The fundamental group of compact manifolds without conjugate points.

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The fundamental group of compact manifolds without conjugate points

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I. Introduction

The purpose of this paper is to prove the following:

MAIN THEOREM. *Let N be a compact manifold that admits an analytic Riemannian metric without conjugate points. Then every abelian subgroup of $\pi_1(N)$ is straight.*

A Riemannian metric is said to have no conjugate points if the exponential map $\exp_x : T_xN \to N$ is non-singular for every $x \in N$. The straightness property of a subgroup of a group is an algebraic property that we define below. First however we mention some consequences of the main theorem.

THEOREM A. *Let N be a compact manifold that admits a C^∞ metric without conjugate points. Then every nilpotent subgroup of $\pi_1(N)$ is abelian.*

THEOREM B. *Let N be as in the main theorem. Then every solvable subgroup Σ of $\pi_1(N)$ is a Bieberbach group. In particular Σ has a finite index abelian subgroup.*

Theorem A is significantly easier to prove than Theorem B, as its proof relies only on the fact that cyclic subgroups are straight. (We prove the straightness of cyclic groups also in the C^∞-case, see Lemma 3.1.)

To define the notion of straightness we introduce the word norm $|\ |_\Gamma$ of a finitely generated group Γ. Let $\{\gamma_1, \ldots, \gamma_p\}$ be a set of generators for Γ. Then for $\gamma \in \Gamma$, $|\gamma|_\Gamma$ is defined as the length of the shortest word in the γ_i's and γ_i^{-1}'s representing γ. Of course, this norm depends on the choice of generators. However, as it is easy to see, different sets of generators give rise to equivalent

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norms. That is, if $| \ |_\Gamma^1$ and $| \ |_\Gamma^2$ are two such norms then there is a constant c such that $(1/c)\,|\gamma|_\Gamma^1 \leq |\gamma|_\Gamma^2 \leq c\,|\gamma|_\Gamma^1$ for all $\gamma \in \Gamma$. A finitely generated subgroup $\Gamma_0 \subset \Gamma$ is called *straight* in Γ if $| \ |_{\Gamma_0}$ and $| \ |_\Gamma$ are equivalent norms on Γ_0. (Note that this notion is independent of the choice of generators on Γ_0 or Γ.)

These types of theorems have been considered under stronger assumptions on the Riemannian metric. The case where M admits a metric of non-positive sectional curvature ($K \leq 0$) was considered by Lawson and Yau [L–Y] and by Gromoll and Wolf [G–W] in the early 1970's. Among other things they prove that a solvable subgroup of $\pi_1(M)$ must be Bieberbach and that M contains a corresponding flat manifold (see [C–E], Theorem 9.1 and Corollary 9.7). This was generalized by O'Sullivan in 1976 [OS] to the case where M has a metric without focal points. In [G] Gromov discusses the question of straightness of abelian subgroups in the general setting of convex length spaces.

All of the above assumptions and proofs involve the convexity or monotonicity of certain functions. If M is a simply connected manifold of non-positive curvature and c_1, c_2 geodesics of M with $c_1(0) = c_2(0)$, then the distance function $f(t) = d(c_1(t), c_2(t))$ is convex; if M has no focal points, then the distance function f is monotone increasing on $[0, \infty)$, compare also the discussion in [E–OS]. The assumption of no conjugate points is equivalent to the synthetic condition that any two points of M can be joined by a unique geodesic. Note that either condition "non-positive sectional curvature" or "no focal points" implies "no conjugate points". The converse is not true by a result of Gulliver [Gul].

Instead of using the convexity or monotonicity of the distance function we use the concept of Busemann functions. These functions are the main tool in our argument.

A major open question about compact manifolds without conjugate points is the following problem, which is often called the "Hopf-conjecture":

CONJECTURE. Any Riemannian n-torus without conjugate points is flat.

E. Hopf solved this problem for the 2-torus [H], compare also the results of Green [Gre]. The general case of the n-torus is easy to see under the stronger condition of non-positive curvature and was proved by Avez [A] under the assumption of no focal points. If the conjecture is true (a question that is still very much open) our theorems would imply that if N is a compact Riemannian manifold without conjugate points such that $\pi_1(N)$ is nilpotent or $\pi_1(N)$ is solvable (and the metric is analytic) then the metric must be flat.

We remark that the compactness condition of the main theorem is crucial even under the assumption of negative curvature. For example, let N be a noncompact quotient with finite volume of the complex hyperbolic plane. Then the subgroup of $\pi_1(N)$ coming from a parabolic end (cusp) of N contains the Heisenberg group and thus a non-straight cyclic subgroup by Lemma 4.1.

We should say a few words about the analytic assumption in the main theorem and in Theorem B. The full strength of the analytic assumption is never used and the theorems should remain true assuming only a C^∞ metric. For our proof we need only assume a weak rectifiability condition (condition *) on the set of points which lie on shortest closed geodesics in a given free homotopy class. This condition is defined precisely before Lemma 2.4, and is clear in the analytic setting.

The paper is organized into four sections the first of which is this introduction. In the second section we prove the main technical result, Proposition 2.5. This says that in the universal covering M of N the Busemann functions associated to two different axis of the same element of $\pi_1(N)$ differ by a constant. In the third section we prove the main theorem. The fourth section contains the algebraic results needed to prove Theorems A and B.

