# Reduction theory using semistability.

# Reduction theory using semistability*

DANIEL R. GRAYSON

## Introduction

The purpose of this paper is to develop some new techniques for proving theorems about arithmetic groups. Strong theorems about these infinite groups of matrices were proved by Borel and Serre [1973], including finite presentation of the group, finite generation of the cohomology and homology, and a form of generalized Poincaré duality. Their technique is to produce a compact aspherical manifold with boundary whose fundamental group is isomorphic to the arithmetic group; then they show the boundary of the universal cover is homotopy equivalent to the Tits building, which yields precise knowledge about its homology groups.

We will produce the required manifold in a slightly different way, still arriving at the same end results. Whereas Borel and Serre adjoin a boundary to an open manifold, we will instead delete an open neighborhood of infinity. Of course, for such a neighborhood we could simply choose a collar of Borel and Serre's boundary, but this choice is not canonical. Our approach appears to be as canonical and explicit as possible, and is independent of Borel and Serre's results. In fact, it amounts to a reworking of part of reduction theory for quadratic forms, as developed by Minkowski [1896, 1911], Hermite [1905], Siegel [1957], and Borel [1966]. We dispense with the "Siegel sets", and replace them with the study of "semistability" for lattices in Euclidean space. Roughly speaking, this involves the part of reduction theory which provides lower bounds on lengths of vectors in lattices. The other, more classical part of reduction theory is concerned with getting upper bounds on lengths of shortest vectors in lattices, and with combining upper and lower bounds to prove finiteness assertions.

The first advance along these lines was made by Stuhler [1976, 1977]. Serre [1977] and Quillen [see Grayson, 1982] used the notion of semistable vector bundle on an algebraic curve to study $Sl_n(\mathcal{O})$ when $\mathcal{O}$ is a Dedekind domain

finitely generated over a finite field. Stuhler knew of the analogy between function fields and number fields formulated by Weil [1939], and was inspired to apply it here. He saw that some work of Harder and Narasimhan [1975] on stable vector bundles carries over, and leads to new facts about lattices in Euclidean space.

One way to reformulate Stuhler's result is this. Let $L$ be a lattice in Euclidean space. The real span of any nonzero subgroup $M$ of $L$ is again an inner product space, so it makes sense to speak of the (nonzero) covolume vol $M$ of $M$ in its span, regardless of the dimension of $M$. Now plot (for all $M$) the points (dim $M$, log vol $M$) in the $(x, y)$ plane. These points are bounded below, so their convex hull is bounded below by a certain convex polygon. The result is this: each vertex of the polygon is represented by a unique subgroup $M$; moreover, the subgroups representing the vertices form a chain. This chain is dubbed the Harder–Narasimhan canonical filtration of $L$ because it is analogous to the filtration Harder and Narasimhan obtain for a vector bundle on a projective algebraic curve.

In this paper we use the canonical filtration to undertake a more detailed study of the structure of the space of lattices in a fixed Euclidean space. The idea is to check that more of the work of Serre and Quillen for function fields can be transferred, by analogy, to number fields. Imagine moving $L$ in the space of lattices; motion towards infinity (towards a cusp) can be detected by a decrease in the angles at the vertices of the convex polygon of $L$, and the canonical filtration itself tells us in which direction we are moving off toward infinity. We make this precise: we get functions which measure the distance from infinity, and use them to determine the open neighborhoods of infinity, the deletion of which gives a manifold. These neighborhoods are small enough so that they do not change the homotopy type of the manifold, and they allow proving that the boundary of the universal cover is homotopy equivalent to the Tits building. This allows recovering all the results of Borel–Serre.

In the first part of the paper we consider $Gl_n\mathcal{O}$, where $\mathcal{O}$ is a ring of integers in a number field. This should be useful to $K$-theorists as a way of simplifying the proof that the higher $K$-groups $K_i\mathcal{O}$ are finitely generated. The reader may easily modify the arguments of this paper to apply to $Sl_n\mathcal{O}$.

In section 7 we consider orthogonal groups. This makes use of some ideas of Atiyah and Bott, who discuss semistability for principal $G$-bundles on a Riemann surface.

In the work of Atiyah and Bott, $G$ is any semisimple Lie group. Thanks to Ofer Gabber, I know how to express the symmetric space of maximal compact subgroups of $G$ as the space of those inner products on the Lie algebra of $G$ which differ from the Killing form by a Cartan involution. I hope to be able to present this case in a future paper.

An interesting incidental consequence of this work is a more "intrinsic" construction of the Borel–Serre "geodesic action" (and their boundary, although we don't pursue that here) for $Gl_n\mathcal{O}$; here intrinsic means without reference to the chosen basis of $\mathcal{O}^n$.

## Acknowledgements

## 1. Lattices

Let $\mathcal{O}$ be the ring of integers in an algebraic number field $F$. Let $\infty$ be an archimedean place of $F$, and let $F_\infty$ be the completion of $F$ at $\infty$. The $\mathbb{R}$-algebra $F_\infty$ is either equal to the real numbers $\mathbb{R}$ or is isomorphic to the complex numbers $\mathbb{C}$ (in one of two ways).

If $V_\infty$ is a finite dimensional $F_\infty$-vector space, then an inner product on $V_\infty$ is a positive definite bilinear form $V_\infty \times V_\infty \to F_\infty$, which is required to be symmetric if $\infty$ is real and to be hermitian if $\infty$ is complex. When equipped with an inner product, $V_\infty$ is called an inner product space.

We define an $\mathcal{O}$-lattice $L$ to be a projective $\mathcal{O}$-module $P$ of finite rank equipped with an inner product on each of the vector spaces $V_\infty = P \otimes_\mathcal{O} F_\infty$. We adopt the notations $V = L \otimes_\mathbb{Z} \mathbb{R}$ and $V_\infty = L \otimes_\mathcal{O} F_\infty$. We will maintain this notation later, and the addition of subscripts or superscripts will not interfere with the meanings of the letters $L$, $P$, and $V$. [The other logical (but too bulky) choice for the notation would be to mimic the notation for global sections of a sheaf over various open sets, for what we have resembles a sheaf on the topological space $\mathrm{Spec}\,(\mathcal{O}) \cup \mathrm{Spec}\,(\mathcal{O} \otimes_\mathbb{Z} R)$, the collection of all primes of $\mathcal{O}$, finite, infinite, and zero. Then $L$ would be the "sheaf" on the whole space, $P$ would be the sections over the finite primes, and $V_\infty$ would appear as the "stalk" at $\infty$.]

If $V_\mathbb{C}$ is a hermitian (complex) inner product space, then there is a procedure called *restriction of scalars* which makes it into a symmetric (real) inner product space $V_\mathbb{R}$. One simply takes the real part of the inner product and forgets the scalar multiplication by complex, nonreal, numbers. If $\{v, w, \ldots\}$ is an orthonormal basis for $V_\mathbb{C}$, then $\{v, iv, w, iw, \ldots\}$ is an orthonormal basis for $V_\mathbb{R}$.

There is also a procedure called *restriction of scalars* which makes an $\mathcal{O}$-lattice into a $\mathbb{Z}$-lattice. Notice that $V = \prod V_\infty$. Define an inner product on the real vector space $V$ by

$$\langle v, w \rangle = \sum_{\text{real}} \langle v_\infty, w_\infty \rangle + \sum_{\text{complex}} \text{Re} \langle v_\infty, w_\infty \rangle.$$

Let $f_* L$ denote the $\mathbb{Z}$-lattice obtained by equipping $L$, regarded as a $\mathbb{Z}$-module, with this inner product (at the unique infinite place of $Q$).

Let $L$ be an $\mathcal{O}$-lattice, and let $P$ denote the underlying $\mathcal{O}$-module. Any submodule $P_1 \subseteq P$ can be made into an $\mathcal{O}$-lattice by restricting the inner product on each $V_\infty$ to $V_{1,\infty} = P_1 \otimes_\mathcal{O} F_\infty$; call the resulting $\mathcal{O}$-lattice $L_1$, and write $L_1 \subseteq L$. We will use the notation $L_1 = L \cap P_1$. Assume now that $P/P_1$ is projective; then we say that $L_1$ is a sublattice of $L$. The orthogonal projections $p_\infty \colon V_\infty \to V_{1,\infty}^\perp$ provide isomorphisms $(P/P_1) \otimes_\mathcal{O} F_\infty \to V_{1,\infty}^\perp$ which can be used to make $P/P_1$ into an $\mathcal{O}$-lattice which we will call $L/L_1$. We say that $L/L_1$ is a quotient lattice of $L$, and that $E \colon 0 \to L_1 \to L \to L/L_1 \to 0$ is an exact sequence of lattices.

We do not define a notion of "morphism" of $\mathcal{O}$-lattices. The arrows in our diagrams will be simply maps of the underlying $\mathcal{O}$-modules. An isomorphism of $\mathcal{O}$-lattices is an isomorphism of underlying $\mathcal{O}$-modules which preserves the inner products at the infinite places.

Recall that a finitely generated $\mathcal{O}$-module is projective if and only if it has no torsion, and it doesn't matter whether we look for torsion by elements of $\mathcal{O}$ or of $\mathbb{Z}$. From this observation comes the following collection of trivia.

LEMMA 1.1. *Suppose $L$ is an $\mathcal{O}$-lattice.*

(a) *If $L_1$ is a sublattice of $L_2$, and $L_2$ is a sublattice of $L$, then $L_1$ is a sublattice of $L$.*

(b) *If $L_1 \subseteq L_2 \subseteq L$, and $L_1$ is a sublattice of $L$, then $L_1$ is a sublattice of $L_2$.*

(c) *If $L_1 \subseteq L \subseteq L_2$ are both sublattices of $L$, then $L_1 \cap L_2$ is a sublattice of $L$.*

(d) *If $L_1 \subseteq L$, then $L_2 \cap (L_1 \otimes_\mathcal{O} F)$ is a sublattice of $L$, and contains $L_1$ as a subgroup of finite index. If $L_1$ is, in addition, a sublattice, then $L_1 = L_2$.*

(e) *If $L_1 \subseteq L$ has volume minimal among volumes of submodules of $L$ of the same rank, then $L_1$ is a sublattice of $L$.*

LEMMA 1.2. (a) *Suppose $L, L'$ are $\mathcal{O}$-lattices with the same underlying module $P$. Then $f_* L = f_* L'$ iff $L = L'$.*

(b) *Suppose $E \colon 0 \to L_1 \to L \to L_2 \to 0$ is an exact sequence of $\mathcal{O}$-lattices. Then the inner products on $L_1$ and $L_2$ are uniquely determined by the inner products on $L$.*

(c) *Suppose $E$ is a sequence of $\mathcal{O}$-lattices, as above. Then $E$ is exact iff $f_* E$ is.*

*Proof.* (a) This part is clear, because a hermitian inner product can be recovered from its real part, or indeed from the associated norm.

(b) This is clear from the definitions.

(c) Suppose $E$ is exact. It is clear that $f_*L_1$ is a sublattice of $f_*L$. We must check that $f_*L \to f_*L_2$ comes from orthogonal projection. This has two ingredients: the first is that if $p_\infty$ is an orthogonal projection at a complex place, then it remains an orthogonal projection after restriction of scalars to $\mathbb{R}$. The second is that $V = \prod V_\infty$ is an orthogonal sum, and similarly for $V_2$; the orthogonal sum of orthogonal projections is still an orthogonal projection.

Now suppose $f_*E$ is exact. Then if $E$ isn't exact, we may modify the inner products on $L_1$ and $L_2$ to make it exact, obtaining a new exact sequence $E': 0 \to L_1' \to L \to L_2' \to 0$, with the same underlying $\mathcal{O}$-modules. Now $f_*E'$ is exact, so $f_*L_i' = f_*L_i$ for $i = 1, 2$. By part (a), $E' = E$, so is exact.  QED

Not all of the usual isomorphism theorems for $\mathcal{O}$-modules go through for $\mathcal{O}$-lattices, so we must be careful at this stage.

