Geometric invariants of link cobordism.

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Geometric invariants of link cobordism

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Abstract. A geometric notion of a "derivative" is defined for 2-component links of S^n in S^{n+2} and used to construct a sequence β^i, $i = 1, 2, \ldots$ of abelian concordance invariants which vanish for boundary links. For $n > 1$, these generalize the only heretofore known invariant, the Sato–Levine invariant. For $n = 1$, these invariants are additive under any band-sum and consequently provide new information about which 1-links are concordant to boundary links. Examples are given of concordance classes successfully distinguished by the β^i but *not* by their $\bar{\mu}$-invariants, Murasugi 2-height, Sato–Levine invariant or Alexander polynomial.

§1. Motivation and summary of results

The most interesting global question in higher-dimensional knot theory is "Is every link $\amalg_{i=1}^{\mu} S^n \hookrightarrow S^{n+2}$, $n \geq 2$ concordant to a boundary link?" [1, 12, 3, 4]. For even n this is equivalent to "Is every link concordant to the trivial link?" [1]. A false proof of an affirmative answer to these questions appeared in [10] and was rebutted in [4]. Thus, in studying the most important geometric equivalence relation on links, the *simplest* possible question has gone unanswered (for partial results, see [3, 4]).

Again, in the classical dimension ($n = 1$), the question of which links are concordant to boundary links is interesting [9]. Briefly, a *boundary link* is one whose components bound *disjoint* Seifert surfaces in S^3 (see §2). For this category of links, connected sum (band-sum) can be made well-defined, and their study reduces largely to the study of the individual components. More importantly, as in higher-dimensions, boundary links seem to be a vital intermediary between the general link and the unlink, when considering concordance questions. For example, the proper analogue of an Alexander polynomial one *knot* (which is known to be concordant to the trivial knot in the topological locally-flat category [6]) seems to be a *good-boundary link* [5].

In this paper, we describe a sequence $\beta^i(L)$ $i = 1, 2, \ldots$ of independent abelian cobordism (concordance) invariants for smooth 2-component links $L = \amalg_{i=1}^{2} S^n \hookrightarrow S^{n+2}$. For $n > 1$, these generalize the only heretofore known invariant

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$\beta(L)=\beta^1(L)$, due to N. Sato [23] and J. Levine (unpublished), and obstruct cobordism to a boundary link. For $n=1$, the invariants are strong, yet are much more computable than many other known cobordism invariants such as the $\bar{\mu}$-invariants of Milnor [18], or the various "covering linkage invariants" of Murasugi, Kojima and Laufer ([14, 15, 19]). Furthermore, the β^i are additive with respect to any band-sum of links (even though the band-sum operation itself is not well-defined on concordance classes). This leads to the following new result which partially answers question 22 of [9].

THEOREM 5.6. *Let $\mathscr{C}_1$ be the set of cobordism classes of 2-component links in S^3 with linking number 0. The invariants β^i $i=1,2,\ldots$ define a function $\phi:\mathscr{C}_1\to \times_{i=1}^{\infty}\mathbb{Z}$ such that:*

a) *the image of ϕ is an infinitely-generated abelian group,*
b) *ϕ is additive on any band-sum of links,*
c) *the first coordinate of ϕ is the Sato–Levine invariant,*
d) *the class of a boundary link vanishes under ϕ.*

Property *d*) distinguishes our invariants from the signature invariants of Tristram, which can be used to construct a function satisfying *a*) and *b*) [27]. In addition, we show that all of our invariants, *including that of Sato–Levine are invariants of I-equivalence.*

In higher dimensions, if there is a single link with a non-vanishing β^i, then a corresponding theorem holds for $\mathscr{C}_n$. To this date, no such link is known, although these new invariants provide new hope (see 5.10, 6.10) for detecting the first higher-dimensional link which is not cobordant to a boundary link.

In accomplishing the afore-mentioned we define a "derivative" and "antiderivative" on the set of links. These notions promise to be of significant interest beyond their immediate application in this paper, and are related to work of R. Hain, R. Porter, and D. Sullivan [11, 20, 25].

§2. The basic definitions and notation

A (*spherical*) *n-link* is an ordered pair (M, K) of disjoint, oriented, smooth submanifolds of S^{n+2}, each component of which is diffeomorphic to S^n. A (*manifold*) *n-link* is the obvious generalization where (M, K) may be any ordered pair of closed, oriented, connected n-manifolds. Two n-links $L_0=(M_0, K_0)$ and $L_1=(M_1, K_1)$ are *CAT-cobordant* if there exists a proper, locally-flat, oriented CAT-submanifold (Y, W) of $S^{n+2}\times I$ which is CAT-homeomorphic to $(M_0, K_0)\times I$ and such that $\partial(Y, W)=L_0\cup(-L_1)$. (Note: CAT = DIFF, PL, or TOP). When we write simply *cobordant* we shall mean DIFF-cobordant. TOP-cobordism without the local flatness is called *I-equivalence. Null-cobordant* will mean cobordant to the trivial link. A *Seifert surface* V for a component M of an n-link (M, K) is an orientable, connected, smooth submanifold of S^{n+2} whose boundary is M. A *boundary link* is one whose components bound *disjoint* Seifert surfaces.

We shall work in the smooth (DIFF) category unless specifically noted (although the invariants, at least for $n=1$, apply to PL-links and are invariant under I-equivalence). A small open regular neighborhood of A in B will be denoted $\mathcal{N}(A)$ and the exterior of A $(B-\mathcal{N}(A))$ will be denoted $E(A)$.

§3. Admissible links and weak cobordism

We are primarily and eventually concerned with *spherical* links and cobordism classes of such. Nonetheless certain types of *non-spherical* links and certain weaker "cobordism" relations arise naturally *from* spherical links. In fact, in §4, we shall define a derivation process $D(\ )$ which will often take us out of the spherical category, and which will be invariant under a weaker "cobordism" relation than that of link cobordism.

The type of manifold link which arises shall be called *admissible.*

DEFINITION. *A (manifold) link $L=(M, K)$ is an* admissible link *if K is an n-sphere and if K has a Seifert surface which misses M.*

PROPOSITION 3.1 (2.1 of [23]). *Let $L=(M, K)$ be a manifold link in S^{n+2}. The following are equivalent:*

i) *K has a Seifert surface which misses M*
ii) *$[K]=0$ in $H_n(S^{n+2}-M)$.*

Note that a link (M, K) in S^3 is admissible if and only if $\text{lk}(M, K)=0$. A higher-dimensional link (M, K) is admissible if and only if K is a sphere and the "inclusion" $H_1(M)\to H_1(E(K))\cong\mathbb{Z}$ is the zero map. (Sato's 2.1 also proves this [23]). Note also that if (M, K) is admissible, then M automatically has a Seifert surface in $E(K)$ since $H_n(E(K))\cong 0$ when K is a sphere $(n>1)$. This motivates the following notion (used also in [2, 23]).