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II. Technical lemmas

Let N be a compact C^∞-smooth n-dimensional Riemannian manifold without conjugate points. We consider N as M/Γ, where M is the universal covering space and $\Gamma \simeq \pi_1(N)$ is the group of deck transformations, acting as isometries on M.

The exponential map $\exp_x : T_xM \to M$ is a diffeomorphism for every point $x \in M$. As a consequence, any two points in M can be joined by an (up to parametrization) unique geodesic. Because there are no cut points, the distance function $d : M \times M \to \mathbb{R}$ is smooth outside the diagonal. All geodesics $c : \mathbb{R} \to M$ will be parametrized by arc length.

For a geodesic c the *Busemann-function* b_c of c is defined by

$$b_c(x) := \lim_{t \to \infty} (d(x, c(t)) - t).$$

Note that the function $t \mapsto (d(x, c(t)) - t)$ is monotone non-increasing by the triangle inequality. Therefore b_c is well defined. We recall the following properties of Busemann functions (cf. [E]).

(i) Busemann functions are C^1-smooth.

(ii) The gradient ∇b_c has norm equal to 1. In particular, b_c is Lipschitz with constant 1.

(iii) For given $x \in M$, there is a unique geodesic g with $g(0)=x$ and $b_c(g(t)) = b_c(g(0)) - t$. The geodesic g is determined by $\dot{g}(0) = -\nabla b_c(x)$.

(iv) It follows from (ii) and (iii) that for given $x \in M$ and $r > 0$ there is a unique point y with $d(x, y) = r$ such that $b_c(y) - b_c(x) = -r$. The point y equals $g(r)$ in the notion of (iii).

Remarks. (a) If $\bar{c}(t) := c(t+a)$ is another parametrization of c, then $b_{\bar{c}}(x) = b_c(x) - a$. Thus, using an orientation preserving reparametrization of c we can normalize b_c to be zero for a given point x_0.

(b) The geodesic g given by (iii) is called *asymptotic* to c. Thus through every point $x \in M$ there is a unique geodesic asymptotic to c. The notion of asymptotic depends only on the oriented geodesic c and not on the particular parametrization of c.

WARNING. It is not known in general, whether the relation "asymptotic" is symmetric (cf. [E]). The purpose of this section is to prove, in the special case that both c and g are axes of the same isometry γ, the even stronger statement that $b_g - b_c$ is constant.

For an isometry $\gamma : M \to M$ we define the *displacement function* $d_\gamma : M \to \mathbb{R}$ by $d_\gamma(x) := d(x, \gamma x)$. Since d is differentiable outside the diagonal, d_γ is differentiable, if γ has no fixed points. A geodesic $c : \mathbb{R} \to M$ is called an axis of γ, if there is a constant $L > 0$ such that $\gamma c(t) = c(t+L)$ for all $t \in \mathbb{R}$. For $\gamma \in \Gamma$ we define $Ax(\gamma)$ to be the set of all points which are contained in an axis of γ.

LEMMA 2.1. *Let $N = M/\Gamma$ be compact and $\gamma \in \Gamma$ be a nontrivial element.*

(1) *Then d_γ assumes a positive minimum,* $\min d_\gamma$.

(2) *The set $Ax(\gamma)$ is equal to the set of critical points of d_γ. Furthermore $Ax(\gamma)$ is the set, where d_γ assumes the minimum.*

(3) *For $m \geq 0$ we have* $\min d_{\gamma^m} = m \cdot \min d_\gamma$.

Remarks. (a) Note that (2) implies in particular that every axis of γ is translated by the same amount, namely by $\min d_\gamma$.

(b) For every $x \in M$ and every $\gamma \in \Gamma$ the displacement $d_\gamma(x)$ is the length of an essential (non-contractible) geodesic loop in N. Namely the projection of the geodesic from x to $\gamma(x)$. Hence in particular $\min d_\gamma \geq \operatorname{sys}(N)$, where $\operatorname{sys}(N)$ represents the length of the shortest essential geodesic in N. The proof of (2) also implies that $b_c(\gamma x) - b_c(x) = \min d_\gamma \geq \operatorname{sys}(N)$ for all axis c of γ and all $x \in M$.

Proof. (1) Let $x_i \in M$ be a sequence with $d_\gamma(x_i) \to \inf_{x \in M} d_\gamma(x)$. Since $N = M/\Gamma$ is compact, Γ has a compact fundamental domain. Hence there are elements $\gamma_i \in \Gamma$ such that the sequence $\gamma_i(x_i)$ is bounded. Choosing a subsequence, we can assume that $\gamma_i(x_i)$ converges to $y \in M$. Since $d_{\gamma_i \gamma \gamma_i^{-1}}(\gamma_i(x_i)) = d_\gamma(x_i)$ which is bounded and $d_{\gamma_i \gamma \gamma_i^{-1}}(\gamma_i(x_i))$ is approximately $d_{\gamma_i \gamma \gamma_i^{-1}}(y)$ for large i, we see $d_{\gamma_i \gamma \gamma_i^{-1}}(y)$ is bounded. Since Γ operates discretely, there exists $\alpha \in \Gamma$ such that $\gamma_i \gamma \gamma_i^{-1} = \alpha \gamma \alpha^{-1}$ for a subsequence. Thus $\lim_{i \to \infty} d_\gamma(x_i) = d_{\alpha \gamma \alpha^{-1}}(y) = d_\gamma(\alpha^{-1}(y))$. Therefore d_γ assumes the minimum, $\min d_\gamma$.