LEMMA 1.3. *If $L_1$ is a sublattice of $L$, and $L_2$ is a sublattice of $L_1$, then $L_1/L_2$ is a sublattice of $L/L_2$, and $(L/L_2)/(L_1/L_2) = L/L_2$.*

*Proof.* By Lemma 1.2, we may as well apply restriction of scalars to everything in sight, achieving the case $\mathcal{O} = \mathbb{Z}$. The first assertion is clear, and the second amounts to the fact that the composite of two orthogonal projections $V \to V/V_2 \to (V/V_2)/(V_1/V_2)$ is again an orthogonal projection.  QED

We let $rk\,(L)$ denote the $\mathcal{O}$-module rank of $L$, and let $\dim\,(L)$ denote the rank of $L$ as $\mathbb{Z}$-module. Of course, $\dim\,(L) = rk\,(L)\dim\,(\mathcal{O})$.

We define the volume of $L$, $\mathrm{vol}\,(L)$, to be the covolume of the lattice $f_*L$ inside its inner product space $V$. This may be computed as $|\det\,\langle l_i, e_j \rangle|$, where $\{l_i\}$ is a $\mathbb{Z}$-basis of $f_*L$, and $\{e_j\}$ is an orthonormal basis of $V$. Thus if $\dim L = 0$, then $\mathrm{vol}\,L = 1$. If $\dim L = 1$, then $\mathrm{vol}\,L$ is the length of a generator of $L$. If $\dim L = 2$, then $\mathrm{vol}\,L$ is the area of a fundamental parallelogram, and so on. It is worth reiterating that the volumes are not measured with respect to a fixed dimension, and they are always nonzero.

FACT 1.4. *If $L'$ is a submodule of finite index in $L$. then $\mathrm{vol}\,L' = [L:L']\,\mathrm{vol}\,L$.*

EXAMPLE 1.5. Take $L = \mathcal{O}$, and for each place $\infty$ declare $\{1\}$ to be an orthonormal basis of $V_\infty = F_\infty$. This makes $\mathcal{O}$ into an $\mathcal{O}$-lattice in a natural way, and it turns out that $\mathrm{vol}\,\mathcal{O} = 2^{-r_2}\sqrt{|d|}$, where $r_2$ is the number of complex places of $F$, and $d$ is the discriminant of $\mathcal{O}$. See Lang [1970, p. 115] for a proof.

EXAMPLE 1.6. Take $L = \mathcal{O}^n$, and give it inner products at each place. Let $z_\infty$ be the matrix of the inner product at $\infty$; then

$$\text{vol } L = \prod (\det z_\infty)^{e(\infty)/2} (\text{vol } \mathcal{O})^n$$

where $e(\infty) = [F_\infty : R]$.

*Proof.* If each $z_\infty = 1$, then the direct sum $L = \mathcal{O}^n$ is orthogonal, and the result is immediate.

Choose an orthonormal basis $\{e_{i,\infty}\}$ for $V_\infty = F_\infty^n$, and let $y_\infty$ be the $F_\infty$-automorphism of $V_\infty$ such that the standard basis vectors $b_i$ are given by $b_i = y \cdot e_{i,\infty}$. Let $y$ be the direct product of the $y_\infty$'s it is an $R$-automorphism of $V = \prod V_\infty$. If we let $L' = y^{-1}L$, then by the first line of the proof, we know $\text{vol}(L') = \text{vol}(\mathcal{O})^n$. Thus $\text{vol } L = |\det y| (\text{vol } \mathcal{O})^n = \prod |\det y_\infty|^{e(\infty)} (\text{vol } \mathcal{O})^n = \prod (\det z_\infty)^{e(\infty)/2}(\text{vol } \mathcal{O})^n$. The last equality comes from $z_\infty = {}^t Y_\infty \cdot Y_\infty$, where $Y_\infty$ denotes the matrix of $y_\infty$ with respect to the basis $\{e_{i,\infty}\}$. Notice also that $\det z_\infty > 0$. The middle equality makes use of the formula

$$\det_R h = |\det h|^2$$

which holds when $h$ is an endomorphism of a complex vector space, and $\det_R h$ denotes its determinant when considered as an endomorphism of the underlying real vector space; to prove it, one reduces to the case where the complex dimension is 1 by row and column reduction.

*Remark* 1.7. For any $\mathcal{O}$-lattice $L$, we can find its volume as follows. There is a sublattice $L' \subseteq L$ with the same rank as $L$, and which is free. Fix a basis for it, and for each $\infty$ let $z_\infty$ be the matrix of the inner product with respect to it. Then

$$\text{vol } L = \prod (\det z_\infty)^{e(\infty)/2}(\text{vol } \mathcal{O})^n/[L : L'].$$

LEMMA 1.8. *If $L$ is an $\mathcal{O}$-lattice and $L'$ is a sublattice of $L$, then* $\text{vol}(L) = \text{vol}(L') \cdot \text{vol}(L/L')$.

*Proof.* By restriction of scalars, we may assume that $\mathcal{O} = \mathbb{Z}$. Choose bases $\{l_i\}$ for $L'$ and $\{l_i\} \cup \{m_j\}$ for $L$. Choose orthonormal bases $\{e_i\}$ for $V'$ and $\{e_i\} \cup \{f_j\}$ for $V$, and let $p$ be the orthogonal projection onto $V'^\perp$. Then, omitting all subscripts

for clarity, we have

$$\text{vol}\,(L) = \left| \det \begin{pmatrix} \langle l, e \rangle & \langle l, f \rangle \\ \langle m, e \rangle & \langle m, f \rangle \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} \langle l, e \rangle & 0 \\ * & \langle pm, f \rangle \end{pmatrix} \right|$$

$$= \text{vol}\,(L') \cdot \text{vol}\,(L/L'). \quad \text{QED}$$

DEFINITION 1.9 [Stuhler, 1976, Definition 1]. The slope of a nonzero lattice $L$ is the number $(\log \text{vol}\,(L))/\dim L$, and can be thought of as the log of an average length. The log is thrown in solely to convert the multiplicativity of the volumes (provided by Lemma 1.8) into additivity. The slope is undefined when $L = 0$.

DEFINITION 1.10. Suppose we plot all submodules of a nonzero lattice $L$ as points in the plane, where the horizontal axis is the dimension, and the vertical axis is log vol. Call this plot the *canonical plot* of $L$. The slope of $L$ appears in this plot as the slope of the line segment joining 0 and $L$. The import of Lemma 1.8 is that slope $(L/L')$ appears in this plot as the slope of the line segment joining $L'$ to $L$ (see Figure 1.11). In fact, the canonical plot for $L/L'$ appears (translated) in the canonical plot for $L$ as those points represented by sublattices of $L$ containing $L'$.

If $A$ and $B$ are subgroups of an abelian group $C$, then a basic fact is that $A/A \cap B = A + B/B$. For lattices this is false, as can be seen in easy examples. Nevertheless, we can make do with a certain inequality for the volumes, which we now derive.
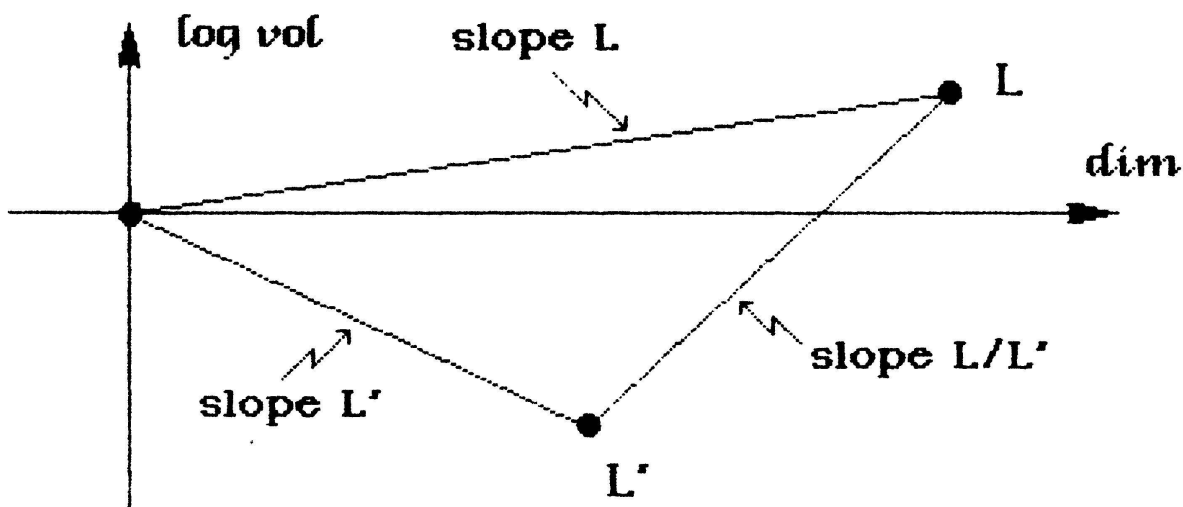


Figure 1.11

In the notation of the proof of Lemma 1.8, there is a formula

$$p\left(\sum a_i e_i + \sum b_j f_j\right) = \sum b_j f_j$$

for the orthogonal projection $p$. It follows that orthogonal projection is length decreasing, i.e. for all $v$, $|v| \geqslant |pv|$; but it also is volume decreasing in the following sense.

THEOREM 1.12 [Stuhler 1976, Proposition 2]. *Suppose $L$ is an $\mathcal{O}$-lattice, and $L_1$ and $L_2$ are sublattices. Then*

(i) $\mathrm{vol}\,(L_2/L_2 \cap L_1) \geqslant \mathrm{vol}\,(L_1 + L_2/L_1)$

*and*

(ii) $\mathrm{vol}\,(L_1 \cap L_2)\,\mathrm{vol}\,(L_1 + L_2) \leqslant \mathrm{vol}\,(L_1)\,\mathrm{vol}\,(L_2)$.

*Proof.* Part (ii) follows from part (i) together with Lemma 1.8. Let's show part (i). First, we may assume $\mathcal{O} = \mathbb{Z}$, by restriction of scalars. Let $P_1$ and $P_2$ denote the underlying modules. Now choose a filtration $P_1 \cap P_2 = Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n = P_2$ in which each subquotient $Q_i/Q_{i-1}$ is free with rank 1. By Lemma 1.8, we may replace $P_2$ by $Q_i$ and $P_1$ by $Q_{i-1} + P_1$, thereby achieving $\dim (L_2/L_1 \cap L_2) = 1$. We may also achieve $L_1 \cap L_2 = 0$ by replacing $L_i$ by $L_i/L_1 \cap L_2$ for $i = 1, 2$; this reduction uses Lemma 1.3. Now let $m$ be a generator for $L_2$, and let $p$ be the orthogonal projection fo $V_1$. Then $\mathrm{vol}\,(L_2) = |m| \geqslant |pm| = \mathrm{vol}\,(L_1 + L_2/L_1)$.  QED

*Remark*: One can use 1.4 and 1.1(d) to extend 1.12 to any pair of submodules, as Stuhler does.