DEFINITION. *(V, Z) is a* special Seifert pair *for (M, K) if V is a connected Seifert surface for M in $E(K)$, Z is a connected Seifert surface for K in $E(M)$, and V meets Z transversely.*

Clearly then, special Seifert pairs exist for admissible links.

All of our invariants will be invariants under a weaker equivalence relationship than strict cobordism; it will be called weak-cobordism.

DEFINITION. *Two admissible links* $L_i=(M_i, K_i)$ $i=0,1$ *are* weakly-cobordant *(denoted* $L_0 \sim L_1$*) if there are Seifert surfaces* Z_i *for the* K_i *in* $E(M_i)$ $i=0,1$, *and a proper, oriented,* $(n+1)$*-dimensional submanifold* (Y, W) *of* $S^{n+2}\times I$ *such that*

a) $\partial(Y, W)=L_0 \amalg (-L_1)$

b) $W \cong S^n \times I$

c) *the closed* $(n+1)$*-manifold* $Z_0 \cup W \cup (-Z_1)$ *bounds a compact, orientable* $(n+2)$*-manifold* Z *in* $E(Y)$ *and* $Z \cap E(M_i)$ *is* Z_i *for* $i=0,1$.

Thus, the spherical component K is required to vary by a true concordance but the manifold component M is allowed to vary by an arbitrary "cobordism" subject to c). The proof of the following is similar to that of 2.1 of [23].

PROPOSITION 3.2. *Condition* c) *above is equivalent to either of*:

i) $[Z_0 \cup W \cup (-Z_1)]$ *is zero in* $H_{n+1}(E(Y))$

ii) *the map* $H_1(Y) \to H_1(E(W)) \cong \mathbb{Z}$ *is zero.*

It is convenient to say that L_0 is *weakly-cobordant* to L_1 *via* (Y, W, Z_0, Z_1) where (Y, W, Z_0, Z_1) is as above. Then Proposition 3.2 has the following useful corollary.

PROPOSITION 3.3. *If* $L_0 \sim L_1$ *via* (Y, W, Z_0, Z_1), *and* Z_i' *are Seifert manifolds for the* K_i *in* $E(M_i)$ $i=0,1$, *then* $L_0 \sim L_1$ *via* (Y, W, Z_0', Z_1').

Proof. Since the M_i are connected, $H_{n+1}(E(M_i))=0$ for $i=0,1$. Thus $Z_0 \cup W \cup Z_1$ is homologous to $Z_0' \cup W \cup Z_1'$ in $H_{n+1}(E(Y))$. $\square$

§4. Derivatives of links

We shall define an operation $D(\ )$ on the set of weak-cobordism classes of admissible links which will *a fortiori* be an operation on the cobordism classes. By iterating this *derivation*, we will produce a sequence of links associated to the original link, and our invariants will be the Sato–Levine invariants of this sequence. First, we require some preliminaries. Suppose that (M, K) is an admissible link. A pair of *arcs* (γ, ι) is called *admissible* if they are disjointly embedded in $E(L)$, joining (M, K) to the basepoint $*$.

THEOREM 4.1. *If* (M, K) *is an admissible link in* S^{n+2} *and* (γ, ι) *is an admissible pair of arcs, then a special Seifert pair* (V, Z) *may be chosen so that*

i) $V \cap Z = F$ *is connected, non-empty, and non-separating in* V *unless* L *is a boundary link,*

ii) $(V \cup Z)$ *intersects the arcs only at their initial points.*

Furthermore, in the case $n = 1$, *given* any *Seifert surface* $\bar{Z}$ *for* K *in* $E(M)$ *which misses the arc interiors, there is a special Seifert pair* $(V, \bar{Z})$ *satisfying* ii) *and*

iii) $V \cup \bar{Z}$ *is the union of parallels of a single simple closed curve on* $\bar{Z}$.

Proof. We shall sketch a proof for the case $n = 1$. First note that whenever 2 sheets of an immersed surface meet transversely in a circle, this singularity may be removed by a "cut-and-paste" operation. Now choose any special Seifert pair for (M, K) which meets the arc interiors transversely. For each sheet of V or Z which hits an arc, remove a small disc and run a tube back along the arc to connect-up to a small torus about M or K. These tubes and torii can be nested to avoid intersections. Any singularities created by tubing a sheet of V with a torus about ∂V may be then desingularized as above. Having achieved ii), the further modifications to V and Z will take place in a small neighborhood of $\operatorname{int}(V \cup Z)$ so that ii) will be undisturbed.

The *oriented* 1-manifold $G = V \cap Z$ (we will choose a convention in the paragraph following this proof) corresponds to a homotopy class of maps $[g]: Z \to S^1$ where $g^{-1}(*) = G$ and thus to a cohomology class y. This class is a non-negative multiple of a primitive class x, which is represented by a simple closed curve on Z. Consequently there is a map $f: Z \to S^1$ which is homotopic to g, such that $f^{-1}(*)$ is the appropriate number of parallels of this curve. It follows that there is a compact surface in $Z \times [-1, 1]$ which intersects $Z \times \{\pm 1\}$ in $\pm G$ and intersects $Z \times \{0\}$ in these parallels which we call F. The insertion of *this* $Z \times [-1, 1]$ in place of our present neighborhood of Z results in a new Seifert surface for M (still called V) which meets Z in F. Only the component of V containing M should be retained. This completes the proof of iii). Now perform this procedure on $V \cap Z$ as a submanifold of V. It must "stabilize" at some point because the procedure decreases the number of components of $V \cap Z$. It is then a small exercise (in orientability) that F must be connected. □

The intersection F of the Seifert surfaces guaranteed by 4.1 is somehow characteristic of the link itself, so we shall call it a *characteristic intersection* of the admissible link (M, K). Notice that it is a closed, *connected*, orientable n-manifold like M itself. In fact, it will be viewed as *oriented* according to the following convention. Beginning with orientations on S^{n+2} and the link (M, K), the special Seifert pair (V, Z) acquires an orientation (σ_v, σ_z) using the "inward normal last".

Choose oriented normal one-fields $(\vec{v}, \vec{z})$ to (V, Z) so that $(\sigma_v, \vec{v})$ and $(\sigma_z, \vec{z})$ yield the orientation on the ambient sphere. Now look at F as a submanifold of Z and choose σ_F so that $(\sigma_F, \vec{v})$ gives σ_z at that point. A technical theorem insures that $\vec{v}$ can be chosen tangent to Z over F. Notice that if (M, K) is a boundary link then F can be taken to be a tiny unknotted sphere far away from K.

We can now define our "derivative operation" for admissible links.

DEFINITION. *The* derived link $D(L)$ *of an admissible link* $L=(M, K)$ *is the weak-cobordism class of the (manifold) link* (F, K) *where* F *is any characteristic intersection for* L.