(2) We compute the gradient of d_γ. For $x \in M$, let $V^+(x)(V^-(x))$ be the initial vector of the geodesic from x to $\gamma(x)(\gamma^{-1}(x))$. We claim that $\nabla d_\gamma(x) = -(V^+(x) + V^-(x))$. To see this let $c:[0, d_\gamma(x)] \to M$ be the geodesic from x to $\gamma(x)$. For $w \in T_xM$ let $c_w: \mathbb{R} \to M$ the geodesic with $\dot{c}_w(0) = w$. Using the first variational formula, we compute

$$(d_\gamma \circ c_w)'(0) = \langle \dot{c}(d_\gamma(x)), \gamma_* w \rangle - \langle \dot{c}(0), w \rangle.$$

By applying γ_*^{-1}, we have

$$\langle \dot{c}(d_\gamma(x)), \gamma_* w \rangle = -\langle V^-(x), w \rangle.$$

Thus we see that $\langle \nabla d_\gamma(x), w \rangle = -\langle V^+(x) + V^-(x), w \rangle$ for all $w \in T_xM$ and the claim follows. If x is contained in an axis of γ, then clearly $V^+(x) = -V^-(x)$ and x is a critical point of d_γ. On the other hand, if x is critical, then $V^+(x) = -V^-(x)$ and x, $\gamma(x)$ and $\gamma^{-1}(x)$ lie on a unique geodesic c. Therefore γ leaves c invariant and c is an axis of γ.

If d_γ is minimal at x, then x is critical and hence contained in an axis of γ. It remains to prove that d_γ assumes the minimum on every axis. Therefore let $c: \mathbb{R} \to M$ be an axis with $\gamma c(t) = c(t + L)$ for $L > 0$. Then, for all $x \in M$,

$$\begin{aligned} b_c(\gamma x) &= \lim_{t \to \infty} (d(\gamma x, c(t)) - t) \\ &= \lim_{t \to \infty} (d(x, c(t - L)) - t) \\ &= b_c(x) - L \end{aligned}$$

Since b_c is a Lipschitz function with constant 1, it follows that $d_\gamma(x) \geqslant L = d_\gamma(c(0))$. Thus d_γ assumes the minimum on c.

(3) An axis of γ is also an axis of γ^m, hence $\min d_{\gamma^m}$ is achieved on $Ax(\gamma)$. On an axis it is clear that $d_{\gamma^m} = m \cdot d_\gamma$. □

Now let $\alpha \in \Gamma$ be an element which commutes with γ. Then $d_\gamma(\alpha(x)) = d(\gamma\alpha(x), \alpha(x)) = d(\alpha\gamma(x), \alpha(x)) = d_\gamma(x)$. In particular α leaves $Ax(\gamma)$, the minimal set of d_γ, invariant. This observation implies that d_γ induces a well defined function $\bar{d}_\gamma$ on $M/Z(\gamma)$, where $Z(\gamma)$ is the centralizer of γ in Γ.

LEMMA 2.2. *The function $\bar{d}_\gamma$ on $M/Z(\gamma)$ is proper.*

Proof. Let a be a positive constant. We have to prove that $S_a := \{\bar{x} \in M/Z(\gamma) \mid \bar{d}_\gamma(\bar{x}) \leq a\}$ is compact. Let $\bar{x}_i$ be a sequence of points in S_a and let $x_i \in M$ be such that $\pi(x_i) = \bar{x}_i$ where $\pi : M \to M/Z(\gamma)$ is the canonical projection. Since Γ is cocompact there are $\gamma_i \in \Gamma$ such that $\gamma_i(x_i)$ lie in a fixed compact set D (a fundamental domain). Hence some subsequence converges to $y \in D$. As in the proof of Lemma 2.1 we see $d_{\gamma_i\gamma\gamma_i^{-1}}(\gamma_i(x_i)) = d_\gamma(x_i) \leq a$, hence $d_{\gamma_i\gamma\gamma_i^{-1}}(y) \leq a + 1$ for i large enough. Thus by the discreteness of Γ there are only a finite number of $\gamma_i\gamma\gamma_i^{-1}$. Passing to a subsequence we may assume $\gamma_i\gamma\gamma_i^{-1} = \gamma_j\gamma\gamma_j^{-1}$, hence $\gamma_j^{-1}\gamma_i \in Z(\gamma)$. Since $d(\gamma_j^{-1}\gamma_i(x_i), x_j) = d(\gamma_i(x_i), \gamma_j(x_j)) \leq \text{diameter}(D)$, we see $d(\bar{x}_i, \bar{x}_j) \leq \text{diameter}(D)$. Hence some subsequence converges. □

We denote by $\bar{A}x(\gamma)$ the set $\pi(Ax(\gamma)) \subset M/Z(\gamma)$. Thus $\bar{A}x(\gamma)$ is the set of minimia of $\bar{d}_\gamma$ and also the set of critical points. Since $\bar{d}_\gamma$ is proper, $\bar{A}x(\gamma)$ is compact.

LEMMA 2.3. *The set $\bar{A}x(\gamma) \subset M/Z(\gamma)$ is connected.*

Proof. Let U and V be disjoint open sets with $\bar{A}x(\gamma) \subset U \cup V$. Let a be the infimum of $\bar{d}_\gamma$ on $M/Z(\gamma) - (U \cup V)$. Then $a > \min \bar{d}_\gamma$ and $U_a := \{\bar{x} \in M/Z(\gamma) \mid \bar{d}_\gamma(\bar{x}) < a\}$ contains $\bar{A}x(\gamma)$ and is contained in $U \cup V$. Since $\bar{d}_\gamma$ has no critical points outside U_a, this set is connected by Morse theory. Therefore U_a and hence $\bar{A}x(\gamma)$ is either contained in U or contained in V. □

We define a subset A of a manifold V to be *locally rectifiably path connected*, if for $x \in A$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that if $y \in A$ and $d(x, y) < \delta$, then there is a rectifiable path in A from x to y staying in the ϵ-ball around x.

