$, with no change in directions orthogonal to this. Our main theorem is true regardless of which of these two metrics is used to measure the volume of vector fields.

The Stiefel manifold V_2R^4 sits as a circle bundle over the Grassmann manifold G_2R^4 of oriented two-planes in four-space, with each orthonormal two-frame (x, y) sitting over the oriented two-plane $x \wedge y$ which it spans. The projection map is a Riemannian submersion, that is, its differential preserves lengths of tangent vectors orthogonal to fibres. The Grassmann manifold G_2R^4 is isometric to $S^2 \times S^2$, with each factor a round two-sphere of radius $1/\sqrt{2}$.

In the Grassmann manifold, the base spaces M_H of the various Hopf fibrations H of S^3 appear as $S^2 \times \text{point}$ and as $\text{point} \times S^2$. There is just one of each kind passing through any given point of the Grassmann manifold. See [G–W] for details.

Up in the Stiefel manifold, the Hopf vector fields V_H appear as totally geodesic round three-spheres of radius $\sqrt{2}$, sitting over the base spaces M_H in the Grassmann manifold.

We try to summarize much of this information in the following figure, which also includes an orthonormal set of tangent vectors at one point, to be used in a moment.



Now we set about finding our isometry-invariant closed 3-form μ on the unit tangent bundle. Each isometry $g: S^3 \to S^3$ has an induced action on $T_1S^3 = V_2R^4$,

$$(x, y) \to (g(x), g(y)),$$

which is an isometry in either metric. Such isometries preserve both the S^2 fibres of the unit tangent bundle and the S^1 fibres of the Stiefel bundle. It will be sufficient for our purposes to restrict attention to those g which are orientation preserving. So far this gives us an $SO(4)$ action.

In addition, the circle group $SO(2)$ acts on $T_1S^3 = V_2R^4$ by the geodesic flow, again isometries in either metric. Such isometries preserve the S^1 fibres of the Stiefel bundle, but do *not* preserve the S^2 fibres of the unit tangent bundle.

Since these two actions commute, we get an action of

$$G = SO(4) \times SO(2)$$

on $T_1S^3 = V_2R^4$ by isometries in either metric. G is simply the identity component of the full isometry group.

We next find the G-invariant differential forms on $T_1S^3 = V_2R^4$.

Since G acts transitively, we simply calculate those linear forms on the tangent space to $T_1S^3 = V_2R^4$ at a single point which are invariant under the action of the isotropy subgroup of G. This isotropy subgroup must be two-dimensional, since G is seven-dimensional and $T_1S^3 = V_2R^4$ is five-dimensional. In fact, it is isomorphic to $SO(2) \times SO(2)$, and operates by independently spinning the e_1e_2-plane and the e_3e_4-plane, while keeping the e_0-axis fixed. See figure above.

By abuse of language, we use the symbols e_i to denote both tangent vectors and dual one-forms. We easily get the following table.

Table of invariant forms and their exterior derivatives

Dimension 1	e_0	$\xrightarrow{d}$	$2(e_1 \wedge e_2 + e_3 \wedge e_4)$
Dimension 2	$e_1 \wedge e_2$	$\xrightarrow{d}$	0
	$e_3 \wedge e_4$	$\xrightarrow{d}$	0
Dimension 3	$e_0 \wedge e_1 \wedge e_2$	$\xrightarrow{d}$	$2\,e_1 \wedge e_2 \wedge e_3 \wedge e_4$
	$e_0 \wedge e_3 \wedge e_4$	$\xrightarrow{d}$	$2\,e_1 \wedge e_2 \wedge e_3 \wedge e_4$
Dimension 4	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$\xrightarrow{d}$	0
Dimension 5	$e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$\xrightarrow{d}$	0

Note that the invariant one-form e_0 represents inner product with a unit vector tangent to the Stiefel fibres, and is hence the *connection form* v of the Stiefel bundle.

Note that the invariant forms on $T_1S^3 = V_2R^4$ which occur in dimensions two and four are missing the e_0-factor. They represent the pullbacks to the Stiefel manifold of the corresponding invariant forms down on the Grassmann manifold G_2R^4. Down there, $e_1 \wedge e_2$ and $e_3 \wedge e_4$ are the volume forms of $S^2 \times$ point and point $\times S^2$, respectively, while $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ is the volume form of $S^2 \times S^2 = G_2R^4$.