Since F is naturally embedded in Z, a Seifert manifold for K, it may be pushed off slightly, thereby exhibiting $D(L)$ as an *admissible* link. *Henceforth, the word link will be used for admissible link, and if spherical link is meant, it will be so specified.*

The following theorem, the cornerstone of the paper, shows that $D(L)$ is well-defined *and* that it is *independent of the weak-cobordism* class of L.

THEOREM 4.2. *Suppose that* $L_i=(M_i, K_i)$ $i=0, 1$ *are weakly-cobordant links and that* (V_i, Z_i) *are special pairs yielding characteristic intersections* F_i *for* $i=0, 1$; *then* (F_0, K_0) *is weakly-cobordant to* (F_1, K_1).

Proof. Suppose that $L_0 \sim L_1$ via (Y, W, Z_0', Z_1'). It follows from 3.3 that $L_0 \sim L_1$ via (Y, W, Z_0, Z_1). Referring now to diagram 4.2, let Z be the $(n+2)$-manifold which $Z_0 \cup W \cup (-Z_1)$ consequently bounds in $E(Y)$. Since $W \cong S^n \times I$, $H_{n+1}(E(W))$ is trivial and so, by Proposition 3.2, $V_0 \cup Y \cup (-V_1)$ bounds an $(n+2)$-manifold V in $E(W)$ such that $V \cap E(K_i) = V_i$ for $i=0, 1$. We can assume that V intersects Z transversely in the compact, *oriented* $(n+1)$-manifold F whose oriented boundary is $F_0 \amalg (-F_1)$. Since Z will have a trivial normal bundle, F can be pushed off of Z slightly, thereby exhibiting that $(F_0, K_0) \sim (F_1, K_1)$ via (F, W, Z_0, Z_1). □

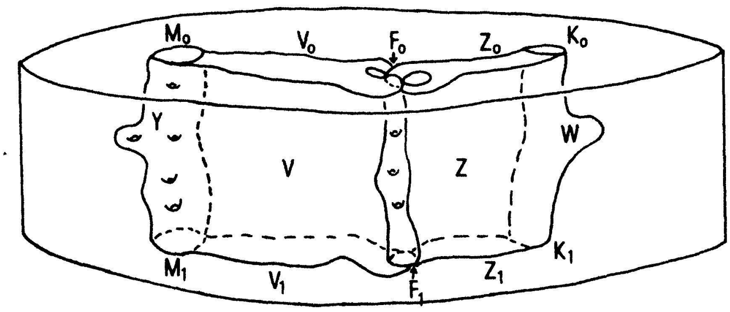


Figure 4.2.

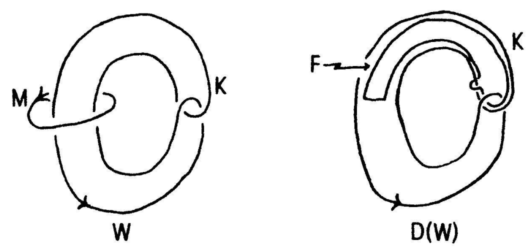


Figure 4.4.

COROLLARY 4.3. *Given any link L, the sequence of links $L, D(L), D(D(L)), \ldots$ (obtained by iterated derivatives) is well-defined in the category of links modulo weak-cobordism.*

Let us examine a few examples of link derivatives. The Seifert surfaces are suppressed but the reader should try to fill them in.

EXAMPLE 4.4 (Figure 4.4). If $W = (M, K)$ denotes the Whitehead link, then $D(W)$ is represented by the unlink (F, K). This may also be observed by noting that (F, M) is the unlink, and $(F, M) = D(K, M) = D(M, K)$ since W is symmetric.

EXAMPLE 4.5 (Figure 4.5). The links $\mathcal{M}_n$ are due to Milnor [18], and illustrate that derived links are often easy to compute. The successive characteristic intersections are shown by the dashed lines. These were obtained by using the obvious genus one surface for M (missing K) and *either* of the fairly natural choices for a Seifert surface for K. It is easier to use the one which is a de-singularization of the apparent ribbon intersections (except the right-most clasp). Since F_n can be isotoped to the right as shown, we see that $D^{n+1}(\mathcal{M}_n)$ is represented by the unlink just as for W above. Notice also that $D(K, M)$ is (F_1, M) which is not (apparently) the same as $D(M, K) = (F_1, K)$. In general, $D(K, M)$ is *not* weakly-cobordant to $D(M, K)$. Thus, since in S^3 a link (M, K) is admissible if and only if (K, M) is, one can generate another invariant sequence $\{D_*^i(M, K)\}$ where $D_*(M, K) \equiv D(K, M)$. One might wonder whether or not "mixed derivatives" are defined. Unfortunately in general they are not, although some interesting positive results concerning these notions are included in §9.

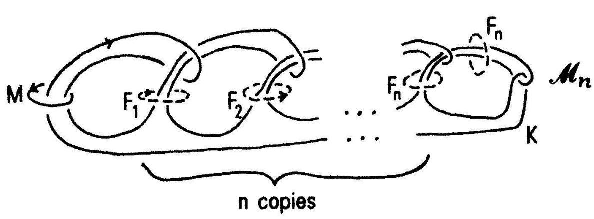


Figure 4.5.

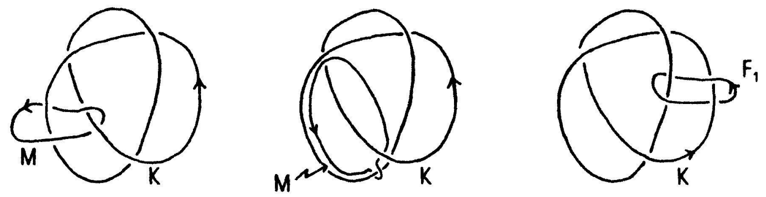


Figure 4.6.

EXAMPLE 4.6 (Figure 4.6). In the previous examples, $D^i(L)$ was eventually represented by the trivial link; whereas here the sequence is constant. Diagram (b) shows that M can actually be viewed as a push-off of a curve on the natural genus one Seifert surface for K. The link $D(L)=(F_1, K)$ is isotopic to (M, K), hence $D^i(L)=L$ for all i. Let us note that the two links in Figure 4.7 have the same $\bar{\mu}$-invariants ([14, 15, 24]) the same Alexander polynomial, and the same Sato–Levine invariant. Yet, their derived series are very different since the right-hand link is identical to 4.6(a). These facts can be used to distinguish their concordance classes quite easily (see §5).

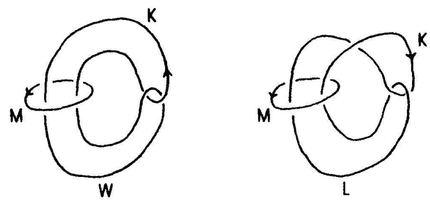


Figure 4.7.

These examples will take on more significance when we have defined our invariants in the next section.