DISCUSSION 1.13. Theorem 1.12 is fundamental – it mimics the equality for the dimensions:

$$\dim (L_1 \cap L_2) + \dim (L_1 + L_2) = \dim (L_1) + \dim (L_2).$$

We can interpret this in terms of the canonical plot of $L$, from Definition 1.10. Consider just the four points obtained from $L_1 \cap L_2$, $L_1 + L_2$, $L_1$, and $L_2$; any three of them determine a parallelogram, and then the theorem can be visualized as an assertion about the relationship of the fourth vertex of that parallelogram to the fourth point. If the fourth point comes from $L_1$ or $L_2$, then it lies at or above the corresponding vertex of the parallelogram (and on the same vertical line, by the equality for the ranks). If the fourth point is from $L_1 \cap L_2$ or $L_1 + L_2$, then it lies at or below the corresponding vertex of the parallelogram (and on the same vertical line). This situation is easily visualized: see Figure 1.14. We will call it the "parallelogram constraint."
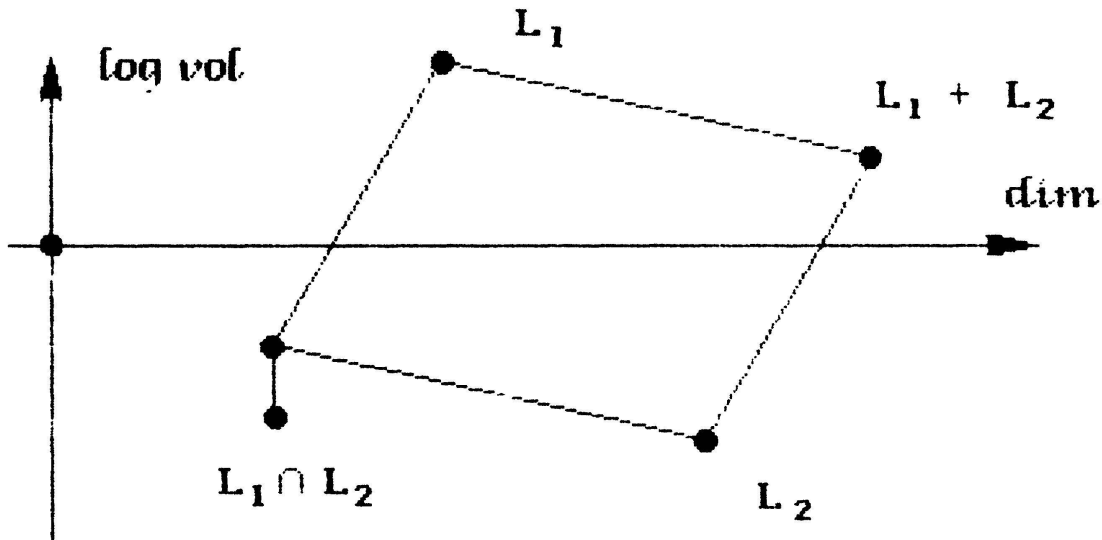
Figure 1.14

**LEMMA 1.15.** *Given a lattice $L$ and a number $c$, there exist only a finite number of submodules $L_1 \subseteq L$ with* $\mathrm{vol}\,(L_1) < c$.

*Proof* [compare Stuhler 1976, Proposition 1]. By restriction of scalars, we may assume $\mathcal{O} = \mathbb{Z}$. Choose an integer $r$ and require also that $\dim L_1 = r$. In case $r = 1$, finiteness follows from the fact that $L_1$ is discrete and the sphere of radius $c$ is compact. For $r > 1$ we may replace $L$ by $\Lambda^r L$ and each $L_1$ by $\Lambda^r L_1$. (As inner product on $\Lambda^r V$ we take the one satisfying $\langle (l_1 \cdots l_r), (m_1 \cdots m_r) \rangle = \det \langle l_i, m_j \rangle$.) The assignment $L_1 \mapsto \Lambda^r L_1$ is finite-to-one because $\Lambda^r L_1$ determines $L \cap (L_1 \otimes \mathbb{Q})$ and the index in it of $L_1$. We have $\dim \Lambda^r L_1 = 1$ and $\mathrm{vol}\,(L_1) = \mathrm{vol}\,(\Lambda^r L_1)$, so the finiteness for $r > 1$ follows from the finiteness for $r = 1$. QED

**DISCUSSION 1.16.** Lemma 1.15 tells us that the canonical plot of $L$ is bounded below. Thus the convex hull of the canonical plot of $L$ will be bounded on left and right by two vertical lines at $0$ and $\dim L$; it is unbounded above unless $L = 0$ (because $L$ has submodules of arbitrarily large finite index). Its lower boundary is a convex polygon stretching from the origin to the point corresponding to $L$: we call it the canonical polygon of $L$. The interesting thing for us will be to study its vertices, each of which, according to 1.15, is represented by a sublattice of $L$. By "vertex", we mean either an endpoint of the polygon, or a point on the polygon where the slope actually changes; points on the polygon represented by submodules may not be at vertices.

Suppose now that $L_1$ and $L_2$, submodules of $L$, are chosen to lie on the canonical polygon of $L$ in such a way that they do not both lie in the interior of the same straight line segment of the boundary (this will happen, for example, if either of them lies on a vertex). Since they have minimal volume for their ranks,
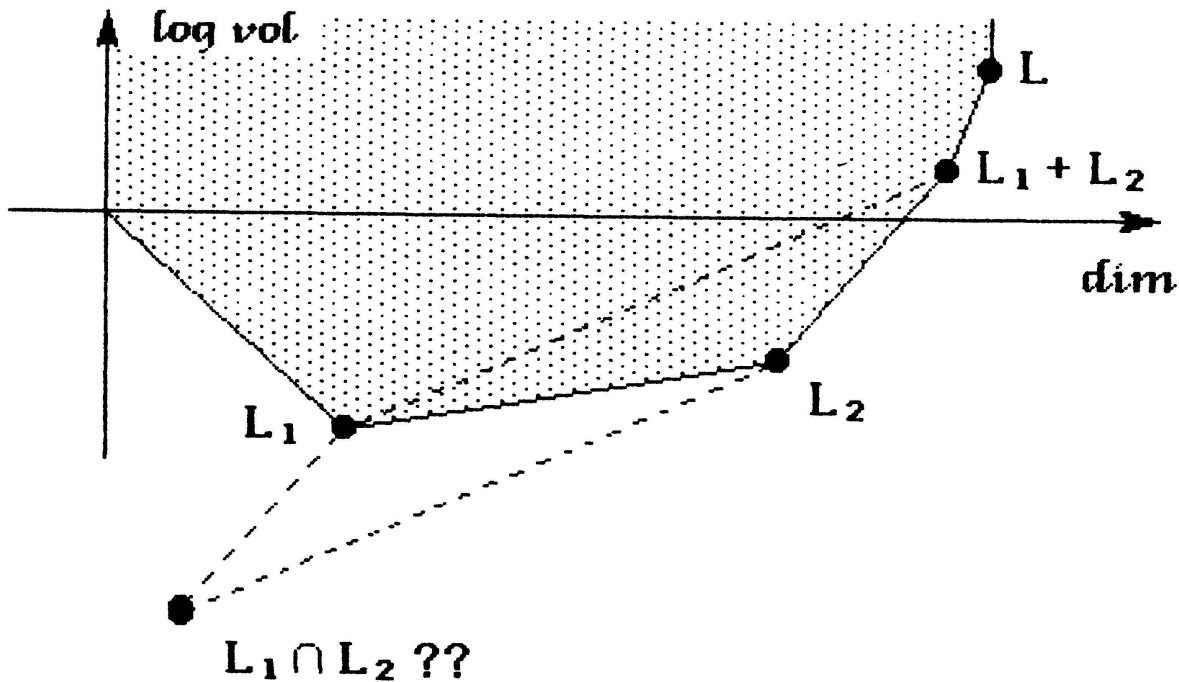
Figure 1.17

both $L_1$ and $L_2$ are actually sublattices of $L$, according to Lemma 1.1(e). Assume for the sake of definiteness, that $\dim L_1 \leq \dim L_2$. Then the slope of the segment of the polygon just to the right of $L_2$ is strictly steeper than the slope just to the left of $L_1$. This means that it is not possible that $\dim L_1 + L_2 > \dim L_2$, without violating the parallelogram constraint from Discussion 1.13, so therefore $\dim L_1 + L_2 = \dim L_2$ (see Figure 1.17). It follows that $L_1 + L_2 = L_2$, for else its volume would be strictly smaller than the volume of $L_2$. Thus we've shown that $L_1 \subseteq L_2$.

Now suppose $L_1$ and $L_2$ represent the same vertex. The preceding argument shows both $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$, so $L_1 = L_2$.

We have proved the following theorem.

THEOREM 1.18. *The vertices of the canonical polygon of $L$ are represented by unique sublattices of $L$, and they form a chain.*

DEFINITION 1.19. The filtration of $L$ consisting of those sublattices of $L$ which represent vertices of the canonical polygon of $L$, is called the *canonical filtration* of $L$. By convention, it always includes 0 and $L$. The canonical filtration is called canonical because it depends only on $L$, and not on any choices.

Theorem 1.18 and Definition 1.19 are roughly equivalent to [Stuhler 1976, Satz 1, Folgerung aus Satz 1].

DEFINITION 1.20. We say that $L$ is *semistable* if its canonical filtration contains only 0 and $L$ (i.e. its canonical polygon is a single line segment). In all other cases we say $L$ is *unstable*.

If rk $(L) = 1$, then $L$ is semistable. The successive subquotients of the canonical filtration of $L$ are all semistable, and their slopes are (strictly) increasing.

$L$ is semistable if and only if it satisfies the inequalities slope $M \geqslant$ slope $L$ for every submodule $M$.

*Remark* 1.21. It follows immediately from the definition that if $h : L \xrightarrow{\sim} M$ is an isomorphism of lattices (i.e. an isometry), then $h$ carries the canonical filtration of $L$ into the canonical filtration of $M$. It is this fundamental fact that enables equivariant constructions in the symmetric space in chapter 2.

OBSERVATION 1.22. The (finite) orthogonal group $G$ of $L$ leaves invariant the canonical filtration of $L$; the same applies if we tensor $L$ with the rationals. If $G$ acts irreducibly on $L$, then $L$ must be semistable. This happens, for example, if we take $L = f_* \mathcal{O}$ where $\mathcal{O}$ is the ring of integers in a cyclotomic field, because the roots of unity of $\mathcal{O}$ are in $G$. This gives an interesting explicit lower bound on volumes of subgroups of $f_* \mathcal{O}$.

DEFINITION 1.23. Let max $L$ denote the largest slope of a segment of the canonical polygon of $L$, and let min $L$ denote the smallest.

DIVERSION 1.24. If $r$ is a positive real number, then from $L$ we may produce a new $\mathcal{O}$-lattice called $L[r]$ by multiplying each of the norms on $L$ by $r$ (or equivalently, by multiplying the inner products by $r^2$). Clearly $f_*(L[r]) = (f_* L)[r]$, thus the following formulas hold.

$$\text{vol } L[r] = r^{\dim L} \text{ vol } L$$

$$\text{slope } L[r] = \log r + \text{slope } L$$

The canonical plot for $L[r]$ can be obtained from the canonical plot for $L$ by applying the affine transformation $(x, y) \mapsto (x, y + x \log r)$. This transformation preserves straight lines, and thus transforms the canonical polygon for $L$ into the canonical polygon for $L[r]$. It follows that $L$ and $L[r]$ have the same canonical filtration (as far as the underlying $\mathcal{O}$-modules are concerned), and the following formulas hold.

$$\max L[r] = \log r + \max L$$

$$\min L[r] = \log r + \min L$$

Another thing to notice is that the rescaling $L \mapsto L[r]$ preserves exact sequences of lattices; if it were a functor, we could call it an exact functor. It is analogous to tensoring a vector bundle with a power of a fixed line bundle (on an algebriac curve).

It is also possible to rescale the norms by independent factors at the infinite places; were we studying $Sl_n\mathcal{O}$ we would do this.

EXAMPLE 1.25. Consider $\mathbb{C} = \mathbb{R}^2$ as the Euclidean plane, and let $\mathcal{H}$ be the upper half plane. For any $t \in \mathcal{H}$ we may form the lattice $L = L(t) = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot t$. Let $\mathcal{D} = \{z : |z| \geqslant 1 \text{ and } |\mathrm{Re}\, z| \leqslant \frac{1}{2}\}$ be the usual fundamental domain for the action of $Sl_2\mathbb{Z}$ on $\mathcal{H}$. Assume that $t \in \mathcal{D}$; then it is clear that 1 is a vector of minimal length in $L$. Since $\mathrm{vol}\, L = \mathrm{Im}\, t$, it follows that $L$ is semistable if and only if $\mathrm{Im}\, t \leqslant 1$. The set $B$ of all $t \in \mathcal{H}$ such that $L$ is semistable is invariant under $\Gamma = Sl_2\mathbb{Z}$; for, given $g \in \Gamma$, we see easily that $L(gt) = zL(t)$ for some complex number $z$. Write $z = ru$, where $r$ is real and $|u| = 1$. Then $uL(t)$ and $L(t)$ are isomorphic lattices (the isomorphism is multiplication by $u$), and $ruL(t)$ has the same canonical filtration as $uL(t)$ by diversion 1.24.