For the rest of this section we make the assumption

(*) The set $\bar{A}x(\gamma)$ is locally rectifiably path connected.

Note that because $\bar{A}x(\gamma)$ is compact, we can choose $\delta(\epsilon)$ uniformly for all $x \in \bar{A}x(\gamma)$.

Remark. If the metric on N is real analytic, then d_γ and $\bar{d}_\gamma$ are analytic and the set $\bar{A}x(\gamma)$ is an analytic variety which clearly satisfies $(*)$.

LEMMA 2.4. *If we assume* $(*)$, *then any two points* x *and* y *in* $Ax(\gamma)$ *can be joined by a rectifiable path in* $Ax(\gamma)$.

Proof. Let $\pi: M \to M/Z(\gamma)$ be the projection and choose $\epsilon > 0$ small enough such that π is a diffeomorphism on each ϵ-ball in M. If $x, y \in Ax(\gamma)$ with $d(x, y) < \delta(\epsilon)$, then $\pi(x)$ and $\pi(y)$ can be joined by a rectifiable path in $\bar{A}x(\gamma)$ contained in the ϵ-ball around $\pi(x)$. We can lift this ball to M and obtain a rectifiable curve from x to y contained in the ϵ-ball around x. One checks easily that $Ax(\gamma) = \pi^{-1}(\bar{A}x(\gamma))$, hence the curve is contained in $Ax(\gamma)$.

We now prove that $Ax(\gamma)$ is connected. Let $L := \min d_\gamma$. Then for $\eta > 0$ the set $\bar{U}_{L+\eta} := \{\bar{x} \in M/Z(\gamma) \mid \bar{d}_\gamma(\bar{x}) < L + \eta\}$ is connected and diffeomorphic to $M/Z(\gamma)$ by Morse theory. By lifting this set to M we see that $U_{L+\eta} = \{x \in M \mid d_\gamma(x) < L + \eta\}$ is connected for all η. For η small enough, $\bar{U}_{L+\eta}$ is contained in the $\delta/2$-neighborhood $T_{\delta/2}(\bar{A}x(\gamma))$ and therefore $U_{L+\eta} \subset T_{\delta/2}(Ax(\gamma))$. Here $\delta = \delta(\epsilon)$ as above.

It follows that any two points in $Ax(\gamma)$ can be joined by a path, which is contained in $T_{\delta/2}(Ax(\gamma))$ and as a consequence $T_{\delta/2}(Ax(\gamma))$ is connected. Thus, if $Ax(\gamma)$ is not connected, then there are two different components of $Ax(\gamma)$ with distance smaller than δ and hence there are points $x, y \in Ax(\gamma)$ in different components with $d(x, y) < \delta$. But this contradicts to the first part of the proof.

Thus $Ax(\gamma)$ is connected. Now, using the fact that nearby points in $Ax(\gamma)$ can be joined by a rectifiable path, it is easy to prove that the set of points in $Ax(\gamma)$ which can be joined to a given $x \in Ax(\gamma)$ by a rectifiable curve is open and closed. Therefore any two points in $Ax(\gamma)$ can be joined by a rectifiable path in $Ax(\gamma)$. □

PROPOSITION 2.5. *We assume* $(*)$. *Let* c_1 *and* c_2 *be axes of an element* $\gamma \in \Gamma$ *and let* b_1 *and* b_2 *be Busemann functions of* c_1 *and* c_2. *Then* $b_1 - b_2$ *is constant on* M.

Proof. Let c be any axis of γ, then $\gamma c(t) = c(t + L)$ for $L = \min d_\gamma$. By the proof of Lemma 2.1(2) we see $b_i(c(L)) = b_i(\gamma c(0)) = b_i(c(0)) - L$ for $i = 1, 2$. Thus the properties (iii) and (iv) of Busemann functions imply $\dot{c}(0) = -\nabla b_i(c(0))$ and hence $\nabla(b_1 - b_2) \equiv 0$ on $Ax(\gamma)$. Furthermore $b_i(c(t)) = b_i(c(0)) - t$ for $t \in \mathbb{R}$ by property (iii). We now claim, that for all $x \in M (b_1 - b_2)(c_2(0)) \geqslant$

$(b_1 - b_2)(x) \geq (b_1 - b_2)(c_1(0))$. To see this we compute for arbitrary s

$$\begin{aligned} b_1(x) &= \lim_{t\to\infty} (d(x, c_1(t)) - t) \\ &\leq \lim_{t\to\infty} (d(x, c_2(s)) + d(c_2(s), c_1(t)) - t) \\ &= d(x, c_2(s)) + b_1(c_2(s)) \end{aligned}$$

The first part of the proof implies

$$b_1(c_2(s)) = b_1(c_2(0)) - s$$

and thus

$$b_1(x) \leq (d(x, c_2(s)) - s) + b_1(c_2(0)).$$

For $s \to \infty$ we obtain

$$b_1(x) \leq b_2(x) + b_1(c_2(0))$$

and because $b_2(c_2(0)) = 0$ we have

$$(b_1 - b_2)(x) \leq (b_1 - b_2)(c_2(0)).$$

By interchanging the roles of c_1 and c_2 we obtain the other inequality of the claim.