In closing, let us remark that it would be computationally convenient to allow an admissible link (M, K) to have a *disconnected* first "component" M, for then the characteristic intersection F could be allowed to be disconnected. All other definitions would remain the same. Besides avoiding the technical Theorem 4.1, this would allow for easier computation of the successive derivatives. Unfortunately, although much of the theory holds in this broader context, there *are* certain problems whose resolution requires more effort than the effort to force F to be connected. Nonetheless, we state the appropriate results in this broader context.

THEOREM 4.8. *Derivatives may be computed for (M, K) using* disconnected *characteristic intersections as long as one uses the same Seifert surface Z for K in each successive computation. In particular, if M is connected, and a generalized derivative $\mathscr{D}^n(L)$ is defined using* disconnected *characteristic intersections, but always using Z as Seifert surface for K, then $\mathscr{D}^n(L)=D^n(L)$, i.e., they are weakly-cobordant.*

Note: Proposition 3.3 fails for M_i disconnected.

§5. The invariants

Given *any* weak-cobordism invariant $\beta(\ )$, we could now produce a *sequence* of weak-cobordism invariants $\{\beta^i(\ )\}$ by setting $\beta^1(L)=\beta(L)$ and $\beta^i(L)=\beta^{i-1}(D(L))$ for $i>1$. We shall investigate such a sequence, with the weak-cobordism invariant $\beta(L)$ being the *Sato–Levine invariant.* To define the latter, consider any characteristic intersection F for L, together with the natural framing of its normal bundle given by the normal 1-fields $(\vec{v}, \vec{z})$. By the Thom–Pontryagin construction $(F, \vec{v}, \vec{z})$ corresponds to an element of $\pi_{n+2}(S^2)$.

DEFINITION. *The Sato–Levine invariant $\beta(L)$ of an admissible link L is the element of $\pi_{n+2}(S^2)$ given by $(F, \vec{v}, \vec{z})$.*

Remarks. It is important to note that *F need not be connected* to compute the Sato–Levine invariant [23]. The invariant β is defined for an even larger class of manifold links than admissible links (Sato calls them *semi-boundary links*). Furthermore, β is invariant under an extremely weak equivalence relationship, called β-equivalence, which essentially allows "weak-cobordisms" on both components. In fact, Sato has shown that the β-equivalence classes of semi-boundary links form a group which is isomorphic to $\pi_{n+2}(S^2)$ via the invariant β [23]. In contrast, we shall show that the set of links in S^3 modulo weak-cobordism maps onto an *infinitely-generated* abelian group, with similar results for the classes of *admissible* links in S^n, $n>3$. These answer a question of Sato (§0. of [23]).

THEOREM 5.1. *If L is admissible, then $\beta(L)$*
a) *is a weak-cobordism invariant (thus a fortiori a cobordism invariant),*
b) *vanishes if L is a boundary link,*
c) *is symmetric (i.e. $\beta(M, K)=\beta(K, M)$).*

Proof. Theorem 4.1 of [23] proves a) since weak-cobordism is a stronger relationship than β-equivalence. Statements b) and c) follow from the definitions.

COROLLARY 5.2. *The sequence $\{\beta^i(\ ), i=1,2,\ldots\}$, given by $\beta^1(-)=\beta(-)$ and $\beta^i(-)=\beta^{i-1}(D(-))$, is a sequence of cobordism invariants on the category of spherical links. These invariants vanish if L is cobordant to a boundary link. More generally, all of the above hold in the category of admissible links modulo weak cobordism.*

Proof. Suppose that m is the least integer such that $\beta^m(\ )$ is *not* a weak-cobordism invariant. Then there are admissible links L, L' which are weakly-cobordant but such that $\beta^m(L)\neq\beta^m(L')$. It follows from 4.3 that $D(L)\sim D(L')$ and thus, by choice of m, that $\beta^{m-1}(D(L))=\beta^{m-1}(D(L'))$. Hence $\beta^m(L)=\beta^m(L')$, contradicting the existence of such an m. The more restricted statement concerning spherical links follows immediately. Finally, if L' is a boundary link, then it is clear that each $D^i(L')$ can be represented by a boundary link (split link for $i>0$). Since $\beta^i(L')$ is equal to the Sato–Levine invariant of $D^{i-1}(L')$, Theorem 5.1b) ensures that it vanishes. □

Let us re-examine the examples of §4 and compute their β^i. Referring to 4.4, $\beta^1(W)$ is seen to be -1 by computing the linking number of F with a push-off of F into the Seifert surface used for K. Thus $\beta^i(W)$ is zero except for $i=1$. On the other hand, the Sato–Levine invariant of $\mathcal{M}_n$ (4.5) is zero if $n\geq 1$, and in fact $\beta^i(\mathcal{M}_n)$ vanishes if $i<n+1$ and $\beta^{n+1}(\mathcal{M}_n)=-1$. Finally, note that $\beta^{n+1}(K, M)$ (example 4.5) is zero, confirming the essential asymmetry of the invariants.

The Sato–Levine invariant of Example 4.6 is calculated by computing the linking of F_1 with its own push-off in the Seifert surface for K. For this link, $\beta^i(L)=-1$ for all i.

The links $-W$ and L of Figure 4.7 have the same $\bar{\mu}$-invariants and the same Sato–Levine invariant, but their weak-cobordism classes are distinguished by β^2.

EXAMPLE 5.3. Suppose that $L=(M, K)$ is a *semi-fibered* (spherical) *link* in S^4. This is one in which K is a fibered knot in S^4 with fiber Z^3 such that M is disjoint from this copy of the fiber. Theorem 5.4 of [2] shows that, if V is a Seifert surface for M, then $V\cap Z$ is the (possibly disconnected) surface F which represents a *spherical* homology class in $H_2(Z)$. As a result it can be arranged that F is actually a union of embedded spheres in Z. Thus, in the sense of 4.8, each $D^i(L)$ will be a semi-fibered spherical link, and since spheres cannot carry the non-zero element of $\pi_4(S^2)$, it follows that $\beta^i(L)=0$ for all i (see Theorems 5.4 of [2] and 4.7 of [3]).

The most striking property of the β^i is that they are additive under band connected-sum of links. This is surprising because, although band-sum makes the set of concordance classes of (spherical) *knots* into a group, it is not well-defined on the set of concordance classes of spherical *links*. We do not even know if band-sum is well-defined on weak-cobordism classes of admissible links. Nonetheless the β^i are additive and it is this fact which allows us to say something about concordance classes of spherical links "modulo" boundary links.

DEFINITION. *If $L_j=(M_j, K_j)$ are admissible links (sitting inside different copies of S^{n+2}) and $b_j=(\gamma_j, \rho_j)$ are pairs of admissible arcs for the L_j $j=0, 1$, then $L_0\#_bL_1$ is defined to be the link $(M_0\#_\gamma M_1, K_0\#_\rho K_1)$ gotten by oriented "band-summing".* (We have suppressed data associated to the "twisting" in the bands.)