We know now that $B$ is $\Gamma$-invariant, and we know its intersection with the fundamental domain $\mathcal{D}$. This allows us to determine $B$ – it is the complement of countably many disjoint open disks, namely all the translates of the half-plane $C = \{t \in \mathcal{H} : \mathrm{Im}\, t > 1\}$. See Figure 1.26: this is the same picture which appears in Rademacher's work on partitions [1973]. Many of these disks are tangent (at points corresponding to those lattices with two independent vectors of minimal length), so clearly $B$ is not a manifold (with boundary). If, however, we shrink $C$ slightly by strengthening the inequality in its definition to $\mathrm{Im}\, t > 1 + \varepsilon$, letting the
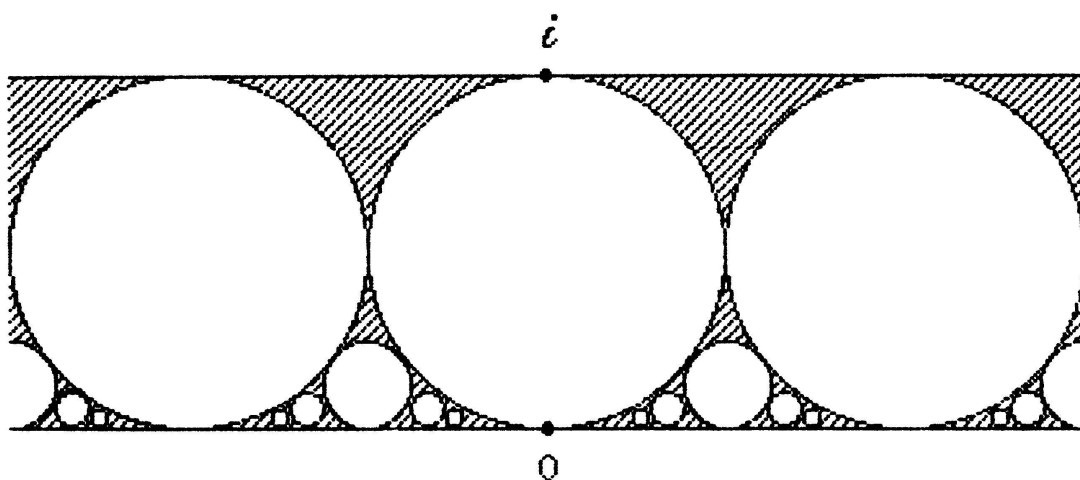


Figure 1.26

other disks shrink the same way, then the tangencies disappear, and this enlarge-ment of $B$ is a manifold with boundary. This was explained by Serre [1979], and it is this which I generalize to $Gl_n$ in the sequel.

DISCUSSION 1.27. Suppose $\mathcal{F}: 0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_s = L$ is a filtration by sublattices of an $\mathcal{O}$-lattice $L$. Consider the plot formed by plotting log vol and dim (as in Definition 1.10) for only those submodules $L'$ of $L$ such that $L_i \subseteq L' \subseteq L_{i+1}$ for some $i$; call it the *canonical plot of $L$ subordinate to the filtration $\mathcal{F}$*. Consider also, as before, the convex hull of this plot, and the corresponding convex polygon $C$ bonding it below. Suppose now that each $L_i$ happens to sit on $C$: I claim then that $C$ actually is the canonical polygon. It is equivalent to show that every sublattice $L'$ of $L$ lies on or above $C$, and this we can do by induction on $s$ (the case $s = 1$ being obvious). Consider $L' + L_{s-1}$ and $L' \cap L_{s-1}$: the former clearly lies on or above $C$, and the latter, by induction, does, too. Now $L_{s-1}$ is on $C$, which is convex, so the parallelogram constraint of discussion 1.13 tells us that $L'$ must be on or above $C$. See Figure 1.28.

The canonical filtration of $L$ will include those $L_i$ which sit at vertices of $C$. Notice that we didn't assume that each $L_i$ occurs at a vertex of $C$; thus this argument might easily lead to the conclusion that $L$ is semistable. Indeed, it follows that $\mathcal{O}^n$ is a semistable $\mathcal{O}$-lattice for any $n$, where $\mathcal{O}^n$ denotes the $n$-fold orthogonal sum of the $\mathcal{O}$-lattice $\mathcal{O}$ from example 1.5.

COROLLARY 1.29. *Suppose $L$ has filtration $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_s = L$ by*
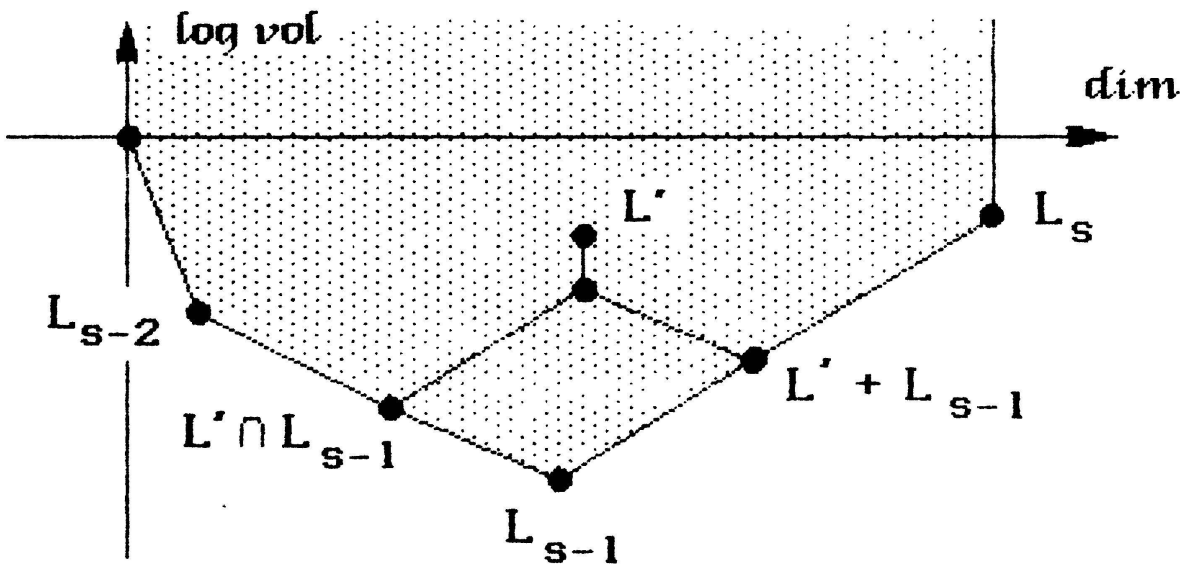


Figure 1.28

*sublattices, so that* $\min L_{i+1}/L_i \geq \max L_i/L_{i-1}$. *Then*

(a) *The canonical polygon of L is formed by laying the canonical polygons of the subquotients* $L_i/L_{i-1}$ *end to end.*

(b) *Each* $L_i$ *lies on the canonical polygon of L.*

(c) $\min L/L_i = \min L_{i+1}/L_i$.

(d) $\max L_i = \max L_i/L_{i-1}$

(e) *If* $\min L_{i+1}/L_i > \max L_i/L_{i-1}$, *then* $L_i$ *is the canonical filtration of L.*

(f) *If* $L_i \subsetneqq L' \subsetneqq L_{i+1}$ *and* $L'$ *is in the canonical filtration of* $L_{i+1}/L_i$, *then* $L'$ *is in the canonical filtration of L.*

(g) *The canonical filtration of L consists solely of sublattices arising as in* (e) *or* (f).

COROLLARY 1.30. *Suppose L has a filtration* $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_s = L$, *whose subquotients* $L_i/L_{i-1}$ *are semistable, with strictly increasing slopes. Then this filtration is the canonical filtration.*

COROLLARY 1.31. *Suppose* $L'$ *is a sublattice of L. Then* $L'$ *is in the canonical filtration of L is and only if* $\max L' < \min L/L'$.

## 2. Spaces of lattices

In this section we investigate the way the canonical filtration behaves when the lattice moves.

Let $P$ be a finitely generated projective $\mathcal{O}$-module of rank $n$, let $\Gamma = Gl(P)$. Let $\tilde{X} = \tilde{X}(P)$ be the space of lattices $L$ whose underlying $\mathcal{O}$-module is $P$. Let $\tilde{X}_\infty$ be the space of inner products on $V_\infty$; if a basis is chosen for $V_\infty$, then $\tilde{X}_\infty$ is seen to be an open subspace of a real or complex vector space. We have $\tilde{X} = \prod \tilde{X}_\infty$, and this provides us with a topology for $\tilde{X}$.

We consider $\Gamma$ to act on $P$ on the left. If a basis is chosen, our vectors will be thought of as column vectors, and matrices of linear maps will be written on the left, as usual.

Given $L \in \tilde{X}$ and $v, w \in V_\infty$, let $\langle v, w; L \rangle_\infty$ denote the value of the inner product on the vectors $v$ and $w$. If $g \in \Gamma$, we define a new lattice $gL$ in $\tilde{X}$ by the formula $\langle v, w; gL \rangle_\infty = \langle g^{-1}v, g^{-1}w; L \rangle_\infty$. This defines an action of $\Gamma$ on $\tilde{X}$ on the left.

It happens that there is an isomorphism $L \xrightarrow{\sim} gL$ of lattices defined by $v \mapsto gv$ which we may as well call g, also.

Conversely, suppose $g : L_1 \xrightarrow{\sim} L_2$ is an isomorphism of $\mathcal{O}$-lattices, each of which is in $\tilde{X}$. Since $P$ is the underlying module for both of them, g gives rise to an element of $\Gamma$, which we may also call g. We see clearly that $L_1 = gL_2$.

Thus the orbit set $\Gamma \backslash \tilde{X}$ can be regarded as the set of isomorphism classes of $\mathcal{O}$-lattices whose underlying $\mathcal{O}$-module is isomorphic to $P$. (It is the analogue of the moduli space for vector bundles on an algebraic curve, and will become compact once we throw out the unstable points.)

Scaling the norms commutes with changing the basis, i.e. if $r > 0$ and $g \in \Gamma$, then $g(L[r]) = (gL)[r]$. Let $X$ be the quotient of $\tilde{X}$ by the equivalence relation $L \sim L[r]$; it is clear that $X$ is a manifold. The difference between $\tilde{X}$ and $X$ becomes important only for assertions about compactness.

DEFINITION 2.1. By an $F$-subspace of $V$, we will mean either an $F$-subspace of $P \otimes_{\mathcal{O}} F$, or the real span of such a subspace in $V$: no confusion should result from this blurring of the distinction between an $F$-subspace and its real span, for each can be recovered from the other. If $L \in \tilde{X}$, then the sublattices $M$ of $L$ are in one-to-one correspondence with the $F$-subspaces $W \subseteq V$. We will use the notation $M = L \cap W$. If $0 \subseteq L \cap W_1 \subseteq \cdots \subseteq L \cap W_s = L$ is the canonical filtration of $L$, then we will refer to $0 \subseteq W_1 \subseteq \cdots \subseteq W_s = V$ also as the canonical filtration of $L$. For $F$-subspaces $W$ of $V$ we may define a real function $d_W$ on $\tilde{X}$ by the formula

$$d_W(L) = d(W, L) = \exp\left((\min L/L \cap W) - (\max L \cap W)\right)$$

This function is concocted so that, by corollary 1.31, $W$ occurs in the canonical filtration is and only if $d(W, L) > 1$ (and in that case, the canonical filtration for $L$ is obtained by splicing the canonical filtrations for $L \cap W$ and $L/L \cap W$). In terms of the polygon, $d(W, L) > 1$ iff $W$ is at a vertex, $d(W, L) = 1$ iff $W$ is in the interior of an edge, and $d(W, L) < 1$ iff $W$ is not on the polygon. A larger value of $d(W, L)$ corresponds to a more acute slope change at the vertex $W$. Moreover, $d(W, L[r]) = d(W, L)$, for any $r > 0$, so $d_W$ descends to a function on $X$. We may imagine that the larger $d(W, L)$ is, the further $L$ is out toward infinity; alternatively, $d(W, L)^{-1}$ measures the distance to the cusp corresponding to $W$.