By Lemma 2.4 we can connect $c_1(0)$ and $c_2(0)$ by a rectifiable path in $Ax(\gamma)$. Since $(b_1 - b_2)$ is C^1 and $\nabla(b_1 - b_2) \equiv 0$ on $Ax(\gamma)$ and hence on the rectifiable path from $c_1(0)$ to $c_2(0)$, it is not difficult to prove, that $(b_1 - b_2)(c_1(0)) = (b_1 - b_2)(c_2(0))$ and therefore $(b_1 - b_2)$ is constant. □

Remark. We used (*) only to prove the existence of a rectifiable path from $c_1(0)$ to $c_2(0)$ in $Ax(\gamma)$. At first glance, it seems that the connectedness of $\bar{A}x(\gamma)$ and the fact that $\nabla(b_1 - b_2)$ is identically 0 on $Ax(\gamma)$ implies that $(b_1 - b_2)(c_1(0))$ equals $(b_1 - b_2)(c_2(0))$. But we only know that the function $(b_1 - b_2)$ is C^1 thus we cannot use Sards theorem to prove that this function is constant on the connected components of the critical set. In fact, Whitney constructed a C^1-function on $\mathbb{R}^2$ which is not constant on the components of its critical set.

For $\gamma \in \Gamma$ we define b_γ to be the Busemann function of an axis of γ. By Proposition 2.5 b_γ is well defined up to a constant. Thus, if we normalize the function such that $b_\gamma(x_0) = 0$ for a given point $x_0 \in M$, then b_γ is well defined. Proposition 2.5 has the following corollary.

COROLLARY 2.6. *If α commutes with γ, then $b_\gamma(\alpha(x)) - b_\gamma(x)$ is independent of x.*

Proof. By definition $b_\gamma(x) = b_c(x)$ for an axis c of γ. Since $b_c(\alpha x) = b_{\alpha^{-1}c}(x)$ and $\alpha^{-1}c$ is also an axis of γ, Proposition 2.5 implies that $b_c(\alpha x) - b_c(x)$ is independent of x. □

III. Proof of the main theorem

In this section we prove the main theorem. But first, we prove the (easy) lemma which will allow us to prove Theorem A.

LEMMA 3.1. *If N is a compact manifold that admits a C^∞ metric without conjugate points then every cyclic subgroup Γ_0 of $\Gamma \equiv \pi_1(N)$ is straight.*

Proof. For any elements $\alpha, \beta \in \Gamma$ we have $d(\alpha\beta(x), x) \leq d(\alpha\beta(x), \alpha(x)) + d(\alpha(x), x) = d(\beta(x), x) + d(\alpha(x), x)$. Hence if $\{\beta_1, \ldots, \beta_p\}$ is a set of generators for Γ and $\gamma = \beta_{i_1}^{\pm 1} \cdot \beta_{i_2}^{\pm 1} \cdot \cdots \cdot \beta_{i_q}^{\pm 1}$, we have $d(\gamma(x), x) \leq \sum_j d(\beta_{i_j}^{\pm 1}(x), x) \leq q \cdot (\max \{d(\beta_i(x), x)\})$. Thus we see that $d(\gamma(x), x) \leq (\max_i \{d(\beta_i(x), x)\}) \cdot |\gamma|_\Gamma$ for all $\gamma \in \Gamma$ and $x \in M$.

Choose a set of generators $\{\alpha, \beta_1, \ldots, \beta_p\}$ for Γ containing a generator α of Γ_0. Since $|\alpha^n|_\Gamma \leq n = |\alpha^n|_{\Gamma_0}$, we need only show that $|\alpha^n|_\Gamma \geq \text{const.}\, n$. Fix $x_0 \in M$. Now $d(\alpha^n(x_0), x_0) \geq \min d_{\alpha^n} = n \cdot \min d_\alpha$ by Lemma 2.1.3. Hence we can take $\text{const} = \min d_\alpha / \max_i \{d(\beta_i(x_0), x_0)\}$. □

For the remainder of this section we assume that the metric satisfies condition *. We also choose, once and for all, a point $x_0 \in M$ and normalize all Busemann functions (by adding a constant) to be 0 at x_0.

We will now consider an arbitrary abelian subgroup Γ_0 of $\Gamma = \pi_1(N)$. Since there are no torsion elements in Γ (from Lemma 2.1.3), Γ_0 is isometric to $\mathbb{Z}^k$ for some k. We choose, once and for all, a basis $\alpha_1, \ldots, \alpha_k$ of Γ_0. Using this basis we can identify Γ_0 with the integer lattice $\mathbb{Z}^k$ in $\mathbb{R}^k$ by the correspondence $\alpha_1^{n_1} \cdot \alpha_2^{n_2} \cdot \cdots \cdot \alpha_k^{n_k} \Leftrightarrow (n_1, \ldots, n_k)$.

