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Parametrization of Möbius groups acting in a disk

MIKA SEPPÄLÄ and TUOMAS SORVALI

Introduction

We consider groups G generated by hyperbolic Möbius transformations acting in the upper half-plane H or in the unit disc D . It is an interesting problem to find invariants which determine G up to conjugation. For properly discontinuous groups G this is the same thing as parametrizing the Teichmüller space of the Riemann surface H/G . In our considerations the discontinuity of G plays no role. Consequently the results hold for rather general groups G .

The proofs are elementary. We use multipliers of transformations of G to parametrize G . In the considerations we suppose that G is finitely generated even though the methods and computations could be easily generalized for infinitely generated groups.

To formulate the main result suppose that the hyperbolic transformations $g_1, h_1, \dots, g_s, h_s$ generate G and satisfy certain technical conditions described in Theorem 2.2. Assume that the commutator $c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$ is hyperbolic and suppose that a relation is given such that c_s has a known representation in the group generated by $g_1, h_1, \dots, g_{s-1}, h_{s-1}$. Then the multipliers of the following $6s - 4$ transformations

$$\begin{aligned} &g_j, h_j \text{ and } g_j \circ h_1 \quad \text{for } j = 1, \dots, s, \\ &h_j \circ h_1 \quad \text{for } j = 2, \dots, s, \\ &g_j \circ g_1 \text{ and } h_j \circ g_1 \quad \text{for } j = 2, \dots, s-1 \text{ and} \\ &g_s \circ h_s \end{aligned}$$

determine G up to conjugation (Corollary to Theorem 2.2). Further, the multipliers of $g_s \circ h_s$ and $g_s \circ h_1$ can both have only two different values for fixed values of the other multipliers.

Note that we allow here, in fact, conjugation by Möbius-transformations mapping the upper half-plane (or the unit disk) onto its complement. Then the conjugacy class of G becomes uniquely determined by the multipliers of the above transformations.

For discontinuous groups G this result can be used to parametrize the Teichmüller space T_S of $S = H/G$ globally by the multipliers of the above $6s - 4$ transformations. This is the same thing as parametrizing T_S by the lengths of $6s - 4$ closed geodesics. Irwin Kra has shown that such a parametrization is not even locally possible by $6s - 6$ fixed curves only. Corollary to Theorem 2.2 gives a set of $6s - 4$ closed curves whose lengths parametrize the Teichmüller space globally. Together with Kra's result this implies that the minimal number of curves parametrizing the Teichmüller space is either $6s - 4$ or $6s - 5$.

Deleting the transformations $g_s \circ h_1$ and $g_s \circ h_s$ from the above list we obtain a set of $6s - 6$ Möbius transformations whose multipliers parametrize an open set of the Teichmüller space. A remarkable fact is that such a parametrization can be obtained by elementary computations concerning the group G and that the discontinuity of G or the geometry of H/G plays no role here.

We start with showing how certain basic results about the geometry of a Riemann surface H/G are actually implied by elementary computations concerning hyperbolic Möbius transformations with intersecting axes. The multipliers of such transformations with a hyperbolic commutator satisfy an inequality (1.6) that, in the case of a Fuchsian group G , implies an inequality between the lengths of closed intersecting geodesics. The considerations here are of the same nature as those in [A], II.3.3, but the result is somewhat stronger.

The parametrization of G is achieved by considering a suitably normalized set of generators of G . That normalization is explained in Section 2 (e.g. Fig. 3). It allows us to avoid huge technical difficulties. For Fuchsian groups a usual standard set of generators satisfies the conditions of the normalization.

Using this set of generators we first parametrize certain free subgroups of G . This is done in detail in Theorem 2.1. These considerations should be contrasted with those in [S-S] where a similar parametrization was obtained for the Teichmüller space of a Riemann surface H/G . There we applied rather strong results of the geometry of H/G . The present paper reflects, in our opinion, the nature of these things more truly. Many of the results which, a priori, look rather deep are actually simple consequences of elementary computations.

Similar questions have been considered by many authors. We would especially like to mention the work of H. Helling [H]. His methods are closely related to those of this paper and his results were a starting point of this investigation.

We thank the referee for valuable comments.

1. A collar inequality

Let g be a hyperbolic Möbius transformation. Denote by $a(g)$ and $r(g)$ its *attracting* and *repelling* fixed points, respectively. Then its *multiplier* $k(g)$ is given by the cross-ratio

$$k(g) = (g(z), z, r(g), a(g))$$

for any $z \in \hat{\mathbb{C}}$ not fixed by g .