For any $t \geqslant 1$, define $\tilde{X}_W(t) = \tilde{X}(W, t) = \{x \in \tilde{X} : d(W, x) > t\}$, and let $\tilde{X}_W = \tilde{X}(W, 1)$. Define $X(W, t)$ and $X_W$ similarly. We let $X_{ss}(t) = X - \bigcup_W X(W, t)$, and $X_{ss} = X_{ss}(1)$. We call $\tilde{X}_{ss}$ the semistable part of $\tilde{X}$, for its points are those $L$ which are semistable. For $t = 1$ and $\mathcal{O} = \mathbb{Z}$, these sets agree with those defined in [Stuhler, 1976]; what we prove in this case was already obtained by Stuhler.

We state the following easy lemma:

**LEMMA 2.1.**

(a) $d(gW, gL) = d(W, L)$, *for* $g \in \Gamma$.

(b) $X(gW, t) = gX(W, t)$, *for* $g \in \Gamma$.

(c) $X_{ss}(t)$ *is closed, and is stable under* $\Gamma$.

(d) *If* $X(W_1, t) \cap \cdots \cap X(W_n, t) \neq \varnothing$, *then* $\{W_1, \ldots, W_n\}$; *ordered by inclusion, is a chain.*

## 2.3. Examples

**EXAMPLE 2.3.1.** Take $\mathcal{O} = \mathbb{Z}$ and $n = 2$. Let $\mathcal{H} \subseteq \mathbb{C}$ be the upper half plane. Given $z \in \mathcal{H}$ we may embed $P = \mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ into $\mathbb{C}$ by sending $e_1$ to $z$ and $e_2$ to 1. The plane $\mathbb{C} = \mathbb{R}^2$, with its standard (real) inner product, makes $P$ into a lattice which we will call $L = L(z)$. The number $z$ can be recovered from $L(z)$ (forgetting the embedding into $\mathbb{C}$) because $\langle e_1, e_1 \rangle = |z|^2$ and $\langle e_1, e_2 \rangle = \mathrm{Re}\, z$. We can also extract $L(z)$ from its equivalence class up to scaling because of the normalization condition $\langle e_2, e_2 \rangle = 1$. This shows that the resulting map $\mathcal{H} \xrightarrow{\sim} X$ is a diffeomorphism. It turns out that $gL(z)$ and $L(gz)$ differ only by scaling, provided we define $gz = (dz - c)/(-bz + a)$. If we replace the usual action of $Sl_2\mathbb{Z}$ on $\mathcal{H}$ by its composite with transpose inverse, we may say that the map $\mathcal{H} \xrightarrow{\sim} X$ is equivariant.

Now suppose a $\mathbb{Q}$-subspace $W = \mathrm{span}\, (re_1 + se_2) \subseteq V$ is given, where $r$ and $s$ are relatively prime integers. Then $L \cap W$ corresponds to the subgroup $\mathbb{Z}(rz + s)$ of $\mathbb{C}$, so

$$\mathrm{vol}\, L \cap W = |rz + s|$$

$$\mathrm{vol}\, L = \mathrm{Im}\, z$$

$$d_W(z) = d(W, L(z)) = (\mathrm{vol}\, L/L \cap W)/(\mathrm{vol}\, L \cap W) = (\mathrm{vol}\, L)/(\mathrm{vol}\, L \cap W)^2$$

$$= (\mathrm{Im}\, z)/|rz + s|^2$$

If $r = 0$, then $|s| = 1$, so $d_W(z) = \mathrm{Im}\, z$ and $X(W, t) = \{z \in \mathcal{H} : \mathrm{Im}\, z > t\}$. If $r \neq 0$, then $X(W, t)$ is the open disk of diameter $1/(tr^2)$ tangent to the real line at the rational number $-s/r$. For $t = 1$, we recover the situation in example 1.25; for $t > 1$, the closed set $X_{ss}(t)$ is a manifold with boundary, and was described by Serre [1979].

The next two examples use Remark 1.7 implicitly for computing volumes.

**EXAMPLE 2.3.2.** This time take $\mathcal{O}$ quadratic imaginary, $n = 2$, and let $\mathcal{H}^3 = \{(z, w) \in \mathbb{C}^2 : \mathrm{Im}\, w = 0, \mathrm{Re}\, w > 0\}$ be the hyperbolic 3-space, sitting in $\mathbb{C}^2$ endowed with the standard hermitian inner product. Choose an embedding $\mathcal{O} \subseteq \mathbb{C}$.

Given $h = (z, w) \in \mathcal{H}^3$ we may embed $P = \mathcal{O}^2 = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ into $\mathbb{C}^2$ by sending $e_1$ to $h$ and $e_2$ to $(1, 0)$. This makes $P$ into an $\mathcal{O}$-lattice we call $L(h)$, and we get a diffeomorphism $\mathcal{H}^3 \overset{\sim}{\to} X$, equivariant for $Sl_2\mathbb{C}$. If we identify $\mathbb{C}^2$ with the quaternions $\mathbb{C} \oplus \mathbb{C}j$, and take

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $Sl_2\mathcal{O}$, the formula $g \cdot h = (dh - c)(-bh + a)^{-1}$ gives the action of $Sl_2\mathcal{O}$ on $\mathcal{H}^3$. We can repeat the discussion from Example 2.3.1 up to a point. We have

$\text{vol } L = w^2(\text{vol } \mathcal{O})^2$

$\text{vol } L \cap W = (\text{vol } L \cap \text{span } (re_1 + se_2))/I = |rh + s|^2 (\text{vol } \mathcal{O})/I$

$d(W, L(h)) = (\text{vol } L/L \cap W)^{1/2}/(\text{vol } L \cap W)^{1/2}$

$\qquad\qquad = (\text{vol } L)^{1/2}/(\text{vol } L \cap W)$

$\qquad\qquad = wI/|rh + s|^2$

Here $r$ and $s$ are chosen from $\mathcal{O}$ so that $L \cap W \supseteq \mathcal{O}(re_1 + se_2) \neq 0$, we identify $s \in \mathcal{O}$ with $(s, 0) \in \mathcal{H}^3$ so the expression $rh + s$ makes sense, and the integer $I$ is defined by $I = [\mathcal{O} : \mathcal{O}r + \mathcal{O}s]$. If $\mathcal{O}$ happens to be a principal ideal domain, we can always take $I = 1$ by choosing $r$ and $s$ properly. This is the same function used by Mendoza [1980], except that he gives a slightly different (but equivalent) definition for $I$.

EXAMPLE 2.3.3. Take $\mathcal{O}$ real quadratic (with its real places labeled 1, 2), $n = 2$, and let $K = \mathcal{H} \times \mathcal{H}$, where $H$ is the upper half plane. A point $(z, z') \in K$ gives two inner products on $\mathbb{R}^2$, each one defined as in Example 2.3.1; this is just what's required to make $P = \mathcal{O}^2$ into a lattice. For $W = \text{span } (e_2)$, we find that

$\text{vol } L = (\text{vol } \mathcal{O})^2(\text{Im } z)(\text{Im } z')$

$\text{vol } L \cap W = \text{vol } \mathcal{O}$

$d(W; z, z') = (\text{vol } L)^{1/2}/(\text{vol } L \cap W) = ((\text{Im } z)(\text{Im } z'))^{1/2}$

The regions $X_W(t)$ turn out to be the same as those used in [Ash, et al., 1975, p. 41–42], where the function $d_W^{-2}$ was called "distance to the cusp".

DISCUSSION 2.4. Now we describe our interpretation of the "geodesic action" of Borel–Serre. Suppose we are given $L \in \tilde{X}$, an $F$-subspace $W \subseteq V$, and

$r > 0$. We construct a new lattice $L[W, r]$ in $\tilde{X}$ by changing the norms of $L$. Writing $V_\infty = L \otimes_\mathcal{O} F_\infty$ and $W_\infty = W \otimes_\mathcal{O} F_\infty$, we can use the inner product on $V_\infty$ provided by $L$ to write $V_\infty = W_\infty \oplus W_\infty^\perp$, an orthogonal sum. Now multiply the norm on $W_\infty^\perp$ by $r$, but leave the norm on $W_\infty$ unchanged; assemble these by orthogonal sum to form a new inner product for $V_\infty$. Doing this for each $\infty$ defines a new lattice $L[W, r] \in \tilde{X}$, there is an obvious exact sequence

$$0 \to L \cap W \to L[W, r] \to (L/L \cap W)[r] \to 0$$

and

$$d(W, L[W, r]) = r \cdot d(W, L).$$

This procedure is easy to visualize as a dilation of $V$ in the directions perpendicular to $W$. Since it commutes with scaling (i.e. $L[W, r_1][r_2] = L[r_2][W, r_1]$), we can also regard it as acting on $X$. For $g \in \Gamma$ we have $g(L[W; r]) = (gL)[gW; r]$.

Now suppose that we have a chain of $F$-subspaces $W_1 \subseteq \cdots \subseteq W_m$ of $V$. Then for $r_1, \ldots, r_m > 0$, let $L' = L[W_1, r_1] \cdots [W_m, r_m]$. Since $L' \cap W_i/L' \cap W_{i-1} = (L \cap W_i/L \cap W_{i-1})[r_1 \cdots r_i]$, 1.29 implies that choosing $r_i$ large enough ensures that $W_i$ is in the canonical filtration of $L'$, or even that $L' \in X(W_i, t)$. This proves the following converse to Lemma 2.2(d).

LEMMA 2.5. Given $F$-subspaces $0 \neq W_1 \subseteq \cdots \subseteq W_s \subseteq V$, the set $X(W_1, t) \cap \cdots \cap X(W_s, t)$ is nonempty.

Remark 2.6. We can also prove that any $P$ can be given norms which make it into a semistable $\mathcal{O}$-lattice. To see this, choose the norms arbitrarily at first, producing an $\mathcal{O}$-lattice $L$. Then let $W_i \subseteq \cdots \subseteq W_m$ be its canonical filtration. Then for suitable choice of numbers $r_i$, the lattice $L' = L[W_1, r_1] \cdots [W_m, r_m]$ will have $L' \cap W_i/L' \cap W_{i-1}$ all of the same slope, and will be semistable by 1.27.

## 3. Continuity

In this section we prove that the functions $d_W$ are continuous.

Suppose $M$ is a topological space. We say that a family $\{f_n\}$ of real functions on $M$ is locally equicontinuous if, for any $\varepsilon$, there is a covering of $M$ by open sets $U$, such that for all $n$ and each $x, y \in U$, $|f_n(x) - f_n(y)| < \varepsilon$. Equivalently, each $x$ in $M$ has a neighborhood $U$ such that for all $n$ and each $y \in U$, $|f_n(x) - f_n(y)| < \varepsilon$. It is

clear that the supremum of a locally equicontinuous family, if finite, is itself a continuous function. The union of a finite number of locally equicontinuous families is locally equicontinuous.

LEMMA 3.1. (a) *For each nonzero F-subspace* $W \subseteq V$ *consider the real function* $\tilde{X}$ *defined by* $L \mapsto \log \mathrm{vol}\,(L \cap W)$; *this family of functions is locally equicontinuous*

(b) $L \mapsto \min L$ *is a continuous function on* $\tilde{X}$.

(c) $L \mapsto \max L$ *is a continuous function on* $\tilde{X}$.

(d) $d_W$ *is a continuous function on* $X$.

(e) $X(W, t)$ *is an open subset of* $X$.