LEMMA 3.2. *For every $\gamma \in \Gamma_0$ we have $b_\gamma(\alpha_1^{n_1} \cdot \alpha_2^{n_2} \cdot \cdots \cdot \alpha_k^{n_k}(x_0)) = \sum_{i=1}^k n_i b_\gamma(\alpha_i)$. In other words the induced map $b_\gamma : \mathbb{Z}^k \to \mathbb{R}$ is linear $(b_\gamma(n_1, \ldots, n_k) \equiv b_\gamma(\alpha_1^{n_1} \cdot \cdots \cdot \alpha_k^{n_k}(x_0))$.*

Proof. Let β_1 and $\beta_2 \in \Gamma_0$. By Corollary 2.6 $b_\gamma(\beta_1\beta_2(x_0)) - b_\gamma(\beta_2(x_0)) = b_\gamma(\beta_1(x_0))$. Hence $b_\gamma(\beta_1\beta_2(x_0)) = b_\gamma(\beta_1\beta_2(x_0)) - b_\gamma(\beta_2(x_0)) + b_\gamma(\beta_2(x_0)) = b_\gamma(\beta_1(x_0)) + b_\gamma(\beta_2(x_0))$. Applying this repeatedly yields the lemma. □

LEMMA 3.3. *There exists a choice of $\gamma_i \in \Gamma_0$, $i = 1, \dots, k$ and a constant $c_1 > 0$ such that for every $\beta \in \Gamma_0$ there is an $i \in \{1, \dots, k\}$ with $|b_{\gamma_i}(\beta(x_0))| \geq c_1 |\beta|_{\Gamma_0}$.*

Proof. By the previous lemma b_γ acts linearly on $\Gamma_0 = \mathbb{Z}^k \subset \mathbb{R}^k$. Hence we can extend b_γ to a linear map $r_\gamma : \mathbb{R}^k \to \mathbb{R}$. The lemma is proved by finding $\gamma_1, \dots, \gamma_k$ such that $r_{\gamma_1}, \dots, r_{\gamma_k}$ are linearly independent.

Assume, for now, that we have found such a $\gamma_1, \dots, \gamma_k$. Then we can construct a linear isomorphism $r : \mathbb{R}^k \to \mathbb{R}^k$ by $v \to (r_{\gamma_1}(v), \dots, r_{\gamma_k}(v))$. On the domain $\mathbb{R}^k$ consider the norm $|(x_1, \dots, x_k)|_D \equiv \sum_{i=1}^k |x_i|$. On the range we take the norm $|(y_1, \dots, y_r)|_R \equiv \max_i |y_i|$. Since r is a linear isomorphism and all norms are equivalent there is a constant c_1 such that for all $v \in \mathbb{R}^k$, $|r(v)|_R \geq c_1 |v|_D$. For v a lattice point we see that this is exactly the conclusion of the lemma.

We now need to find $\gamma_1, \dots, \gamma_k$ such that $r_{\gamma_1}, \dots, r_{\gamma_k}$ are linearly independent. Assume we have found $\gamma_1, \dots, \gamma_j$ with $j < k$ and $r_{\gamma_1}, \dots, r_{\gamma_j}$ linearly independent. We need to find γ_{j+1}. Let K be the common kernel of $r_{\gamma_1}, \dots, r_{\gamma_j}$ (i.e. the intersection of the kernels). K has dimension ≥ 1 (in fact, by the independence of $\{r_{\gamma_j}\}$, $\dim K = k - j \geq 1$).

We claim that there is a constant c_2 such that for every $\gamma \in \Gamma_0$ we have $|r_\gamma(v)| \leq c_2 |v|_E$, where $|v|_E$ represents the Euclidean norm in $\mathbb{R}^k$. To see this let $v = (a_1, \dots, a_k)$. Then, using the fact that $|b_\gamma(y)| \leq d(y, x_0)$ (since $|\nabla b_\gamma| = 1$), we have

$$|r_\gamma(v)| = \left|\sum_{i=1}^n a_i b_\gamma(\alpha_i(x_0))\right| \leq \left(\sum_{i=1}^n |a_i|\right)\left(\max_i |b_\gamma(\alpha_i(x_0))|\right)$$
$$\leq \left(\sum_{i=1}^n |a_i|\right)\left(\max_i d(\alpha_i(x_0), x_0)\right) \leq c_2 |v|_E.$$

We now choose a lattice point γ_{j+1} such that the distance in $\mathbb{R}^k$ between γ_{j+1} and K is less than $\operatorname{sys}(N)/2c_2$. Recall, as mentioned in the remark Lemma 2.1, that $|b_\gamma(\gamma(x_0))| \geq \operatorname{sys}(N)$ for all $\gamma \in \Gamma$. Let $v_0 \in K$ be such that

$$|v_0 - \gamma_{j-1}|_E \leq \frac{\operatorname{sys}(N)}{2c_2}.$$

Then $r_{\gamma_{j+1}}(v_0)=r_{\gamma_{j+1}}(\gamma_{j+1})+r_{\gamma_{j+1}}(v_0-\gamma_{j+1})$, hence:

$$\begin{aligned}|r_{\gamma_{j+1}}(v_0)| &\geq |r_{\gamma_{j+1}}(\gamma_{j+1})|-|r_{\gamma_{j+1}}(v_0-\gamma_{j+1})| \\ &= |b_{\gamma_{j+1}}(\gamma_{j+1}(x_0))|-|r_{\gamma_{j+1}}(v_0-\gamma_{j+1})| \\ &\geq \operatorname{sys}(N)-c_2\,|v_0-\gamma_{j+1}|_E \\ &\geq \operatorname{sys}(N)-c_2\frac{\operatorname{sys}(N)}{2c_2}=\frac{\operatorname{sys}(N)}{2}>0.\end{aligned}$$

In particular $r_{\gamma_{j+1}}(v_0)\neq 0$, while $r_{\gamma_i}(v_0)=0$ for all $i\in\{1,2,\ldots,j\}$. Hence $r_{\gamma_1},\ldots,r_{\gamma_{j+1}}$ are linearly independent and the lemma follows.