For the values $k > 0$ let

$$f(k) = \sqrt{k} + 1/\sqrt{k}$$

and denote

$$f(g) = f(k(g)).$$

Then $f(g) > 2$. For a parabolic g set $k(g) = 1$ and $f(g) = 2$.

If

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc = 1,$$

is any transformation conjugate to g , then

$$f(g) = |a + d|.$$

Let (g, h) be a pair of hyperbolic transformations of the unit disc D onto itself. Suppose that g and h have no common fixed points. Denote

$$k_1 = k(g)$$

$$k_2 = k(h)$$

$$t = (r(g), r(h), a(h), a(g)).$$

Then a calculation yields

$$f(g \circ h) = |tf(k_1 k_2) + (1 - t)f(k_1/k_2)|.$$

Divide the pairs (g, h) in the classes $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{E}$ as follows:

$$\begin{aligned} (g, h) \in \mathcal{H} &\Leftrightarrow f(g \circ h) = tf(k_1 k_2) + (1 - t)f(k_1/k_2) \geq 2 \\ &\Leftrightarrow t \geq t_1 = (2 - f(k_1/k_2))/(f(k_1 k_2) - f(k_1/k_2)) \end{aligned} \quad (1.1)$$

$$\begin{aligned} (g, h) \in \mathcal{P} &\Leftrightarrow f(g \circ h) = -tf(k_1 k_2) - (1 - t)f(k_1/k_2) \geq 2 \\ &\Leftrightarrow t \leq t_2 = (-2 - f(k_1/k_2))/(f(k_1 k_2) - f(k_1/k_2)) \end{aligned} \quad (1.2)$$

$$(g, h) \in \mathcal{E} \Leftrightarrow t_2 < t < t_1. \quad (1.3)$$

Suppose that k_1 and k_2 are kept fixed but t is let to vary. If t increases from $-\infty$ to t_2 , then $(g, h) \in \mathcal{P}$ and $f(g \circ h)$ decreases from ∞ to 2. If $t_2 < t < t_1$ then $(g, h) \in \mathcal{E}$ and $g \circ h$ is elliptic. If t increases from t_1 to ∞ , then $(g, h) \in \mathcal{H}$ and $f(g \circ h)$ increases from 2 to ∞ .

If we conjugate by a Möbius transformation sending $r(h) \rightarrow 1$, $a(h) \rightarrow 0$, $a(g) \rightarrow \infty$ and $r(g) \rightarrow t$, we get Fig. 1. Here the open interval $]t_2, t_1[$ separates the classes $\mathcal{P}$ and $\mathcal{H}$. Note that the pairs (g, h) and (h, g) belong to the same class.

We state two theorems (cf. [S-S], §1).

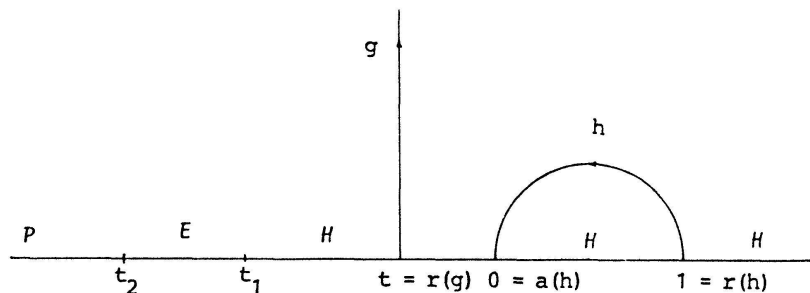


Figure 1.

THEOREM 1.1. *For any $k_1 > 1$, $k_2 > 1$ and $k_3 > 1$ there exist up to conjugation unique pairs $(g, h) \in \mathcal{H}$ and $(g, h) \in \mathcal{P}$ such that $k_1 = k(g)$, $k_2 = k(h)$ and $k_3 = k(g \circ h)$. $\square$*

THEOREM 1.2. *Deform $a(g)$, $r(g)$, $a(h)$, $r(h)$, $k(g)$ and $k(h)$ continuously. If g , h and $g \circ h$ stay hyperbolic during the deformation, then the class of (g, h) is not changed. $\square$*

Theorem 1.2 shows that, in later applications, the class of any pair (g, h) can be kept fixed.