*Proof.* It is enough to prove (a). For example, to see that (a) $\rightarrow$ (c) simply observe that $\max L = \sup \{(\log \mathrm{vol}\,L - \log \mathrm{vol}\,L \cap W)/(\mathrm{rk}\,P - \mathrm{rk}\,L \cap W)\}$.

We may as well restrict scalars, achieving $\mathcal{O} = \mathbb{Z}$, for this only increases the family of functions being considered.

We may choose a number $m$, and restrict attention to $F$-subspaces $W$ of $V$ of dimension $m$. As in the proof of Lemma 1.15, we may apply the $m^{\mathrm{th}}$ exterior power to everything, achieving $m = 1$ (and possibly enlarging the family once again). Now let $n = \dim P$.

Choose a basis for $V$, and identify each $L$ in $\tilde{X}$ with its (positive definite symmetric) matrix $z$. Let $w$ be a generator for $L \cap W$, so $\log \mathrm{vol}\,(L \cap W) = \log |w| = (1/2) \log (w^t z w)$. Enlarge the family of functions once again by dropping the requirement that $w$ be in $W$, and forget $W$; for any nonzero $w$ in $V$ we will consider the function $z \mapsto \log (w^t z w)$ on $\tilde{X}$, (forgetting the factor $1/2$).

Fix a point $z \in \tilde{X}$ and a number $\varepsilon > 0$. We seek a small neighborhood of $z$, but we may as well first change the basis of $V$ to make $z = 1$, the identity matrix. Let $\delta > 0$ be a small number (to be determined later) and consider arbitrary symmetric matrices $\Delta z$ with $|(\Delta z)_{ij}| < \delta$ for each $i, j$. Then

$$|(w^t \cdot \Delta z \cdot w)/(w^t \cdot w)| \leq \delta (\textstyle\sum\sum |w_i| |w_j|)/(\textstyle\sum w_i^2)$$
$$\leq \delta (\textstyle\sum\sum (1/2)(w_i^2 + w_j^2))/(\textstyle\sum w_i^2)$$
$$= n\delta$$

This leads to

$$|\log (w^t(z + \Delta z)w) - \log (w^t \cdot z \cdot w)| = |\log (1 + (w^t \cdot \Delta z \cdot w)/(w^t \cdot w))|$$
$$\leq |\log (1 - n\delta)|,$$

which is smaller than $\varepsilon$ if $\delta$ is chosen small enough.    QED

COROLLARY 3.2. *Given a point* $L \in \tilde{X}$ *and a F-subspace* $W \subseteq V$, *there is a neighborhood of* $L$ *on which* $d_W$ *is the infinum of a finite set of smooth functions.*

*Proof.* We can write $\log d(W, L)$ as the infinum of all functions

$$\text{slope } (L \cap W_2/L \cap W) - \text{slope } (L \cap W/L \cap W_1)$$

where $W_2$ runs over all $F$-subspeaces of $V$ containing $W$ properly, and $W_1$ runs over all those contained properly in $W$. Each one of these functions is smooth, so it is enough to show that in some neighborhood of $L$, only a finite number of them are needed. In fact, only the ones which already achieve the minimum are needed, because by 1.15 and 3.1(a), the others stay far enough away on some small enough neighborhood $U$ of $L$. As for the ones which do achieve the minimum, there are only a finite number of them (by 1.15, again). QED.

COROLLARY 3.3. *Given* $t \geq 1$, *the family of open subsets* $X(W, t)$ *of* $X$ (*one for each F-subspace* $W$ *of* $V$) *is locally finite. Moreover, if* $t > 1$, *each* $L$ *in* $X$ *has a neighborhood* $U$ *so small that* $\{W \mid X(W, t) \cap \neq \varnothing\}$, *in addition to being finite, is a chain.*

*Proof.* Suppose $L \in X$. Consider first those $W \subseteq V$ for which $L \cap W$ lies above the canonical polygon of $L$. They are all further above it than a certain minimum distance, according to 1.15. By 3.1(a), we can find a neighborhood $U$ of $L$ so that whenever $L' \in U$, the points corresponding to the various $L' \cap W$ are still above the canonical polygon of $L'$; thus the only candidates for members of the canonical filtration of such an $L'$ will be those $W$ for which $L \cap W$ was on the canonical polygon of $L$; of these there are only a finite number, by 1.15 again.

For the second statement, the same argument shows that $U$ can be chosen so that for all $W$, $d(W, L') < t$ if $d(W, L) < t$ for all $L'$ in $U$. Q.E.D.

THEOREM 3.4. *Given* $t > 1$, *the spaces* $\tilde{X}_{ss}(t)$ *and* $X_{ss}(t)$ *are manifolds with boundary, and the boundary consists of those points* $x$ *with* $\sup_W d(W, x) = t$.

*Remark.* Example 1.25 shows that the bound on $t$ is sharp, for when $t = 1$, these spaces are not manifolds.

*Proof.* The following proof works equally well for either $\tilde{X}_{ss}(t)$ or $X_{ss}(t)$.

Define $h(x) = \max(1, \sup_W d(W, x))$, so that $X_{ss}(t)$ is $h^{-1}([1, t])$. It follows from 3.3 that $h$ is continuous. Choose a point $x_0$ in $X$ with $h(x_0) = t$. Since $t > 1$, in a small enough neighborhood $U$ of $x_0$ we have $h(x) = \sup_i d(W_i, x)$, where $W_1 \subseteq \cdots \subseteq W_s$; this follows from 3.3. If $U$ is small enough, then any $x$ in $U$ has

each $W_i$ in its canonical filtration, so $d(W_i, x) = d(W_i/W_{i-1}, (x \cap W_{i+1}/x \cap W_{i-1}))$; replace $h(x)$, globally, by the supremum $k(x)$ of these latter functions. Since $h = k$ on $U$, it will be enough to show that $k^{-1}((0, t])$ is a manifold with boundary. For $r > 0$ and $x \in X$, define $r^* x = x[W_1, r] \cdots [W_s, r]$. Our modification of $h$ was rigged to force $k(r^* x) = rk(x)$, for all $x$ and all $r$ (not just those near $x_0$).

Now define a function $g$ on $X$ by $g(x) = \exp(\text{slope}(x \cap W_2/x \cap W_1) - \text{slope}(x \cap W_1))$. This function is smooth (infinitely differentiable), and satisfies $g(r^* x) = r \cdot g(x)$ for all $x$ and all $r$. The differential $dg$ is nonzero everywhere, so the level set $Y = g^{-1}(\{1\})$ is a submanifold of $X$ of codimension 1. Then there is an evident homeomorphism:

$$\tilde{X} \xrightarrow{\sim} Y \times \mathbb{R}^{>0}$$

$$x \mapsto (g(x)^{-1*}x, k(x))$$

$$(k(y)^{-1}r)^* y \mapsto (y, r)$$

This makes it clear that $k^{-1}((0, t])$ is homeomorphic to $Y \times (0, t]$, and thus is a manifold with boundary. Q.E.D.

## 4. Contractibility

4.1. The space $X = X(P)$ is contractible, as is well known. To prove it, one embeds $X$ into $\tilde{X}$ using some section of the map $\tilde{X} \to X$; this makes $X$ into a deformation retract of $\tilde{X}$. Now $\tilde{X} = \prod \tilde{X}_\infty$; each $\tilde{X}_\infty$ can be identified with the set of positive definite (symmetric of hermitian) $n \times n$ matrices, is therefore a convex subset of a vector space, and thus is contractible.

THEOREM 4.2. *For any number $t \geq 1$, the spaces $\tilde{X}_{ss}(t)$ and $X_{ss}(t)$ are contractible.*

*Proof.* It is enough to show that $\tilde{X}_{ss}(t)$ is contractible, for choice of a section of the map $\tilde{X}_{ss}(t) \to X_{ss}(t)$ exhibits the latter space as a deformation retract of the former.

In this proof we use the geodesic action to straighten out all the angles in the canonical polygon; this will give a deformation retraction of $\tilde{X}$ onto its subset $\tilde{X}_{ss}$, as well as a deformation retraction of $\tilde{X}_{ss}(t)$ onto $\tilde{X}_{ss}$. This will be enough because we already know $\tilde{X}$ is contractible.

We would like to define the deformation retraction $H: \tilde{X} \times I \to \tilde{X}$ by setting $H(L, u) = L[W_1, f_1(u)] \cdots [W_s, f_s(u)]$, where $(W_1 \subseteq \cdots \subseteq W_s)$ is the canonical filtration of $L$, and $f_1(u)$ is any monotonic continuous function decreasing from 1 to

$(d(W_i, L))^{-1}$, as $u$ goes from 0 to 1. It is clear from 1.29 that if $L \in \tilde{X}_{ss}(t)$, then so is $H(L, u)$; moreover, $H(L, 1)$ is semistable, and if $L$ is semistable, then $H(L, u) = L$, for all $u$.

The only trick is to see that the function $H$ is continuous, but for this purpose we have the equicontinuity results of the previous section. They imply, that given $L_0$, there is a neighborhood $U$ of $L_0$ so that the members of the canonical filtration for any $L$ in $U$ are drawn from a finite list, say $W_1, \ldots, W_m$. These $F$-subspaces are in fact exactly those $W$ which lie on the canonical polygon of $L_0$. This time, let $f_i(u)$ be $u$ $(\max(1, d(W_i, L)))^{-1} + (1 - u)$. This function is chosen to be continuous as a function of the pair $(L, u)$. If $W_i$ occurs in the canonical filtration of $L$, then as a function of $u$, $f_i$ decreases monotonically from 1 to $d(W_i, L)^{-1}$ as $u$ goes from 0 to 1. In addition, if $W_i$ is not in the canonical filtration of $L$, then $f_i(u) = 1$ for all $u$; notice that $L[W_i, 1] = L$. Now define $H(L, u) = L[W_i, f_1(u)] \cdots [W_m, f_m(u)]$. This agrees with the previous definition for $H$ on the neighborhood $U$, and is clearly continuous. Q.E.D.

We will call a map $h$ between convex subsets of vector spaces *affine* if $h((1 - u)x + ux') = (1 - u)h(x) + uh(x')$ for all points $x, x'$, and all $u, 0 \leq u \leq 1$. A real function $h$ on a convex set is *convex* if $h((1 - u)x + ux') \leq (1 - u)h(x) + uh(x')$. A convex function composed with an affine function is convex. A sum or a supremum of convex functions is convex. A function $h$ is called *concave* if $-h$ is convex.

There are several continuous affine maps involving spaces $\tilde{X}$:

$$\tilde{X}(P) \to \tilde{X}(f_* P) \qquad\qquad L \mapsto f_* L$$

$$\tilde{X}(P) \to \tilde{X}(P') \qquad\qquad L \mapsto L \cap P'$$

$$\tilde{X}(P) \times \tilde{X}(Q) \to \tilde{X}(P \oplus Q) \qquad (L, M) \mapsto L \oplus M$$

Here $f_* P$ denotes the $\mathbb{Z}$-module underlying $P$, and $P'$ is some submodule of $P$.

LEMMA 4.3.

(a) *The function* $L \mapsto -\log \mathrm{vol}\,(L)$ *is a convex function on* $\tilde{X}$.

(b) *For any constant* $c$, *the set* $\{L \in \tilde{X} : \min L > c\}$ *is convex, and thus contractible.*

*Proof.* (a) By restriction of scalars, we may assume $\mathcal{O} = \mathbb{Z}$. If a basis if chosen for $P$, and we let $x$ denote the matrix of the inner product on $L$, then $\log \mathrm{vol}\, L = (1/2)(\log \det x)$. Thus we must check that $-\log \det ((1 - u)x + ux')$ is a convex function of $u$. This becomes clear if we diagonalize $x$ and $x'$ simultaneously, for then we see that we have a sum of functions of the form $-\log ((1 - u)y + uy')$,

where $y$ and $y'$ are positive numbers; each of these is convex because $-\log(u)$ is convex and $(1-u)y + uy'$ is a linear function of $u$.

(b) The proposed condition on $L$ amounts to the conjunction of the inequalities $-\log \text{vol } L \cap W < -c \text{ rk } W$, so (a) yields the result.    Q.E.D.