Proof of the main theorem. Given an abelian subgroup $\Gamma_0\subset\Gamma$ with a fixed set of generators $\alpha_1,\ldots,\alpha_k$, extend this set to a set of generators $\{\beta_i\}$ of Γ. Let $|\ |_{\Gamma_0}$ and $|\ |_\Gamma$ represent the word norms with respect to these sets of generators. By the choice of generators, $\gamma\in\Gamma_0$ implies $|\gamma|_{\Gamma_0}\geq|\gamma|_\Gamma$. Hence to prove Γ_0 is straight in Γ we need only find a constant c_3 such that for every $\gamma\in\Gamma_0$, $|\gamma|_\Gamma\geq c_3\,|\gamma|_{\Gamma_0}$.

To see this let c_1 be the constant from Lemma 3.3 and $c_4=\max_i\{d(\beta_i(x_0),x_0)\}$. By the first paragraph of the proof of Lemma 3.1 for $\gamma\in\Gamma$, $|\gamma|_\Gamma\geq(1/c_4)\,d(\gamma(x_0),x_0)$. Fix $\gamma\in\Gamma_0$ and let γ_p be the element of Γ_0, guaranteed by Lemma 3.3, such that $|b_{\gamma_p}(\gamma(x_0))|\geq c_1\,|\gamma|_{\Gamma_0}$. Combining these yields:

$$|\gamma|_\Gamma\geq\frac{1}{c_4}\,d(\gamma(x_0),x_0)\geq\frac{1}{c_4}\,|b_{\gamma_p}(\gamma(x_0))|\geq\frac{c_1}{c_4}\,|\gamma|_{\Gamma_0}.$$

Hence the theorem is proved.

IV. Theorems A and B

In this section we prove the algebraic results which are needed to prove Theorem A and Theorem B. We need a slight extension of the concept of straight subgroups. Let Γ be any (not necessarily finitely generated) group and $\Gamma_0\subset\Gamma$ a finitely generated subgroup. Then Γ_0 is called *straight* in Γ if for every finitely generated subgroup Γ' with $\Gamma_0\subset\Gamma'\subset\Gamma$, Γ_0 is straight in Γ'.

LEMMA 4.1. *Let Γ be the Heisenberg group. (That is Γ is generated by two elements α and β and further the element $\gamma=[\alpha,\beta]$ commutes with α and with β.) Then the cyclic group generated by γ is not straight.*

Proof. Let $|\ |_\Gamma$ be the norm of Γ defined by the generators α and β. Since β commutes with $[\alpha, \beta]$, β also commutes with $[\beta, \alpha^{-1}]$. Therefore we see:

$$\begin{aligned}\gamma^4 = [\alpha, \beta]^4 &= [\alpha, \beta][\alpha, \beta]\alpha\beta\alpha^{-1}\beta^{-1}[\alpha, \beta]\\ &= \alpha[\alpha, \beta][\alpha, \beta]\beta\alpha^{-1}[\alpha, \beta]\beta^{-1}\\ &= \alpha^2[\beta, \alpha^{-1}]\beta\alpha^{-1}\beta\alpha^{-1}\beta^{-2}\\ &= \alpha^2\beta[\beta, \alpha^{-1}]\alpha^{-1}\beta\alpha^{-1}\beta^{-2}\\ &= [\alpha^2, \beta^2]\end{aligned}$$

By induction we have $\gamma^{4^n} = [\alpha^{2^n}, \beta^{2n}]$, and hence $|\gamma^{4^n}|_\Gamma \leq 4 \cdot 2^n$. It follows that $\lim_{m\to\infty} (1/m)\,|\gamma^m|_\Gamma = 0$ and therefore the group generated by γ is not straight. □

Because every non-abelian and torsion free nilpotent group contains a Heisenberg group, we have the following consequence.

LEMMA 4.2. *Let Γ be a torsion free nilpotent group. If every cyclic subgroup of Γ is straight, then Γ is abelian.*

We now focus on solvable groups.

LEMMA 4.3. *Let Γ be a torsion free solvable group. If every abelian subgroup of Γ is finitely generated and straight, then Γ is a Bieberbach group.*

Before we prove the theorem, we recall some facts about crystallographic groups. A group Γ is called *crystallographic* if Γ is isomorphic to a discrete cocompact subgroup of the isometry group Iso $(\mathbb{R}^n)$. A crystallographic group Γ is called *Bieberbach group* if Γ is torsion free. Then a group Γ is a Bieberbach group if an only if Γ is isomorphic to the fundamental group of a compact flat Riemannian manifold. Crystallographic groups can be characterized algebraically (see [W], Theorem 3.2.9).

(**) A group Γ is crystallographic, if and only if Γ has a normal free abelian subgroup Γ^* of finite rank and finite index in Γ which is maximal abelian in Γ. In that case, Γ^* is unique.

Proof (of Lemma 4.3.). By induction on the length of the derived series we can assume that the commutator subgroup $[\Gamma, \Gamma]$ is a Bieberbach group. By the characterization (**) there is a unique maximal abelian normal subgroup A of finite index in $[\Gamma, \Gamma]$.