Embed the Grassmann manifold G_2R^4 in CP^3 in the usual way by sending $x \wedge y$, where x and y are orthonormal, to the complex line through $x + iy$ in C^4. The image is the complex hyperquadric

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0,$$

where $z = x + iy$. In this way the Grassmann manifold inherits the complex structure J and the corresponding Kähler 2-form ω from CP^3. It is easy to check that

$$J(e_1) = e_2 \quad \text{and} \quad J(e_3) = e_4.$$

Hence

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4$$

is the Kähler 2-form on the Grassmann manifold.

Refer again to the above table and note that the even-dimensional forms, which are pulled back from the Grassmann manifold, are already closed. This happens because the Grassmann manifold is a symmetric space, and hence every invariant form is closed. By contrast, the invariant forms on the Stiefel manifold in dimensions one and three are not all closed. But clearly the cohomology computed from the invariant forms is the same as the deRham cohomology.

Given the preceding table of G-invariant forms and their derivatives on $T_1S^3 = V_2R^4$, we naturally choose

$$\mu = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4.$$

This form is closed and generates the 3-dimensional cohomology. We can write

$$\mu = \nu \wedge \lambda,$$

where $\nu = e_0$ is the connection form of the Stiefel bundle and where $\lambda = e_1 \wedge e_2 - e_3 \wedge e_4$ generates the 2-dimensional cohomology of the Stiefel manifold, and is the pullback of a closed form which together with the Kähler form ω generates the 2-dimensional cohomology of the Grassmann manifold. Finally, note that

$$d\nu = 2\omega$$

up in the Stiefel manifold.

2. Finding what μ calibrates

Having selected the closed G-invariant 3-form

$$\mu = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4$$

on $T_1S^3 = V_2R^4$, we face the following tasks:

A) Show that $\mu(u \wedge v \wedge w) \leq \mathrm{vol}\,(u \wedge v \wedge w)$. Then we will know that μ is a calibrating form.

B) Find out what μ calibrates infinitesimally. This means finding those oriented 3-planes in 5-space for which the above inequality is actually an equality.
C) Find out what μ calibrates globally. This means finding those oriented 3-manifolds in $T_1S^3 = V_2R^4$ which are tangent to such 3-planes at each point.

To begin, we write

$$\mu = e_0 \wedge (e_1 \wedge e_2 - e_3 \wedge e_4) = v \wedge \lambda,$$

as before.

It is straightforward linear algebra to check that μ is calibrating, and that infinitesimally it calibrates precisely the 3-planes which contain the e_0-axis and which meet the $e_1e_2e_3e_4$-space in the graph of an anticonformal map from the e_1e_2-plane to the e_3e_4-plane (including the e_3e_4-plane itself). After all, except for the minus sign, $\lambda = e_1 \wedge e_2 - e_3 \wedge e_4$ is the usual Kähler 2-form in real 4-space, and multiplication by the new variable e_0 has the expected effect.

Suppose the oriented 3-manifold M^3 in $T_1S^3 = V_2R^4$ is calibrated by our form $\mu = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4$. Infinitesimally, this means that each tangent space to M^3 contains the e_0-axis, which is itself tangent to the Stiefel fibres. Globally, this means that M^3 is a union of Stiefel fibres, and hence the inverse image of a submanifold M^2 down in the Grassmannian G_2R^4. And this submanifold M^2 must in turn be calibrated by the invariant 2-form $\lambda = e_1 \wedge e_2 - e_3 \wedge e_4$ on G_2R^4.

The usual complex structure J on the Grassmann manifold G_2R^4 is defined by $J(e_1) = e_2$ and $J(e_3) = e_4$. Define another complex structure J^* there by $J^*(e_1) = e_2$ and $J^*(e_3) = -e_4$. Then the 2-form λ is the Kähler form of the complex structure J^*, and hence calibrates the J^*-complex submanifolds of G_2R^4. Each such J^*-complex submanifold M^2 minimizes area in its homology class. Its inverse image M^3 in the Stiefel manifold is calibrated by our 3-form μ and minimizes volume in its homology class. In fact, the volume of M^3 is simply the length of a Stiefel fibre times the area of M^2.