Theorem 1.1 shows that the numbers $k_1 > 1$, $k_2 > 1$ and $k_3 > 1$ can be given

freely in the classes $\mathcal{P}$ and $\mathcal{H}$. We consider next the class $\mathcal{H}$, impose some natural restrictions on (g, h) and show that k_1 and k_2 cannot then be simultaneously ≈ 1 .

Consider the commutator

$$c = h \circ g^{-1} \circ h^{-1} \circ g.$$

A straightforward calculation (cf. [L], p. 100) yields

$$f(c) = |t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) - 2|.$$

Suppose that the axes of g and h intersect, i.e., $0 < t < 1$ (cf. Fig. 1). If c is hyperbolic, then either

$$f(c) = t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) - 2 > 2 \quad (1.4)$$

or

$$f(c) = 2 - t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) > 2$$

Since $0 < t < 1$, $t(1-t)$ is positive. Hence also $t(1-t)(f(k_1^2) - 2)(f(k_2^2) - 2) > 0$ which implies that (1.4) holds. Since $0 < t(1-t) \leq \frac{1}{4}$ for $0 < t < 1$, it follows from (1.4) that

$$2 < f(c) \leq \frac{1}{4}(f(k_1^2) - 2)(f(k_2^2) - 2) - 2$$

or

$$(f(k_1^2) - 2)(f(k_2^2) - 2) > 16$$

or

$$(\sqrt{k_1} - 1/\sqrt{k_1})(\sqrt{k_2} - 1/\sqrt{k_2}) > 4. \quad (1.5)$$

If $k_2 \rightarrow 1$ then $k_1 \rightarrow \infty$. We state our result as a theorem.

THEOREM 1.3. *Suppose that g and h are hyperbolic Möbius transformations of the unit disc D onto itself. If the axes of g and h intersect and the commutator $h \circ g^{-1} \circ h^{-1} \circ g$ is hyperbolic, then*

$$k(g) > 16/(k(h) - 1)^2. \quad (1.6)$$

Proof. Under the hypotheses of the theorem, (1.5) holds with $k_1 = k(g)$ and $k_2 = k(h)$. Since $0 < 1/\sqrt{k_j} < 1$, $j = 1, 2$, we get

$$\sqrt{k_1} > \frac{4}{(1/\sqrt{k_2})(k_2 - 1)} > \frac{4}{k_2 - 1}. \quad \square$$

To (1.6) analogous collar inequalities are proved e.g. in [K] and [A]. Retaining the assumptions of Theorem 1.3 we derive an equation satisfied by the multipliers

$$\begin{aligned} k_1 &= k(g), \\ k_2 &= k(h), \\ k_3 &= k(g \circ h), \\ k_4 &= k(h \circ g^{-1} \circ h^{-1} \circ g). \end{aligned}$$

From (1.1) we get

$$t = \frac{f(k_3) - f(k_1/k_2)}{f(k_1 k_2) - f(k_1/k_2)}.$$

Hence by (1.4)

$$\frac{f(k_4) + 2}{(f(k_1^2) - 2)(f(k_2^2) - 2)} = t(1 - t) = \frac{(f(k_3) - f(k_1/k_2))(f(k_1 k_2) - f(k_3))}{(f(k_1 k_2) - f(k_1/k_2))^2}. \quad (1.7)$$

With respect to $f(k_3)$, the equation (1.7) is of second degree. This proves the following theorem:

THEOREM 1.4. *Suppose that g and h are hyperbolic Möbius transformations of the unit disc D onto itself such that the axes of g and h intersect and the commutator $c = h \circ g^{-1} \circ h^{-1} \circ g$ is hyperbolic. If $k_1 = k(g)$, $k_2 = k(h)$ and $k_4 = k(c)$ are known, then $k_3 = k(g \circ h)$ has two possible values except in the case*

$$f(k_4) + 2 = \frac{1}{3}(f(k_1^2) - 2)(f(k_2^2) - 2) \quad (1.8)$$

when k_3 is uniquely determined. $\square$

By (1.4), the case (1.8) occurs if and only if $t = \frac{1}{2}$.

Denote by k_3 and k'_3 and two possible values in Theorem 1.4. If

$$f(k_3) = tf(k_1 k_2) + (1 - t)f(k_1/k_2), \quad (1.9)$$

then, by (1.7),

$$f(k'_3) = (1 - t)f(k_1 k_2) + tf(k_1/k_2). \quad (1.10)$$

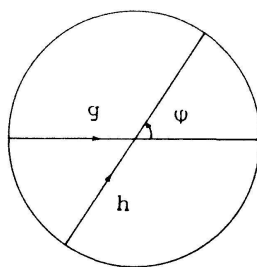


Figure 2. The angle between the axes of g and h is φ .