We first claim, that A is a normal subgroup of Γ. Since $[\Gamma, \Gamma]$ is normal, any conjugate subgroup $\gamma A \gamma^{-1}$ is contained in $[\Gamma, \Gamma]$. Since A is the unique maximal abelian normal subgroup of $[\Gamma, \Gamma]$ and $\gamma A \gamma^{-1}$ satisfies the same properties, it follows that $\gamma A \gamma^{-1} = A$ for all $\gamma \in \Gamma$ and hence A is normal. Therefore the centralizer $Z(A)$ of A in Γ is also a normal subgroup. Since A is maximal abelian in $[\Gamma, \Gamma]$, we see that $Z(A) \cap [\Gamma, \Gamma] = A$. As a consequence $[\alpha, [\beta, \gamma]]$ is trivial for $\alpha, \beta, \gamma \in Z(A)$. Thus $Z(A)$ is nilpotent and indeed abelian by Lemma 4.2.

The group Γ acts by conjugation on $Z(A)$. Let us consider the map $K: \Gamma \to \mathrm{Aut}(Z(A))$, where $K(\gamma)$ is the conjugation $\alpha \mapsto \gamma \alpha \gamma^{-1}$. Note that the kernel of K equals $Z(A)$. We will show that $K(\gamma)$ has finite order for all $\gamma \in \Gamma$. Assuming this result for a moment, we prove that Γ is Bieberbach.

By our assumption, $Z(A)$ is isomorphic to $\mathbb{Z}^k$ for some k and hence $\mathrm{Aut}(Z(A))$ is isomorphic to $\mathrm{SL}_k(\mathbb{Z})$. By Selbergs-Lemma, (cf. [B], p. 38), $\mathrm{SL}_k(\mathbb{Z})$ has a torsion free subgroup of finite index. Because every $K(\gamma)$ is torsion, we conclude that $K(\Gamma)$ is finite and $Z(A) = \text{kernel } K$ has finite index in Γ. Clearly $Z(A)$ is maximal abelian. By (**), Γ is crystallographic and because it is torsion free, a Bieberbach group.

Thus it remains to show that $K(\gamma)$ has finite order or equivalently, that there exists $m \in \mathbb{N}$ with $\gamma^m \in Z(A)$. Let $Z(A)$ be free abelian of rank k and choose generators $\alpha_1, \ldots, \alpha_k$ of $Z(A)$. We denote by $| \ |_{Z(A)}$ the word norm with respect to these generators. Let Γ' be the group generated by $\alpha_1, \ldots, \alpha_k$ and γ with word norm $| \ |_{\Gamma'}$. The straightness of $Z(A)$ implies that there is a constant C such that for $\alpha \in Z(A)$.

$$|\alpha|_{Z(A)} \leq C\, |\alpha|_{\Gamma'}.$$

Furthermore by the choice of the generators of $Z(A)$ we have

$$|\alpha^m|_{Z(A)} = m\, |\alpha|_{Z(A)}.$$

Now let $\beta \in \Gamma'$ and $\alpha \in Z(A)$. Then it is easy to check that $[\beta, \alpha] \in Z(A) \cap [\Gamma, \Gamma] = A$. Therefore α commutes with $[\beta, \alpha]$ and hence also with $[\alpha, \beta^{-1}]$. Hence we compute:

$$\begin{aligned}[\beta, \alpha]^2 &= \beta[\alpha, \beta^{-1}]\alpha\beta^{-1}\alpha^{-1} \\ &= \beta\alpha[\alpha, \beta^{-1}]\beta^{-1}\alpha^{-1} \\ &= [\beta, \alpha^2]\end{aligned}$$

and by the same argument $[\beta, \alpha]^m = [\beta, \alpha^m]$. We now apply these formulas for $\beta = \gamma^p$ and $\alpha = \alpha_i$

$$\begin{aligned}
\|[\gamma^p, \alpha_i]\|_{Z(A)} &= \frac{1}{m} \|[\gamma^p, \alpha_i]^m\|_{Z(A)} \\
&= \frac{1}{m} \|[\gamma^p, \alpha_i^m]\|_{Z(A)} \\
&\leq C \frac{1}{m} \|[\gamma^p, \alpha_i^m]\|_{\Gamma'} \\
&\leq C \frac{1}{m} (2m + 2p)
\end{aligned}$$

Because this formula is true for all $m \in \mathbb{N}$, we see $\|[\gamma^p, \alpha_i]\|_{Z(A)} \leq 2C$. Since there are only finitely many elements in $Z(A)$ with norm $\leq 2C$ there are $p < q$ with

$$[\gamma^p, \alpha_i] = [\gamma^q, \alpha_i] \quad \text{for all} \quad i = 1, \ldots, k.$$

It follows that $\gamma^m = \gamma^{q-p}$ commutes with all generators α_i, thus $K(\gamma^m)$ is trivial.

Proof of Theorems A and B. Theorem B follows from the main theorem and Lemma 4.3. To see this, one notes that our solvable subgroup $\Sigma \subset \Gamma = \pi_1(N)$ is torsion free since Γ is torsion free. Hence to apply Lemma 4.3 we need to show every abelian subgroup $\Gamma_0 \subset \Sigma$ is straight. But the main theorem tells us Γ_0 is straight in Γ and it is easy to see, by choosing appropriate generating sets, that if $\Gamma_0 \subset \Gamma' \subset \Gamma$ and all are finitely generated then Γ_0 straight in Γ implies Γ_0 straight in Γ'.

Theorem A follows similarly from Lemma 3.1 and Lemma 4.2.

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