So we come to the conclusion: *our 3-form μ calibrates those oriented 3-manifolds in $T_1S^3 = V_2R^4$ which are inverse images under the Stiefel projection of the J^*-complex submanifolds of G_2R^4.*

It is clear that the submanifold M^2 of G_2R^4 has minimum area (over *all* nontrivial homology classes) precisely when it equals $S^2 \times$ point or point $\times S^2$, in which case its inverse image M^3 is a Hopf vector field V_H. Since all unit vector fields V on S^3 represent the same non-trivial 3-dimensional homology class when viewed as submanifolds of $T_1S^3 = V_2R^4$, this gives the desired result:

The unit vector fields of minimum volume on S^3 are the Hopf vector fields, and no others.

3. Why the method of calibrated geometries fails in higher dimensions

In this section we will see that there is a 5-dimensional submanifold of $T_1S^5 = V_2R^6$ in the same homology class as $2V_H$, but with less volume. If there were any closed 5-form on the unit tangent bundle T_1S^5 (isometry-invariant or not) which calibrated the Hopf vector fields V_H, then automatically kV_H would be the "manifold" of minimum volume in the homology class $k[V_H]$. Since this is not the case for $k = 2$, the method of calibrated geometries can not be used to show that the Hopf vector fields on S^5 have minimum volume. The same holds on S^7, S^9, $S^{11}, \ldots$.

To produce this 5-manifold inside T_1S^5, start with a single fibre F^4 of the unit tangent bundle $T_1S^5 \to S^5$. It is a totally geodesic round 4-sphere of radius 1. Flow it by the geodesic flow g_t to produce the 5-dimensional submanifold L^5 of T_1S^5.

We can see L^5 another way. Take the fibre F^4 in T_1S^5 and view it in V_2R^6, where it now appears horizontal. Use the Stiefel projection to project it to a totally geodesic round 4-sphere L^4 of radius 1 in the Grassmann manifold G_2R^6. L^4 represents the set of all oriented 2-planes in 6-space which can be obtained from a given one by rotating it about a given line therein. Then L^5 is simply the inverse image of L^4 under the Stiefel projection, because the orbits of the geodesic flow on T_1S^5 are the same as the fibres of the Stiefel bundle V_2R^6.

In the Stiefel manifold, L^5 is isometric to $S^4(1) \times S^1(\sqrt{2})$. In the unit tangent bundle, it is isometric to $S^4(1) \times S^1(1)$. These isometries follow immediately from the parametrizaton of L^5 given below.

We claim that L^5, properly oriented, represents the 5-dimensional homology class $2[V_H]$ in the unit tangent bundle T_1S^5.

Suppose that F^4 is the fibre of the unit tangent bundle over the point x_0 on S^5. Thus

$$F^4 = \{(x_0, y) : y \in S^5, \langle x_0, y \rangle = 0\}.$$

Applying the geodesic flow, L^5 can be viewed as the image of $S^4 \times S^1$ under the map

$$(y, \theta) \to (x_0 \cos\theta + y \sin\theta, -x_0 \sin\theta + y \cos\theta).$$

The projection $T_1S^5 \to S^5$ is an isomorphism on 5-dimensional homology, so we simply need to check the degree of the map

$$(y, \theta) \to x_0 \cos\theta + y \sin\theta.$$

Clearly the 5-sphere is covered once for $0 \leqslant \theta \leqslant x$. Note that the above map takes (y, θ) and $(y, \theta + \pi)$ to antipodal points. Since the antipodal map on S^5 has degree 1, our map must have degree 2. The corresponding map for a Hopf vector field V_H in place of L^5 has degree 1, and the claim follows: $[L^5] = 2[V_H]$.