**Remark 4.4.** If we use geodesic paths instead of affine paths in 4.3 we get concavity instead of convexity.

**Remark 4.5.** Lemma 4.3 implies that the function on $\tilde{X}$ which assigns to a point $x$ its canonical polygon, is itself a concave function. To make this precise, one considers the polygon as a real function on the set $\{1, 2, \ldots, n\}$, where $n = \dim P$.

Suppose $q = (W_1 \subseteq \cdots \subseteq W_k)$ is a strictly increasing chain of $F$-subspaces of $V$; we call $q$ an $F$-*flag* on $V$. Let $t$ be a fixed real number $\geq 1$. Let $\tilde{X}_q = \tilde{X}_q(P) = \tilde{X}(W_1, t) \cap \cdots \cap \tilde{X}(W_k, t)$; we don't claim that this set is convex, but nevertheless we can prove the following theorem by mimicking Quillen's argument in [Grayson, 1980].

**THEOREM 4.6.** $\tilde{X}_q$ *is contractible.*

*Proof.* We construct a map to a contractible space, whose fibers are convex, as follows. Take $q- = (W_1 \subseteq \cdots \subseteq W_{k-1})$, $W' = W_k$, $P' = P \cap W'$, and $P'' = P/P'$. Regard $q-$ as a flag on $P'$. Let $\text{Com}(W'_\infty, V_\infty)$ denote the set of subspaces $T_\infty$ of $V_\infty$ such that $W'_\infty \oplus T_\infty = V_\infty$. Let $\text{Com}(W', V)$ denote $\prod_\infty \text{Com}(W'_\infty, V_\infty)$; it is a real affine space, thus is contractible. We have a homeomorphism $\tilde{X}(P) = \tilde{X}(P') \times \tilde{X}(P/P') \times \text{Com}(W', V)$ defined by $L \mapsto (L \cap W', L/L \cap W', W'^\perp)$, where we take the collection of orthogonal complements $W'^\perp$ with respect to the inner products provided by $L$ at each infinite place. Use the notation $b(L', L'', T)$ to denote the inverse map. Let $g$ be the continuous map $\tilde{X}_q(P) \rightarrow \tilde{X}_{q-}(P') \times \text{Com}(W', V)$ defined by $g(L) = (L \cap W', W'^\perp)$.

By induction on the length $k$ of the flag, we may assume that $\tilde{X}_{q-}(P')$ is contractible, so it is enough to show that $g$ is a homotopy equivalence. We will do this by constructing a section $h$ for $g$, and a homotopy between $h \circ g$ and the identity which respects the fibers of $g$, so $g$ also has the property that any map obtained from $g$ by pullback is a homotopy equivalence, too.

Choose norms for $P''$ at infinity arbitrarily, yielding a fixed lattice $L''$ on $P''$. We construct a section $h$ of $g$ by defining $h(L', T) = b(L', L''[r], T)$, where $r = t \exp(\max L' - \min L'' + 23)$. (Here 23 is a random positive number.) Check that $h(L', T) \in \tilde{X}_q(P)$ by computing $d(W', h(L', T)) = \exp(\min L''[r] - \max L') =$

$\exp(\log r + \min L'' - \max L') > t$. Notice that $r = r(L')$ is a function of $L'$, but by 3.1 it is continuous, and thus $h$ is continuous, too. It is clear that $g(h(L', T)) = (L', T)$.

Now suppose $x, y \in \tilde{X}_q$, $0 \le u \le 1$, and let $z = (1 - u)x + uy$. We claim that if $g(x) = g(y)$, then $g(z) = g(x)$, and $z \in \tilde{X}_q$. The first part is clear, and the second follows from 4.3(b). Thus there is a homotopy from $h \circ g$ to the identity function on $\tilde{X}_q$, defined by $(1 - u)h \circ g + u \cdot id$.    Q.E.D.

**COROLLARY 4.7.** *For* $t > 1$ *the open set* $U = X - X_{ss}(t)$ *has the homotopy type of the Tits building of F-subspaces of* $V$, *as does the boundary of the manifold* $X_{ss}(t)$. *In either case, the homotopy equivalence can be chosen so it respects the action of* $\Gamma$ *(up to homotopy).*

*Proof.* The open set $U$ is the union of the sets $X(W, t)$, where $W$ runs over all proper nontrivial $F$-subspaces of $V$. The closure of $U$ or of any finite intersection $X(W_1, t) \cap \cdots \cap X(W_n, t)$ is a manifold with boundary; this follows from arguments like those in the proof of 3.4. Moreover, the closure of any finite intersection is the intersection of the closures. Thus we may replace each $X(W, t)$ by its closure without changing the nerve of the cover, or the fact (provided by 4.6) that the intersections are contractible or empty. Now that we have a closed cover, we may apply Theorem 8.2.1 of [Borel–Serre] to produce the desired homotopy equivalence. (We could have made a shortcut here by using Mayer–Vietoris to produce a homology equivalence, which is all that is required later.) By 2.2(d) and 2.5, the nerve is the simplicial complex whose vertices are the proper nontrivial $F$-subspaces of $V$, and whose simplices are the chains of such subspaces; this is one of the definitions of the Tits building. It follows from 2.2(b) that the homotopy equivalence respects the action of $\Gamma$.

In order to prove the same thing about the boundary of $X_{ss}(t)$, one simply shows it is a deformation retraction of $X - X_{ss}$, by using a retraction similar to the one used in the proof of 4.2.    Q.E.D.

## 5. Compactness

In this section we show that the quotient spaces $\Gamma \backslash X_{ss}(t)$ are compact. We follow a suggestion of Borel, and derive the compactness directly from the Mahler criterion. As in the previous sections, $P$ is a fixed projective $\mathcal{O}$-module, and we study lattices on $P$.

**THEOREM 5.1.** *Given* $t \ge 1$, *there is a constant* $c$ *so that any* $L \in \tilde{X}_{ss}(t)$, *normalized so* $\mathrm{vol}\, L = 1$, *has every nonzero vector* $v \in f_* L$ *of length* $> c$.

*Proof.* Let $W_i \subseteq \cdots \subseteq W_s = V$ be the canonical filtration of $L$, and write $L_i = L \cap W_i$. Then

$$c_1 = \log t \geqslant \log d(W_i, L) = \text{slope } L_{i+1}/L_i - \text{slope } L_i/L_{i-1}$$

Adding these inequalities up for $i = 1, 2, \ldots, s - 1$ yields

$$c_2 = (n-1)c_1 \geqslant (s-1)c_1 \geqslant \text{slope } L/L_{s-1} - \text{slope } L_1$$

where $n = \text{rk } L$. Since slope $L = 0$, and $L_{s-1}$ is below the line connecting 0 and $L$, we have slope $L/L_{s-1} \geqslant 0$. Thus

$$c_3 = -c_2 \leqslant \text{slope } L_1$$

Now if $v \in f_* L$ is a nonzero vector, then we can consider the submodule $\mathcal{O}v$ of $L$. We have

$$(1/m) \log \text{vol } \mathcal{O}v = \text{slope } \mathcal{O}v \geqslant \text{slope } L_1 \geqslant c_3$$

where $m = \dim \mathcal{O}$, so

$$c_4 = \exp(mc_3) \leqslant \text{vol } \mathcal{O}v$$

Now by 1.6

$$\text{vol } \mathcal{O}v = \text{vol } \mathcal{O} \cdot \prod |v|^{e(\infty)}$$
$$\leqslant \text{vol } \mathcal{O} \, |v|^m$$

Thus

$$|v| \geqslant ((\text{vol } \mathcal{O}v)/(\text{vol } \mathcal{O}))^{1/m} \geqslant (c_4/\text{vol } \mathcal{O})^{1/m}$$

Q.E.D.

COROLLARY 5.2. *For any* $t \geqslant 1$, *the space* $X_{ss}(t)$ *is compact modulo* $\Gamma$.

*Proof.* Let $Z \subseteq \tilde{X}$ be the set of lattices on $P$ whose volume is one. The intersection $B = Z \cap \tilde{X}_{ss}(t)$ maps onto $X_{ss}(t)$ because any lattice can be normalized by scaling to make its volume one; it is enough to show that $B$ is compact modulo $\Gamma$. Let $G$ be the Lie group $\prod Gl(V_\infty)$, fix an $L \in Z$, and choose orthonormal bases

for $L$ on each $V_\infty$. We write all matrices and vectors with respect to the union of these bases. Now consider the continuous map $p: G \to \tilde{X}$ defined by $p(g) = {}^t g^{-1} \cdot g^{-1}$, the matrix of the inner product on $f_*(gL)$. This is the map which expresses $\tilde{X}$ as a homogeneous space of $G$. It will be enough to show that the set $M = p^{-1}(B)$ is compact modulo $\Gamma$. (A point of $M$ may be thought of as a lattice $L'$ on $P$ together with a choice of orthonormal basis at each infinite place.) Restriction of scalars gives an embedding of $G$ into $Gl_n(\mathbb{R})$, where $n = \dim P$. Keeping in mind that $|\det g|^{-1}$ is the volume of the lattice $p(g)$, and that if $x$ is a vector in $P$, then its length as a vector of the lattice $f_*p(g)$ is $|g^{-1}x|$, we may apply [Borel, 1966, Proposition 8.2], because the group $G$ is reductive. It says that $\Gamma\backslash M$ is compact (our set M is closed) if there is an upper bound on the volumes of the lattices in $M$, and a lower bound on the lengths of nonzero vectors in the restriction-of-scalars of the lattices of $M$. The upper bound is clear, because all our lattices have volume 1, and the lower bound is provided by 5.1.   Q.E.D.


# 6. Consequences

From 4.2, 3.4, 4.7, and 5.2 we know that the space $X_{ss}(t)$, for $t > 1$, is a contractible manifold with boundary, its boundary is homotopy equivalent to the appropriate Tits building, via a homotopy equivalence compatible with the action of $\Gamma$, and it is compact modulo $\Gamma$. At this point we have acquired, for our manifold, all the same topological information as Borel and Serre acquired for their manifold with corners. (Notice that $X_{ss}(t)$ inherits from $X$ the property that some subgroup of $\Gamma$ of finite index acts freely, properly, and discontinuously on it.) The only difference is that our manifold does not have a differentiable structure on the boundary. Let $\Gamma' \subset \Gamma$ be a torsion-free subgroup of finite index acting freely on $X$. The manifold $M = X_{ss}(t)/\Gamma'$ is not known to be triangulable because its boundary is not smooth. Nevertheless, it follows from the "local finiteness theorem" of Kirby–Siebenmann, p. 123, that $M$ (which is metrizable because it has a countable basis) is homotopy equivalent to a finite simplicial complex; this property can be used in place of triangulability in Borel–Serre [1973, §11.1]. It follows, for example, that $\Gamma' = \pi_1 M$ is finitely presented.

All the qualitative results about $\Gamma$ follow immediately, using the same techniques Borel–Serre use. The list of results is rather lengthy, so we refer the reader to [Borel–Serre, 1973, section 11] or to [Bieri–Eckmann, 1973]. The results include finite presentation for $\Gamma$ and its integral homology and cohomology, a determination of its cohomological dimension, and a sort of generalized Poincaré duality for torsion-free subgroups of finite index. The latter duality is an essential ingredient in Quillen's proof [1973] of the finite generation of the higher

$K$-groups $K_i(\mathcal{O})$, for it gives the finite generation of the homology of $\Gamma$ with coefficients in the Steinberg module (since the Steinberg module is not a finitely generated group, compactness alone of the manifold with boundary doesn't suffice).

## 7. Orthogonal groups

We now proceed to the case of arithmetic groups which occur as subgroups of orthogonal groups of symmetric or alternating nondegenerate bilinear forms. We begin by defining the dual of a lattice, and then express the relevant symmetric space $X$ in terms of inner products, following Siegel [1957, Chapter III]. For simplicity's sake, from now on $\mathcal{O} = \mathbb{Z}$, and a lattice is a $\mathbb{Z}$-lattice.