In the case (1.9), conjugate in such a way that $r(g) \rightarrow -1$, $a(g) \rightarrow 1$, $a(h) \rightarrow e^{i\varphi}$ and $r(h) \rightarrow -e^{i\varphi}$, $0 < \varphi < \pi$ (Fig. 2). It follows that $t = \cos^2(\varphi/2)$. If we denote by φ' the corresponding angle of (1.10), then

$$\cos^2 \frac{\varphi'}{2} = 1 - t = \sin^2 \frac{\varphi}{2}.$$

Hence $\varphi' = \pi - \varphi$. In both cases the acute angle between the axes of g and h is the same. We have proved the following corollary to Theorem 1.4:

COROLLARY. *The axes of g and h are orthogonal if and only if (1.8) holds. In other cases $k(g)$, $k(h)$ and $k(h \circ g^{-1} \circ h^{-1} \circ g)$ determine uniquely the acute angle between the axes of g and h . $\square$*

Analogous results for Fuchsian groups representing compact Riemann surfaces have been applied e.g. in [S-S].

2. Parametrization of a Möbius group with a relation

Let $\mathcal{H} = \{g_1, h_1, g_2, h_2, \dots, g_s, h_s\}$ be a set of hyperbolic Möbius transformations of D onto itself. Suppose that the cyclic order of the fixed points is given by Fig. 3 for any $\gamma \in \{g_2, h_2, \dots, g_{s-1}, h_{s-1}\}$. Then $(g_1, h_1) \in \mathcal{H}$ and $(g_s, h_s) \in \mathcal{H}$.

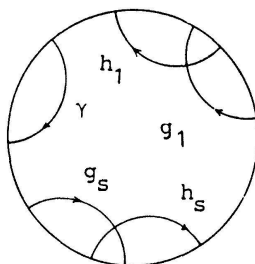


Figure 3.

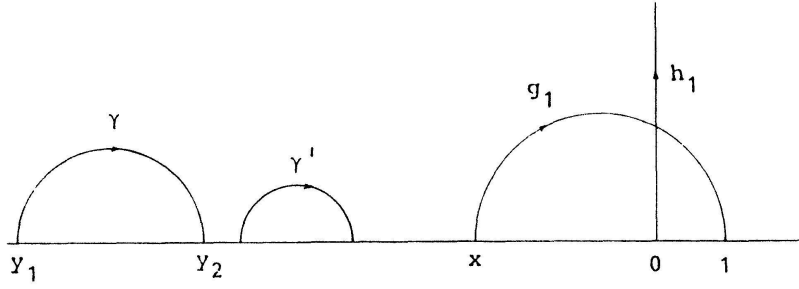


Figure 4.

Suppose the classes of (γ, g_1) and (γ, h_1) are fixed for all $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$.

Conjugate to the upper half-plane H in such a way that $r(h_1) \rightarrow 0$, $a(h_1) \rightarrow \infty$, $a(g_1) \rightarrow 1$ and $r(g_1) \rightarrow x = 1 - 1/t < 0$ (Fig. 4).

Fix the multipliers $k_1 = k(g_1)$, $k_2 = k(h_1)$ and $k_3 = k(g_1 \circ h_1)$. Then, by Theorem 1.1, x is uniquely determined.

LEMMA 2.1. *Under the above assumptions, the numbers $k(\gamma)$, $k(\gamma \circ g_1)$ and $k(\gamma \circ h_1)$ determine $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$ uniquely.*

Proof. Let γ' be another candidate for γ . By Theorem 1.1, the pairs (γ, g_1) and (γ', g_1) as well as the pairs (γ, h_1) and (γ', h_1) are conjugate. Hence we have Möbius transformations ψ and σ such that

$$\gamma' = \sigma \circ \gamma \circ \sigma^{-1}, \quad \sigma(0) = 0, \quad \sigma(\infty) = \infty$$

$$\gamma' = \psi \circ \gamma \circ \psi^{-1}, \quad \psi(1) = 1, \quad \psi(x) = x.$$

Moreover, ψ and σ are hyperbolic. Let y_1 and y_2 denote the fixed points of γ . Then

$$\psi(y_1) = \sigma(y_1)$$

$$\psi(y_2) = \sigma(y_2)$$

or

$$\frac{\psi(y_1)}{\psi(y_2)} = \frac{y_1}{y_2}$$

since $\sigma(z) = kz$, for $k = k(\sigma)$ or $k^{-1} = k(\sigma)$.