In contrast to this, we claim that

$$\operatorname{vol} L^5 < 2 \operatorname{vol} V_H.$$

First we compute vol V_H. For each complex structure J on R^6 we have a Hopf fibration of S^5 by the unit circles on the corresponding complex lines, and a Hopf vector field $V_H = \{(x, Jx) : x \in S^5\}$. Since J is an isometry, V_H is a round 5-sphere of radius $\sqrt{2}$ in the Stiefel manifold V_2R^6. One easily calculates that the unit 5-sphere has volume π^3, and hence vol $V_H = 4\sqrt{2}\,\pi^3$.

Next we compute vol L^5. Viewed in the Stiefel manifold, this submanifold is isometric to $S^4(1) \times S^1(\sqrt{2})$. Since vol $S^4 = (8/3)\pi^2$, we have

$$\operatorname{vol} L^5 = (8/3)\pi^2 \times 2\pi\sqrt{2} = 5\tfrac{1}{3}\sqrt{2}\,\pi^3,$$

verifying the claim.

Thus we have found a submanifold L^5 of T_1S^5 in the same homology class as $2V_H$, but with less volume. Hence the method of calibrated geometries can not be used to show that the Hopf vector fields on S^5 have minimum volume. The same holds on $S^7, S^9, S^{11}, \ldots$.

Nevertheless we can carry out the search for G-invariant forms on $T_1S^{2n+1} = V_2R^{2n+2}$ for all n, where now G is the group $SO(2n+2) \times SO(2)$ of isometries.

In our earlier problem on S^3, there was no ambiguity (except for sign) in the choice of G-invariant calibrating 3-form

$$\mu = \nu \wedge \lambda = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4.$$

The same thing happens on S^{2n+1}: there is a unique (up to sign) G-invariant calibrating $2n+1$ form μ on the Stiefel manifold V_2R^{2n+2}, and it too can be written as $\nu \wedge \lambda$, where ν is the connection form of the Stiefel manifold and where λ is the pullback of a G-invariant $2n$-form (also written λ) from the Grassmann manifold G_2R^{2n+2}. Down there, λ represents the "other" generator in the middle dimensional cohomology $H^{2n}(G_2R^{2n+2}) \cong Z + Z$, that is, other than the nth power ω^n of the Kähler form. This becomes precise when we ask in addition that λ be in the kernel of exterior multiplication by the Kähler form ω. And this in turn is what makes the $2n+1$ form $\mu = \nu \wedge \lambda$ closed:

$$d\mu = d(\nu \wedge \lambda) = d\nu \wedge \lambda = 2\omega \wedge \lambda = 0.$$

The $2n+1$ form μ provides a calibrated geometry on the Stiefel manifold, while the $2n$-form λ provides one on the Grassmann manifold. We know from our previous discussion that μ can not calibrate the Hopf vector fields when $n \geq 2$. Defining the submanifolds $L^{2n+1} = S^{2n}(1) \times S^1(\sqrt{2})$ of the Stiefel manifold V_2R^{2n+2}, and $L^{2n} = S^{2n}(1)$ of the Grassmann manifold G_2R^{2n+2} just as we did above for $n = 2$, we will prove in [G–M–Z] the

PROPOSITION. *For* $n \geqslant 2$: *The* $2n+1$ *form* μ *on the Stiefel manifold calibrates the submanifolds* L^{2n+1} *and nothing else. The* $2n$*-form* λ *on the Grassmann manifold calibrates the submanifolds* L^{2n} *and nothing else.*

Note that the subgroup $G' = SO(2n+2)$ of $G = SO(2n+2) \times SO(2)$ still acts transitively on $T_1S^{2n+1} = V_2R^{2n+2}$. If we look for G'-invariant closed $2n+1$ forms, the choice is much wider, and includes e.g. the pullback to the unit tangent bundle of the volume form on S^{2n+1}. Choosing an appropriate G'-invariant calibrating $2n+1$ form, one easily obtains

$$\text{vol } V \geq 2 \text{ vol sphere}$$

for a unit vector field V on a unit sphere.

We can do a little better. Let V be a unit vector field on S^{2n+1} and μ the calibrating $2n+1$ form on T_1S^{2n+1} mentioned above. Then

$$\text{vol } V \geq \int_V \mu = \int_{V_H} \mu = c(n) \text{ vol } S^{2n+1}.$$

It follows from an explicit formula for μ given in [G–M–Z] that

$$c(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \Big/ \binom{2n}{2k}$$

For example, $c(1) = 2$, $c(2) = 2\frac{2}{3}$, $c(3) = 3\frac{1}{5}, \ldots$.