THEOREM 5.4. *If $L_j=(M_j, K_j)$ are admissible links and b_j are pairs of admissible arcs, for $j=0,1$, then $\beta^i(L_0\#_b L_1)=\beta^i(L_0)+\beta^i(L_1)$ for $i\geq 1$.*

Proof. The theorem is true for $i=1$ because Theorem 4.1 shows that special Seifert surfaces may be chosen to avoid the arcs, and Sato has shown under these conditions that his invariant is additive [23]. Lemma 5.5 below will show that $D(L_0\#_b L_1)\sim D(L_0)\#_c D(L_1)$ for some other band data "c". Then

$$\beta^n(L_0\#_b L_1)=\beta^{n-1}(D(L_0\#_b L_1))=\beta^{n-1}(D(L_0)\#_c D(L_1))$$

is the induction step in a proof that each β^i is additive.

LEMMA 5.5. *With L_j, b_j as above, $D(L_0\#_b L_1)$ is represented by $D(L_0)\#_c D(L_1)$ where $c_0=(q_0,\rho_0)$ and $c_1=(q_1,\rho_1)$.*

Proof. Suppose (V_j, Z_j) are the special Seifert pairs for $L_j=(M_j, K_j)$ $j=0,1$ ensured by Theorem 4.1. Assume that neither L_j is a boundary link since those other cases follow easily. Then the characteristic intersections F_j are non-separating in their respective V_j. A special Seifert pair for $L_0\#_b L_1$ may be taken to be $(\bar V,\bar Z)=(V_0\#_\gamma V_1, Z_0\#_\rho Z_1)$ where these denote the boundary-connected-sums along the arcs $\gamma=\gamma_0\cup\gamma_1$ and $\rho=\rho_0\cup\rho_1$; but $\bar V\cap\bar Z=F_0\amalg F_1$ does not satisfy the conditions to be a characteristic intersection since it is disconnected. However, since the F_j are non-separating, they can be joined by arcs ω_j (in V_j) to the initial points of their respective γ_j, in such a way that an *oriented* band-sum can be carried out along the union of $q_0=\gamma_0\cup\omega_0$ and $q_1=\gamma_1\cup\omega_1$. As in the proof of 4.1, the Seifert manifold $\bar Z$ may now be surgered along this arc so that the new $\bar V\cap\bar Z$ is $F_0\#_q F_1$. By definition then, $D(L_0\#_b L_1)=(F_0\#_q F_1, K_0\#_\rho K_1)=(F_0,K_0)\#_c(F_1,K_1)$ as desired. □

We can now say something about the structure of $\mathscr{C}_n$, the set of cobordism classes of spherical n-links (if $n=1$, require $\operatorname{lk}(M,K)=0$), by saying something about $\mathscr{W}\mathscr{C}_n$, the set of weak-cobordism classes of *spherical* n-links (same restriction). Let $\mathscr{P}_n(\pi_{n+2}(S^2))$ be the abelian group of formal power series in the variable x with vanishing constant term and with coefficients in $\pi_{n+2}(S^2)$. There is a natural "derivation" $\partial:\mathscr{P}_n\to\mathscr{P}_n$ given by $\partial(x)=0$ and $\partial(x^{i+1})=x^i$ if $i>0$. The following theorem expresses our major results for $n=1$ and gives partial answers to Cameron Gordon's questions 21 and 22 of [9].

THEOREM 5.6. *There are commutative diagrams of functions:*

$$\begin{array}{ccccccc} \mathscr{C}_1 & \twoheadrightarrow & \mathscr{W}\mathscr{C}_1 & \xrightarrow{B} & \mathscr{P}_1(\mathbb{Z}) & \xrightarrow{\pi}\!\!\!\!\rightarrow & \pi_3(S^2)\cong\mathbb{Z} \\ & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \partial} & & \\ & & \mathscr{W}\mathscr{C}_1 & \xrightarrow{B} & \mathscr{P}_1(\mathbb{Z}) & & \end{array}$$

where $B(L)=\sum_{i=1}^{\infty}\beta^i(L)x^i$ *and* π *is projection to the first factor, such that*

a) B *is additive with respect to any band-sum of links,*

b) *the image of* B *is an infinitely-generated, torsion-free abelian group,*

c) B *sends any boundary link to zero.*

Proof. The links $\mathcal{M}_n$ of 4.5 form a sequence $L_j=\mathcal{M}_{j-1}$ for $j=1,2,\ldots$ such that $\beta^i(L_j)=-\delta_{ij}$ (-1 or 0 according as $i=j$ or not). Since B is additive, it follows that, for any $r\in\mathbb{Z}^+$, the image of B contains a copy of $\times_{i=1}^{r}\mathbb{Z}$. □

COROLLARY 5.7. *There is a surjection* $\mathscr{C}_1\xrightarrow{B}G$ *where* G *is an infinitely-generated torsion-free abelian group, such that* B *is additive with respect to any band-sum and sends any class containing a boundary link to zero.*

Remarks

(5.8) In the next section it will be shown there are representative links in $\mathscr{C}_1$ with *unknotted* components whose images generate the $\mathbb{Z}^\infty$ subgroup.

(5.9) The image of B cannot be all of $\mathscr{P}(\mathbb{Z})$ for cardinality reasons, and indeed we shall see that $B(L)$ must be the power series expansion of a *rational* function in x.

(5.10) If $n\geq 2$, there is no known spherical link L with $\beta(L)\neq 0$; however there *are admissible* links (M,K) in each dimension realizing non-zero classes under the Sato–Levine invariant [23]. Thus, even if β vanishes for all higher-dimensional spherical links, the higher β^i may not, since $\beta^2(L)\equiv\beta(D(L))$, and $D(L)$ is often *non-spherical.* As of this writing, we have been unsuccessful in realizing the aforementioned (M,K) as the derived link of a *spherical* link. To read more on the Sato–Levine invariant of spherical 2-links, the reader may consult [2].

§6. Antiderivatives and realizability

We shall show that *antiderivatives* exist in the category of weak-cobordism classes of links and that these behave as expected. Antiderivatives of links are then used in realization theorems involving our invariants, and in proving theorems like 5.6.