The next Lemma shows that $\psi = id$ and $\gamma' = \gamma$.

LEMMA 2.2. Let $\psi : H \rightarrow H$,

$$\psi(z) = \frac{(ak - r)z - ar(k - 1)}{(k - 1)z - kr + a},$$

$k > 0$, $k \neq 1$, be a hyperbolic Möbius transformation with the fixed points a and r and with the multiplier $\max(k, 1/k)$. If there exist real numbers y_1, y_2 such that $y_2\psi(y_1) = y_1\psi(y_2)$, then

$$k = \frac{r(y_1 - a)(y_2 - a)}{a(y_1 - r)(y_2 - r)}. \quad \square \quad (2.1)$$

The proof of Lemma 2.2 is a direct calculation. To prove Lemma 2.1 assume that ψ is not the identity. Then we can apply, to this ψ , Lemma 2.2 with $a = 1$, $r = x < 0$, $y_1 < x$ and $y_2 < x$. Hence $k < 0$ which is impossible. Lemma 2.1 is hereby proved. $\square$

At this stage we have proved the following result:

THEOREM 2.1. Let $\mathcal{K} = \{g_1, h_1, \dots, g_s, h_s\}$ be a set of hyperbolic Möbius transformations of D onto itself. Suppose that the cyclic order of the fixed points is given by Fig. 3 and the classes of the pairs (γ, g_1) and (γ, h_1) are fixed for all $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$. Then the multipliers of the following $6s - 3$ Möbius transformations determine $\mathcal{K}$ uniquely up to conjugation:

$$\begin{aligned} g_j, h_j, g_j \circ h_1, & \quad j = 1, \dots, s \\ h_j \circ h_1, g_j \circ g_1, h_j \circ g_1, & \quad j = 2, \dots, s. \quad \square \end{aligned}$$

Suppose next that the commutator

$$c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$$

is hyperbolic and that c_s has a given representation in the group generated by the set $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$. Suppose also that c_s and h_1 have no common fixed points. Let the multipliers of the following $6s - 9$ Möbius transformations be given:

$$\begin{aligned} g_j, h_j, g_j \circ h_1, & \quad j = 1, \dots, s - 1. \\ h_j \circ h_1, g_j \circ g_1, h_j \circ g_1, & \quad j = 2, \dots, s - 1. \end{aligned}$$

Then, by Theorem 2.1, the set $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$ is determined up to conjugation. Hence $k(c_s)$ is uniquely determined.

Assume that also $k(g_s)$ and $k(h_s)$ are given. Then, by Theorem 1.4, $k(g_s \circ h_s)$ has at most two possible values. Choose one of these. By Theorem 1.1,

$$t_s = (r(g_s), r(h_s), a(h_s), a(g_s))$$

is uniquely determined.

Finally, give $k(h_s \circ h_1)$. Let (g'_s, h'_s) be another candidate for the pair (g_s, h_s) . Since both have the same cross-ratio t_s , there exists a Möbius transformation ψ such that

$$h'_s = \psi \circ h_s \circ \psi^{-1}$$

$$g'_s = \psi \circ g_s \circ \psi^{-1}.$$

Since (g_s, h_s) and (g'_s, h'_s) have the same commutator c_s , ψ is hyperbolic and has the same axis as c_s . Denote by a and r the common fixed points of c_s and ψ .

Denote $y_1 = r(h_s)$, $y_2 = a(h_s)$. Since $k(h_s \circ h_1) = k(h'_s \circ h_1)$, it follows as in Lemma 2.1 that $y_2\psi(y_1) = y_1\psi(y_2)$. By Lemma 2.2, we have two alternatives. Either $\psi = id$ or

$$\psi(z) = \frac{(ak - r)z - ar(k - 1)}{(k - 1)z - kr + a},$$

$$k = \frac{r(y_1 - a)(y_2 - a)}{a(y_1 - r)(y_2 - r)}. \quad (2.2)$$

Hence we have in general two alternatives for the pair (g_s, h_s) .