By contrast

$$\operatorname{vol} V_H = 2^n \operatorname{vol} S^{2n+1},$$

so that starting on S^5, the above inequalities are much weaker than the expected ones.

Nevertheless, the inequalities report that all unit vector fields on a round sphere of any dimension fail to be parallel by at least a certain minimum amount.

4. An inequality for the volume of a fibre bundle map over a surface

Our goal here is to prove the following

PROPOSITION. *Suppose $f: M^m \to N^2$ is a fibration of the compact Riemannian manifold M^m over the compact surface N^2. Then*

$$\operatorname{vol} f \geq \operatorname{vol} M + (\text{average vol fibre})\ (\text{area } N),$$

with equality if and only if f is a conformal submersion.

Recall that we defined the *volume* of a map $f: M \to N$ between Riemannian manifolds to be the volume of its graph in $M \times N$.

EXAMPLE. Among all maps $f: S^2 \to S^2$ of nonzero degree, the conformal and anticonformal homeomorphisms have minimum volume.

EXAMPLE. Among all maps $f: S^3 \to S^3$ of nonzero degree, there are *none* of minimum volume. All have volume $> 2 \operatorname{vol} S^3$. Some maps have volumes approaching this lower limit, but none equal it. The identity map $S^3 \to S^3$ has volume $= 2\sqrt{2} \operatorname{vol} S^3$.

Homotopically nontrivial maps $f: S^3 \to S^2$ behave as in the previous example: there are *none* of minimum volume. All have volume $> \operatorname{vol} S^3$. Using the *equality* in the above proposition, we will see some very beautiful maps which have volumes approaching this lower limit. By contrast, the Hopf map $h: S^3 \to S^2$ has volume $= 2 \operatorname{vol} S^3$.

Let $f: M^m \to N^n$ again be a smooth map between Riemannian manifolds. For each $x \in M$, we define a pseudo-norm $|df_x|$ for the differential of f at x, as follows. Pick an orthonormal basis $e_1, e_2, \dots, e_m$ for the tangent space TM_x. If $m \leq n$, define

$$|df_x| = |df_x(e_1) \wedge \cdots \wedge df_x(e_m)|,$$

the usual norm in $\bigwedge^m TN_f(x)$. But if $m > n$, pick the basis for TM_x so that $e_{n+1}, \ldots, e_m$ belong to the kernel of df_x, which is at least $m - n$ dimensional. Then define

$$|df_x| = |df_x(e_1) \wedge \cdots \wedge df_x(e_n)|.$$

Note that $|df_x|$ is nonzero when the dimension of $\ker df_x$ is exactly $m - n$, and is zero when the dimension of the kernel is $> m - n$.

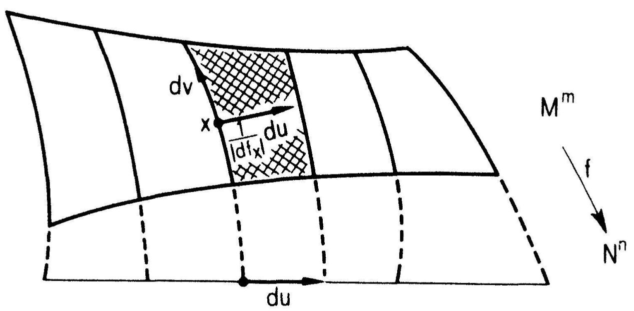
Now define the *image volume* of the map f by integrating the above pseudo-norm of its differential over the domain M:

$$\text{image vol} f = \int_M |df_x| \, d \text{ vol}.$$

EXAMPLE. If $f: M \to N$ is an embedding, then the image volume of f is simply the volume of $f(M)$ as a submanifold of N.