DEFINITION. *L' is an antiderivative of the admissible link L if $D(L') \sim L$, and then L' is written $\int L$. The link L' is called an α-antiderivative of L if $L' = \int L$ and $\beta^1(L') = \alpha$ in $\pi_{n+2}(S^2)$. This is denoted $\int_\alpha L$. As with $D(L)$, $\int L$ should be understood to be a weak-cobordism class.*

THEOREM 6.1. *If $L = (M, K)$ is an admissible n-link with special Seifert pair (V, Z), then L has a 0-antiderivative $\int_0 L = (M', K)$. Furthermore,*

1) *if $n = 1$, L has an m-antiderivative $\int_m L = (M', K)$ for any $m \in \pi_3(S^2)$,*

2) *if L is spherical then the above (M', K) is spherical with M' unknotted and with special Seifert pair $((S^1 \times S^n)^0, (Z \# S^1 \times S^n)^0)$ where ^{0} denotes the punctured manifold.*

Proof. The desired link (M', K) is obtained by a procedure akin to "doubling" the component M (see [21]). M bounds V and this induces, as in §4, a normal 1-field $\vec{v}$ to V which can be completed to a trivialization $(\vec{v}, \vec{w})$ of the normal-bundle of M in S^{n+2}. Let M_+ be a push-off of M along $\vec{v}$ (for part one, use an "m" push-off instead of this "0" push-off). Let A be the manifold thus spanned ($A \cong M \times I$). Referring now to Figure 6.2, choose an arc γ which begins on M_+, runs along A to M, intersects Z transversely in a single point very near K, and then ends shortly thereafter. The direction from which γ cuts Z involves the orientation convention and we will not belabor this. Now thicken γ along its entire length by a factor of B^n, remaining tangent to A and Z where they intersect. Then thicken this $\gamma \times B^n$ by a final $I = [-1, 1]$ factor. We will only be concerned with the "top half", or $\gamma \times B^n \times [0, 1]$ as shown in 6.3. There is now an embedded $(n+1)$-manifold V' in $E(K)$ consisting of $A \cup (\gamma \times B^n \times \{0\}) \cup (\gamma \times B^n \times \{1\}) \cup (\partial\gamma \times B^n \times [0, 1])$. In case M is a sphere, V' is a punctured $S^1 \times S^n$, but in any case the boundary of V' is a connected n-manifold M' in $E(K)$. Notice that $Z \cap M'$ is $(\gamma \cap Z) \times \partial B^n$, an $(n-1)$-sphere which we call S. There is another manifold $\bar{M}$ whose boundary is S, which is constructed by taking the boundary

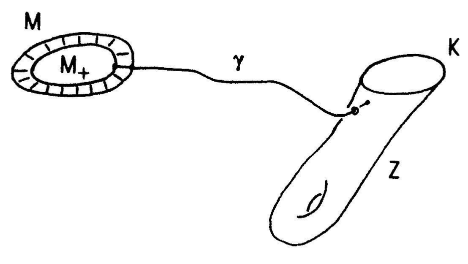


Figure 6.2.

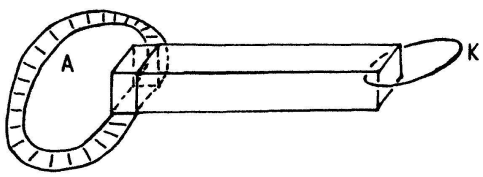


Figure 6.3.

$(n-1)$-sphere of $M-(\gamma\times B^n\times\{0\})$ and running it along $\gamma\times\partial B^n\times\{0\}$ until it hits S. A thickening $B^2\times \bar{M}$ may then be used to perform an ambient "surgery" on Z. Let $Z'=Z-(B^2\times S)\cup(\partial B^2\times \bar{M})$, a Seifert surface for K which misses M' and which, if M were a sphere, would be homeomorphic to a punctured $Z\#(S^1\times S^n)$. Refer to Figure 6.4. The link $L'=(M',K)$ is thus admissible and $D(L')$ is represented by $(V'\cap Z',K)$. Upon examination, the first component of this link is easily seen to be isotopic (fixing K) to a push-off of M into A. Hence, L' is an antiderivative of L and $\beta^1(L')$ is the class of $(M,\vec{v},\vec{w})$ in $\pi_{n+2}(S^2)$. Since M bounds V in S^{n+2}, this class is certainly zero. If $n=1$ and we had used the "m" push-off, then β^1 would be m. □

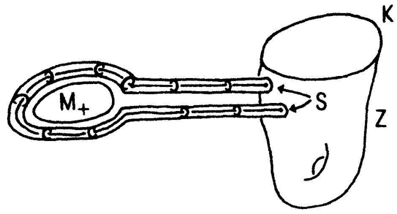


Figure 6.4.

The next theorem, concerning links in S^3, is philosophically satisfying. It shows that *antiderivatives are* "unique up to their Sato–Levine invariants", that is that any two $\int_c L$ are weakly-cobordant. This implies that, *on the category of links whose derivatives are eventually boundary links, the invariants* β^i *determine the weak-cobordism class.* The proof fails for $n>1$.

THEOREM 6.5. *If* $D(L)\sim D(L')$ *and* $\beta^1(L)=\beta^1(L')$ *for* 1-*links* L, L' *then* $L\sim L'$.

Proof. The method of proof is embodied in Figure 6.6, which is valid if neither L nor L' is a boundary link. The idea is to use copies C_+ and C_- of the given weak-cobordism (C,W) to "surger" the Seifert surfaces for M and M' along copies of the characteristic intersections F and F'. The linking number of a *push-off* of F into the Seifert surface with F *itself* is given by $\beta^1(L)$. Thus $\beta^1(L)=\beta^1(L')$ is used to extend the choices of push-off (F_+,F'_+) to a push-off C_+. □

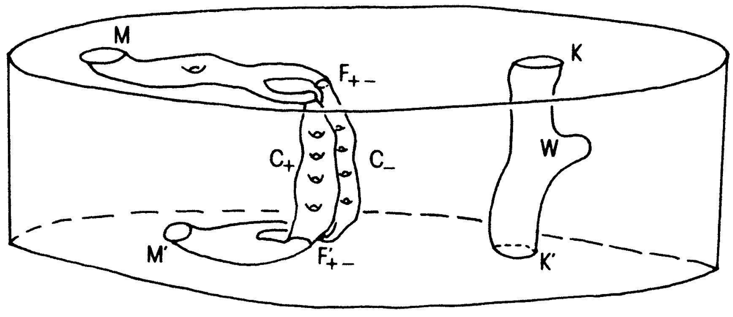


Figure 6.6.

Theorems 6.1 allows for realizability results which imply, in the presence of *any* non-vanishing β^i, the same sort of infinitely-generated behavior that we observed in Theorem 5.6 and Corollary 5.7.

COROLLARY 6.7. *If L is a spherical n-link with $\beta^k(L)=C$, then there is a sequence of spherical links L_j, $j=k, k+1, k+2, \ldots$, each with one component unknotted, such that*

$$\beta^i(L_j)=\begin{cases}0 & i<j\\ C & i=j\\ \beta^{i-j}(L) & i>j.\end{cases}$$

Proof. Simply let $L_{k+r}=\int_0\int_0\cdots\int_0 L$ (the r^{th} antiderivative).

COROLLARY 6.8. *There are links W_n, $n=1,2,\ldots$, in S^3 with unknotted, unlinked components such that $\beta^i(W_n)=-\delta_{in}$ (delta function).*

Proof. Let W_1 be the Whitehead link of Figure 4.4 and $W_n=\int_0 W_{n-1}$ thereafter. Figure 6.9 shows the 0-antiderivative of the Whitehead link (constructed as in the proof of 6.1).