THEOREM 2.2. *Let $\mathcal{H} = \{g_1, h_1, \dots, g_s, h_s\}$ be a set of hyperbolic Möbius transformations of D onto itself. Suppose that the cyclic order of the fixed points is*

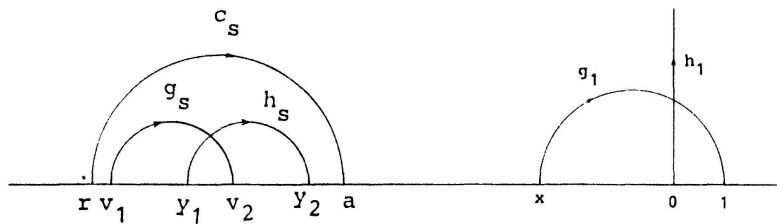


Figure 5.

given by Fig. 3 and the classes of the pairs (γ, g_1) and (γ, h_1) are fixed for all $\gamma \in \{g_2, h_2, \dots, g_s, h_s\}$. Suppose that the commutator $c_s = h_s \circ g_s^{-1} \circ h_s^{-1} \circ g_s$ is hyperbolic and has a given representation in the subgroup generated by $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$. Suppose that c_s and h_1 have no common fixed points. Let the multipliers of the following $6s - 6$ Möbius transformations be given:

$$\begin{aligned} g_j \circ h_j, & \quad j = 1, \dots, s, \\ g_j \circ h_1, & \quad j = 1, \dots, s-1, \\ h_j \circ h_1, & \quad j = 2, \dots, s, \\ g_j \circ g_1 \quad \text{and} \quad h_j \circ g_1, & \quad j = 2, \dots, s-1. \end{aligned} \quad (2.3)$$

Then the set $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$ is uniquely determined up to conjugation. If $\{g_1, h_1, \dots, g_{s-1}, h_{s-1}\}$ is fixed then (g_s, h_s) has at most four possible alternatives. $\square$

The $6s - 6$ multipliers in Theorem 2.2 give in fact local parameters for the Teichmüller space of the group generated by $\mathcal{H}$. To obtain a global parametrization for the Teichmüller space, additional multipliers are needed.

We retain the assumptions of Theorem 2.2. We add to the $6s - 6$ multipliers (2.3) also $k(g_s \circ h_1)$ and suppose that the cyclic order of the fixed points of c_s is given by Fig. 5. Denote $v_1 = r(g_s)$ and $v_2 = a(g_s)$. Then similarly as for h_s , $v_2\psi(v_1) = v_1\psi(v_2)$. Hence we get a second expression for the number k in (2.2):

$$k = \frac{r(v_1 - a)(v_2 - a)}{a(v_1 - r)(v_2 - r)}.$$

Hence

$$\frac{v_1 - a}{v_1 - r} \cdot \frac{v_2 - a}{v_2 - r} = \frac{y_1 - a}{y_1 - r} \cdot \frac{y_2 - a}{y_2 - r}$$

or

$$(v_1, y_1, a, r) = (y_2, v_2, a, r).$$

But this is impossible, since $(v_1, y_1, a, r) > 1$ and $(y_2, v_2, a, r) < 1$. Hence $\psi = id$.

The set $\mathcal{H}$ is not uniquely determined up to conjugation by the $6s - 5$ multipliers (i.e. the $6s - 6$ multipliers of the transformations (2.3) plus $k(g_s \circ h_1)$ since $k(g_s \circ h_s)$ still has in general two possible values. Theorem 1.4 and its Corollary give a clear picture of these two alternatives.

If we finally give $k(g_s \circ h_s)$, then $\mathcal{K}$ is uniquely determined up to conjugation.

COROLLARY. *Suppose that $\mathcal{K}$ satisfies the hypotheses of Theorem 2.2. If the cyclic order of the fixed points of c_s is given by Fig. 5, then the multipliers of the following $6s - 4$ Möbius transformations determine $\mathcal{K}$ uniquely up to conjugation:*

$$\begin{aligned} g_j, h_j, & \quad j = 1, \dots, s \\ g_j \circ h_1, & \quad j = 1, \dots, s \\ h_j \circ h_1, & \quad j = 2, \dots, s \\ g_j \circ g_1, h_j \circ g_1, & \quad j = 2, \dots, s-1 \\ g_s \circ h_s. \end{aligned}$$

Note that the $6s - 3$ multipliers in Theorem 2.1 give global parameters also for a set $\mathcal{K}$ without any relation.

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