LEMMA. *Suppose $f: M^m \to N^n$ is a fibration between compact Riemannian manifolds. Then*

$$\textit{image vol} f = (\textit{average vol fibre}) \, (\textit{vol} \, N).$$



Referring to the picture above, we have

$$d \text{ vol}_x = |df_x|^{-1} \, du \, dv,$$

where du and dv are the volume forms on base and fibre, respectively.

Integrating over m, we get

$$\begin{aligned}\text{image vol} f &= \int_M |df_x| \, d\,\text{vol}_x = \int_M du\,dv \\ &= \int_N \left(\int_{\text{fibre}} dv \right) du = \int_N (\text{vol of fibre})\, du \\ &= (\text{average vol of fibre})\,(\text{vol}\, N),\end{aligned}$$

as claimed.

Again let $f: M^m \to N^n$ be a smooth map between compact Riemannian manifolds, with $m \geq n$. Given $x \in M$, let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an orthonormal basis for the tangent space TM_x, chosen so that $e_{n+1}, \ldots, e_m$ belong to the kernel of df_x. Suppose that on the n-plane spanned by $e_1, \ldots, e_n$, the map df_x is conformal, and suppose this is true for each $x \in M$. Then we call f a *conformal submersion*. If the constant of conformality is never zero, then f is a submersion in the usual sense, and hence a fibration.

EXAMPLE. A Riemannian submersion $f: M^n \to N^n$ is a submersion whose differential is an isometry on subspaces orthogonal to the fibres. An example is the Hopf fibration $h: S^3 \to S^2(1/2)$. Any Riemannian submersion is also a conformal submersion.

EXAMPLE. Consider the composite map

$$S^3 \xrightarrow{g} S^3 \xrightarrow{h} S^2 \xrightarrow{g'} S^2,$$

where g is a conformal homeomorphism of the three-sphere, h is the Hopf map, and g' is a conformal homeomorphism of the two-sphere. Any such map is a conformal submersion.

LEMMA. *Let $f: M^m \to N^2$ be a smooth map of a compact Riemannian manifold to a compact surface. Then*

$$\text{vol} f \geq \text{vol}\, M^m + \text{image vol} f,$$

with equality if and only if f is a conformal submersion.

Given $x \in M^m$, we choose an orthonormal basis $e_1, e_2, e_3, \ldots, e_m$ for the tangent space TM_x so that $e_3, \ldots, e_m$ belong to the kernel of df_x. Then the volume element of the graph of f is

$$\begin{aligned} &|(e_1 + df_x e_1) \wedge (e_2 + df_x e_2) \wedge e_3 \wedge \cdots \wedge e_n| \\ &= \sqrt{1 + |df_x e_1|^2 + |df_x e_2|^2 + |df_x e_1 \wedge df_x e_2|^2} \\ &\geq 1 + |df_x e_1 \wedge df_x e_2|, \end{aligned}$$

by elementary linear algebra, with equality if and only if df_x is a conformal map of the $e_1 e_2$-plane to the tangent plane $TN_{f(x)}$.

Integrating this inequality over M^m proves the lemma.

Putting the preceding two lemmas together, we get the proposition stated at the beginning of this section.

If we apply the *equality* in the proposition to the conformal submersion

$$f = g'hg : S^3 \to S^2$$

defined above, we get

$$\text{vol}\, f = \text{vol}\, S^3 + (\text{average length fibre})\ (\text{area } S^2).$$

If g and g' are the identity maps, then f is the Hopf map h and we get

$$\text{vol}\, h = 2\, \text{vol}\, S^3.$$

Now choose the conformal homeomorphism g of S^3 so that it takes a very small circle to one of the Hopf circles. Then choose the conformal homeomorphism g' of S^2 so that it spreads a small neighborhood of the point corresponding to this Hopf circle over most of the two-sphere. The composite map $f = g'hg$ then has average fibre length very small. As a result, $\text{vol}\, f$ is very close to $\text{vol}\, S^3$. If we keep the homeomorphisms g and g' orientation preserving, then f is in the same homotopy class as the Hopf map h, yet has smaller volume. Amusingly, $\text{vol}\, h$ is the *maximum* volume among all maps f of this type. The limiting value of $\text{vol}\, f$, namely $\text{vol}\, S^3$, can never be achieved for a map homotopic to h.

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