Since Sato and Levine have shown that *any* $\alpha\in\pi_{n+2}(S^2)$ can be realized as $\beta(L)$ for a non-spherical, admissible link L [23], we can use Theorem 6.1 to get the following higher-dimensional analogue of Theorem 5.6.

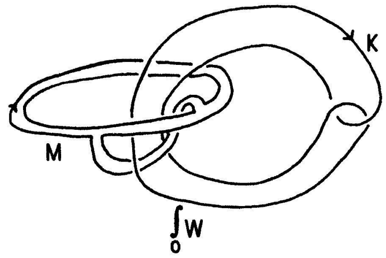


Figure 6.9.

COROLLARY 6.10. *Let $\mathcal{W}\mathcal{C}\mathcal{A}_n$ be the weak-cobordism classes of admissible n-links for $n \geq 1$. Then there is a map $B: \mathcal{W}\mathcal{C}\mathcal{A}_n \to \mathcal{P}_n(\pi_{n+2}(S^2))$ which is defined as in* 5.6 *and enjoys properties* a) *and* c) *of that theorem. Furthermore there are links L_j^α, $j = 1, 2, \dots$ such that $B(L_j) = \alpha x^j$ where $\alpha \in \pi_{n+2}(S^2)$.*

§7. Kojima's η-function and invariance under I-equivalence

Since our invariants $\beta^i(L)$ $i = 1, 2, \dots$ naturally yield a power series $B(L) = \sum_{i=1}^{\infty} \beta^i(L)x^i$, one wonders if they are related to other known "polynomial invariants". In fact, the theorem below shows that, for links in S^3, $B(L)$ is identical to the η-*function* of Kojima [14] after a non-trivial change of variable. This observation allows us to use Kojima's result that the η-function is invariant under TOP-cobordism (I-equivalence) to deduce that the β^i are also. Since Kojima's function is a generalization of invariants considered by D. Goldsmith [8], and has a (rather abstruse) connection to some of Laufer's invariants [15], the same may be said of our invariants. This relationship also implies that $B(L)$ is the power series of a *rational* function of x and that if K is unknotted, for example, then $B(L)$ is a polynomial (finite). Conversely, Kojima's function can now be seen to be additive on band-sums and to be an invariant only of weak-cobordism. Finally, in general it is difficult to calculate the η-function. (§2. of [14]), in as much as it involves constructing an infinite cyclic cover of $E(K)$ and the Alexander polynomial of K. On the contrary, the β^i may be calculated in S^3 and the complexity of calculation, we feel, grows much more slowly (with link complexity) than for the η-function and similar covering invariants.

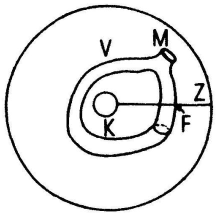
Let us define the η-function of an admissible 1-link $L = (M, K)$. Let Y be the infinite cyclic cover of $E(K)$, $\lambda(t)$ be the symmetrized Alexander polynomial of K, z be a lift of M to Y, z_0 be a nearby lift of the zero push-off of K, and t_* be a generator of the covering translations. Then $\lambda(t_*)$ annihilates the class of z in $H_1(Y)$, so $\lambda(t_*)(z) = \partial d$ for some 2-chain d in Y. The η-function is:

$$\eta(L) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda(t)} (z_0 \cdot t_*^n d) t^n.$$

The following theorem holds for links in S^3.

THEOREM 7.1. *Kojima's η-function may be expanded in powers of $x = (1-t)(1-\bar{t})$ so that $\eta(L) = \sum_{i=1}^{\infty} a_i x^i$ where $\beta^i(L) = a_i$ for all i. Thus*

a) *the η-function is an invariant of weak-cobordism and is additive on any band-sum,*



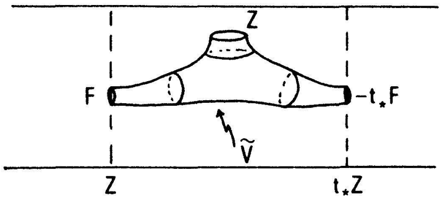


Figure 7.2.

b) *$B(L)$ is an invariant of I-equivalence and is the power series of a rational function,*

c) *If $L=(M,K)$ and a lift of M vanishes in $H_1(Y;\mathbb{Q})$, then $B(L)$ is a (finite) polynomial.*

Proof. We only sketch the proof. First one shows that for any embedded circles w, y in Y satisfying $w \cap t_*^n y = \phi$, there is a "linking" $\langle w, y\rangle$ defined as above so that $\eta(L) = \langle z_0, z\rangle$, and that this "pairing" is "sesqui-linear" and conjugate-symmetric (with respect to $t \to \bar{t}$). Figure 7.2 shows that $z \sim (1-t_*)F$ in $H_1(Y)$ where F is a lift of the characteristic intersection curve. Therefore

$$\langle z_0, z\rangle = (1-t)(1-\bar{t})\langle F_+, F\rangle = x\langle F_+, F\rangle$$

where F_+ is the push-off of F *in* Y normal to $\tilde{V}$. If F_0 is a lift of the zero push-off of F in S^3, then $\langle F_+, F\rangle$ can be seen to equal $\langle F_0, F\rangle + \beta^1(L)$. Once having shown that any $\langle w_0, w\rangle$ can be written as a power series in positive powers of x, it follows that $\beta^i(L) = a_i$ for all i. Theorem 2 of [14] insures that each β^i is invariant under I-equivalence (although defined only for PL links). It can be shown that both $\sum_{n=-\infty}^{\infty}(z_0 \cdot t_*^n d)t^n$ and $\lambda(t)$ are *polynomials* in x with integral coefficients, which shows that $B(L)$ is rational. If c) holds then there is a constant a such that $d = (\lambda(t)/a)d'$ where $\partial d' = a(z)$. Hence, $\eta(L)$ is a polynomial. □

EXAMPLE 7.3 (Figure 7.3). The link pictured has the same Murasugi "2-height" as the unlink (see [19]) and has vanishing Sato–Levine invariant. In fact, it is a $\mathbb{Z}_2$-boundary link. Yet it is not I-equivalent to a boundary link since $\beta^2(M,K) = -4$. Its derivative is shown on the right.

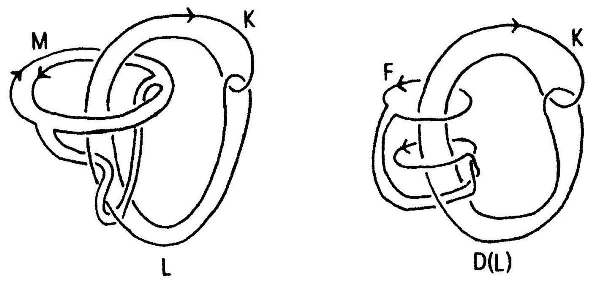


Figure 7.3.

§8. Calculations in S^3

There is an algebraic method of computing β^i for large i which is amenable to computer calculations. Suppose that $[M]=x$ is a primitive class in $H_1(S^3-F)$ where F is a Seifert manifold for K (there always exists such an F by 4.1). Let Q be the matrix representing the Seifert pairing (p. 200 of [21]) corresponding to a symplectic basis for $H_1(F)$. Let A be the inverse of the Mayer–Vietoris isomorphism:

$$H_1(F) \xrightarrow{v_+ - v_-} H_1(S^3-F).$$

Then the n^{th} characteristic intersection can be taken to be an embedded curve on F representing $A^n x$ and $\beta^n(L)=\pm(A^n x \cdot A^{n+1}x)$. Furthermore, if $P=Q-Q^T$ is the block diagonal matrix then $A=-PQ$, so β^i can be computed *solely from* $x=[M]$ and Q.

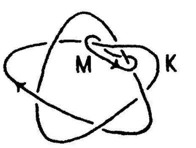


Figure 8.1.

It follows that if F is genus one then $\beta^i(L)=\pm\beta^1(L)$ for all i. It can also be calculated that the absolute value of β^{49} of the link in Figure 8.1 is greater than 10^{20}, so it appears unlikely that $\{\beta^i\}$ is always bounded. This could be confirmed by calculating the η-function of the link.

§9. Generalizations and further applications

The invariants we have discussed may be generalized in several directions. The first of these would be to links of 3 components. Here, there is a Sato–Levine invariant associated to the 3-component link, as well as an invariant associated to each 2-component sub-link. The generalization of our notion of derivative could take several forms, and we shall not pursue this.

Secondly, there is the possibility of more (and deeper) invariants. Specifically, the reader has no doubt noticed that the asymmetry of the derivation D leads to two independent sequences of invariants. For if $L=(X, Y)$ is a link, then we could define $D_Y(L)=(V_X\cap V_Y, Y)$ and $D_X(L)=(X, V_X\cap V_Y)$. Iterations of either can be used to define a sequence of concordance invariants. But what about "mixed derivatives"? Unfortunately $D_XD_Y(L)$ is *not*, in general, invariant under concordance of L; but this seems to be because this is *not* the proper generalization. It seems to be more productive to fix L and compute successive intersections always using one of either V_X or V_Y. This leads to a sequence of characteristic intersections which is indexed by the set of sequences of X's and Y's. The self-linkings and linkings of these characteristic intersections are invariants, in a certain sense, of the original link. In a subsequent paper, we shall fully develop these invariants and shall relate them to Milnor's $\bar{\mu}$-invariants. For now, we include the following result, in order to under-score the relationship of our invariants with the lower central series of the fundamental group G of the exterior of a classical link L whose components (M, K) have zero linking number. Recall that the lower central series $G_n, n = 1, 2, \ldots$ of a group G is defined inductively by $G_1 \equiv G$ and $G_i \equiv [G, G_{i-1}]$. Milnor defines his $\bar{\mu}$ invariants in [18].

THEOREM 9.1. *The following are equivalent*:
a) $\beta^1(L) = 0$,
b) *the longitudes of* L *lie in* $G_4 = [G, [G, [G, G]]]$,
c) *the first* $\bar{\mu}$*-invariant*, $\bar{\mu}(1122)$, *vanishes.*

Proof. We shall prove only a)$\Rightarrow$b). We shall need a small lemma. Suppose $\alpha \in G$ is represented by an embedded curve in $S^3 - V_M - V_K$ and that F is a characteristic intersection of L.

LEMMA 9.2. *If* $\operatorname{lk}(\alpha, F) = 0$ *then* $\alpha \in G_3$.

Proof of Lemma 9.2. The hypotheses insure that α (as a curve) has a Seifert surface V in $S^3 - L - F$. It follows that $H_1(V)$ has a symplectic decomposition $A \oplus B$ where $i_*(B)$ is in G_2 (ignoring basepoints). Thus $\alpha \in G_3$.

From Figure 9.3 we see that $l = (l\gamma^{-1})\gamma = (l\gamma^{-1})[m, F]$ where $\operatorname{lk}(m, K) = 1$. If we ignore basepoints and apply the lemma to F^+, we see that F^+ (and hence F) lies in G_3 if $\operatorname{lk}(F^+, F) = \beta^1(L) = 0$. Thus we need only show that $l\gamma^{-1}$ lies in G_4. There is an obvious embedded curve ω representing $l\gamma^{-1}$ which bounds a surface S in $S^3 - V_M - V_K$. The surface S is simply a push-off of a sub-surface of V_M. Examining the homomorphism $\phi: H_1(S) \to \mathbb{Z}$ given by $\phi([a]) = \operatorname{lk}(a, F)$, it can be deduced that there exists a set $\{a_1, a_2, \ldots, a_n\}$ of embedded curves on S which are in the kernel of ϕ, and such that $l\gamma^{-1} = \prod_{i=1}^n [b_i, a_i]$. By the preceding lemma, each a_i lies in G_3 so $l\gamma^{-1}$ is in G_4. □

The final generalization of the invariants would be to study classes of boundary links modulo "boundary cobordism" (see [1]). For example, if $W(M)$ stands

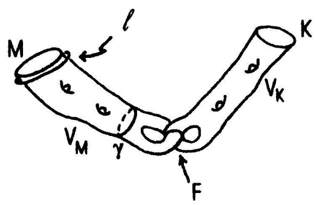


Figure 9.3.

for either 0-twisted Whitehead double of M (a knot in S^3), and K is such that $\mathrm{lk}(K, M) = 0$, then for the boundary-link $(W(M), K)$ to be boundary-null-cobordant, all of the $\beta^i(M, K)$ must vanish. It is not clear to me that the invariants in this context would provide any more information than various signature invariants.

§10. Questions

1. Is there a link $L = (M, K)$ in S^3 such that both (M, K) and (K, M) are weakly-cobordant to boundary links but L is not cobordant to a boundary link?

2. Do the vanishing of the $\beta^i(L)$ imply that L is weakly-cobordant to a boundary link? (True if some $D^i(L)$ is a boundary link.)

3. Is band-sum well-defined on weak-cobordism classes of 1-links? If so, and 2) is true, then the monomorphism $\psi: \mathcal{W}\mathcal{C}_1 \to \mathbb{Z}^\infty \times \theta_1$ given by $\psi(L) = (\beta^i(L)$, cobordism class of 2$^{\text{nd}}$ component of L) tells us *exactly* what $\mathcal{W}\mathcal{C}_1$ is.

4. Is there a higher-dimensional spherical link with a non-vanishing β^i?

5. Is there a classical link L which is not weakly-cobordant to a boundary link but such that $L \#_b L$ is? (Yes here implies No to 2).)

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