DISCUSSION 7.1. The dual of an inner product space is naturally an inner product space, because the inner product itself provides an isomorphism $H : V \xrightarrow{\sim} V^*$ which can be used to transport the inner product from $V$ to $V^*$. The dual basis of an orthonormal basis of $V$ is an orthonormal basis for $V^*$. We have $V^{**} = V$, and have compatibility with orthogonal sum: $(V \oplus W)^* = V^* \oplus W^*$.

Choose a basis for $V$, and form the dual basis for $V^*$. The matrix of the inner product on $V$ is then the same as the matrix of the map $H$, formed with respect to these two bases: let $H$ also denote this matrix. Letting $J = H^{-1}$, we see that the matrix of the inner product on $V^*$ (with respect to the dual basis) is ${}^tJHJ = {}^tH^{-1} = H^{-1}$, according to the standard formula for transport of matrix of bilinear form.

We define the dual lattice $L^*$ to be $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ equipped with the dual inner product just described. The discussion above about matrices shows that $\mathrm{vol}(L^*) = (\mathrm{vol}\, L)^{-1}$, because $\mathrm{vol}(L) = (\det H)^{1/2}$ if our basis for $V$ is chosen to be a $\mathbb{Z}$-basis for $L$.

The duality $L \mapsto L^*$ preserves exact sequences: it does so as far as the underlying abelian groups are concerned, because the underlying sequence of abelian groups splits; to check this assertion we may therefore forget the underlying abelian groups, retaining only the inner product spaces; but at this level the exact sequences are also split (canonically), and come from orthogonal sum. Duality is compatible with orthogonal sum, converts inclusions to projections, and vice versa; thus it preserves exact sequences.

Now it is easy to see the relation between the canonical filtrations of $L$ and $L^*$. The sublattices of $L$ are in one-to-one correspondence with the sublattices of $L^*$: corresponding to $L_1$ is $L_1^{\#}$, defined as the image of $(L/L_1)^*$ in $L^*$. Letting $n = \dim L$ and $v = \log \mathrm{vol}\, L$, we see that the transformation $(x, y) \mapsto (n - x, y - v)$ of the plane transforms the canonical plot for $L$ into the canonical plot for $L^*$,
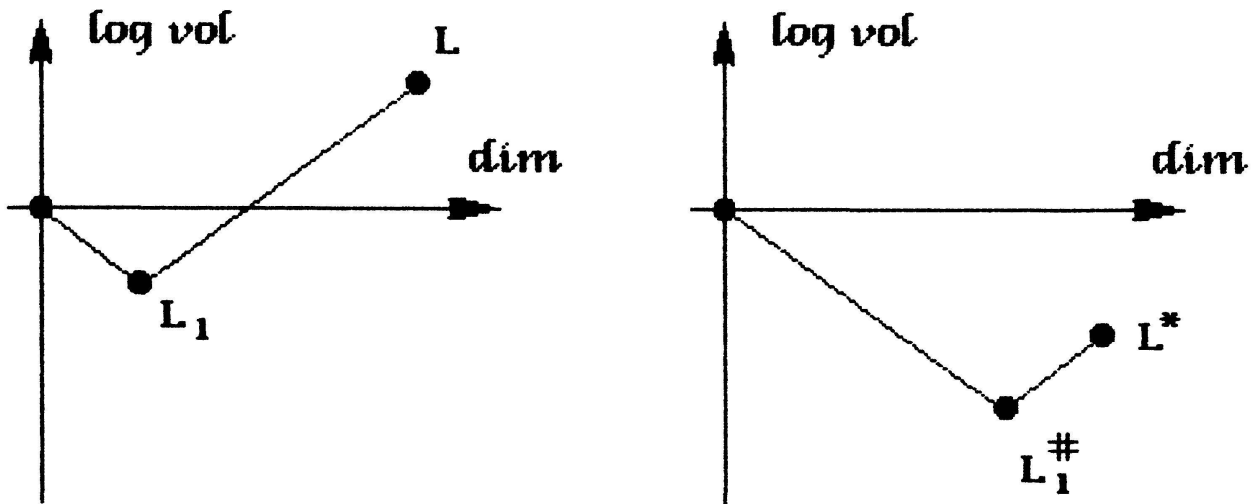
Figure 7.2

and the canonical polygon for $L$ into the canonical polygon for $L^*$. Therefore, a sublattice $L_1$ of $L$ is in the canonical filtration of $L$ if and only if $L_1^{\#}$ is in the canonical filtration for $L^*$. Moreover, $\max(L^*) = -\min(L)$, and for any $Q$-subspace $W$ of $V$, $d(W, L) = d(W^{\#}, L^*)$. See figure 7.2.

DEFINITION 7.3. Suppose $S$ and $H$ are two nondegenerate bilinear forms on a vector space $V$, each of which is either alternating or symmetric. As described above, we use $H$ to transport $H$ to $V^*$, yielding a bilinear form ${}^tH^{-1}$ there. We say that $S$ and $H$ are *compatible* if $S$, regarded as a map $V \to V^*$, is an isometry for $H$ and ${}^tH^{-1}$. The corresponding equation is ${}^tS^tH^{-1}S = H$. If a basis is chosen for $V$, and its dual basis is used for $V^*$, then this equation can be regarded as an equation on the corresponding matrices. It happens that this condition is symmetric with respect to interchanging $S$ and $H$, for it can be written as $H^{-1}\,{}^tS = S^{-1}\,{}^tH$. If $W$ is a subspace of $V$, we will use $W^{\vee}$ (resp. $W^{\perp}$) to denote the orthogonal complement of $W$ with respect to $S$ (resp. to $H$). We say that $W$ is *totally isotropic* if $W \subseteq W^{\vee}$, and is *coisotropic* if $W^{\vee} \subseteq W$.

EXAMPLE 7.4. If $S$ is definite, then there is a unique inner product $H$ compatible with $S$, for $S$ may be assumed to be positive definite symmetric, and then with respect to an orthonormal basis of $S$ the matrix of $H$ must be orthogonal and symmetric. We may choose the basis to diagonalize $H$, also, so then $H$ must be the identity matrix.

LEMMA 7.5. *If $S$ and $H$ are compatible, then $W^{\perp\vee} = W^{\vee\perp}$. If, in addition, $W$ is coisotropic (resp. totally isotropic) then $W^{\perp}$ is totally isotropic (resp. coisotropic).*

*Proof.* It is easy to see that $W^{\vee\perp} = H^{-1} \cdot {}^tS \cdot W$, because $({}^ty)^tS(z) =$

$({}^t y)H(H^{-1} \cdot {}^t S \cdot z)$; similarly, $W^{\perp \vee} = S^{-1} \cdot {}^t H \cdot W$, quite generally. The desired equality then follows from the definition of compatibility. The second statement is clear.   Q.E.D.


DISCUSSION 7.6.   Let $\bar{X}(V)$ denote the set of inner products on $V$, and let $X(V, S)$ denote the set of inner products $H$ on $V$ which are compatible with $S$.

If $H$ is compatible with $S$, and $W^\vee \subseteq W \subseteq V$, then let $U = W \cap W^{\vee \perp}$, and let $T = W^\perp$. We get a decomposition of $V$ into subspaces orthogonal for $H$, namely $V = W^\vee \oplus U \oplus T$. It turns out that $U$ can be recovered from $T$ because $U = W \cap W^{\vee \perp} = W \cap W^{\perp \vee} = W \cap T^\vee$. For this reason we focus attention on $T$. By 7.5 $T$ is totally isotropic.

Recall that Com $(W, V)$ denotes the affine space of complements for $W$ in $V$, and let Com $(W, V, S)$ denote the space of totally isotropic complements $T$ for $W$ coisotropic with respect to $S$ in $V$.


LEMMA 7.7.   Com $(W, V, S)$ *is contractible (and nonempty).*

*Proof.* Consider the map $h : \text{Com}(W, V, S) \to \text{Com}(W^\vee, W)$ defined by $T \mapsto U = W \cap T^\vee$. Since $W \oplus T = V$, we have $W^\vee \oplus T^\vee = V$, and intersection with $W$ gives $W^\vee \oplus U = W$; this shows $U \in \text{Com}(W^\vee, W)$.

Now we show that $h$ is surjective: fix a $U$, and we construct $T$ by induction on dim $W^\vee$. If $W^\vee$ is nonzero, the write $W^\vee = W_1^\vee \oplus Rx$, with $x \neq 0$, and choose $y$ so $W_1 = W \oplus Ry$. Since $S|_U$ is nonsingular, there is a unique $u \in U$ such that $y - u \in U^\vee$. Replace $y$ by $y - u$, so now $y \in U^\vee$. If $S$ is alternating, then ${}^t y S y = 0$; if $S$ is symmetric, we can arrange ${}^t y S y = 0$ by adding a suitable (unique) multiple of $x$ to $y$. Let $U_1 = Rx \oplus U \oplus Ry$. By induction, we find $T_1 \in \text{Com}(W_1, V)$ totally isotropic and orthogonal to $U_1$. Let $T = Ry \oplus T_1$; it works.

The fibers of $h$ are real affine subspaces of Com $(W, V)$, as we see now. Fix $U$, and choose a basis for one of the spaces $T \in \text{Com}(W, V, S)$ which is orthogonal to $U$. The other complements $T'$ to $W$ can be obtained by adding arbitrary elements of $W$ to the basis vectors. The complements $T'$ which are orthogonal to $U$ can be obtained by adding arbitrary elements of $W \cap U^\vee = W^\vee$ to the basis vectors. The condition that $T'$ be totally isotropic imposes additional conditions on those elements which happen to be linear equations (rather that quadratic) because $W^\vee$ is totally isotropic.

The fibers of $h$ are nonempty affine spaces, and therefore are contractible, so it is reasonable that $h$ is a homotopy equivalence. The proof we gave for the surjectivity of $h$ actually gives a recipe for constructing a continuous section $g$ of $h$, provided we fix in advance all the $x$'s and $y$'s to be used inductively. Since the fibers of $h$ are all affine subspaces of Com $(W, V)$, the formula $g \circ h + (1 - t)id$, for

$0 \le t \le 1$, defines a homotopy from $g \circ h$ to the identity, and shows that $h$ is a homotopy equivalence.

Since $\text{Com}(W^\vee, W)$ is contractible, it follows that $\text{Com}(W, V, S)$ is too.  Q.E.D.

In the following lemma we learn how to reconstruct inner products compatible with $S$, from data on the pieces of the decomposition of 7.6.

LEMMA 7.8. *Let* $W \subseteq V$ *be a coisotropic subspace, and let* $S'$ *denote the* (*nonsingular*) *form induced by* $S$ *on* $W/W^\vee$.

Consider the map $X(V, S) \to X(W/W^\vee, S') \times \tilde{X}(V/W) \times \text{Com}(W, V, S)$ which sends $H \mapsto (K, J, W^\perp)$, where $K$ and $J$ are induced by $H$ (by orthogonal complement and restriction) and the orthogonal complement $W^\perp$ is formed with respect to $H$.

This map is a homeomorphism.

*Proof.* We show surjectivity of the map, so suppose we are given $(K, J, T) \in X(W/W^\vee, S') \times \tilde{X}(V/W) \times \text{Com}(W, V, S)$. Let $U = W \cap T^\vee$: the proof of 7.7 shows that $V = W^\vee \oplus U \oplus T$. We regard $K$ as an inner product on $U$ and $J$ as an inner product on $T$. The map $S$, by restriction, gives an isomorphism $S_1 : W^\vee \xrightarrow{\sim} T^*$, and $J^{-1}$ is an inner product on the target of this map. Thus $J' = {}^t S_1 J^{-1} S_1$ is an inner product on $W^\vee$. Now let $H$ be the orthogonal sum $J' \oplus K \oplus J$: it is easy to check that $H$ is compatible with $S$, because the matrix of $S$ (with respect to the triple direct sum) is antidiagonal.

Injectivity and continuity of the map and its inverse are clear.  Q.E.D.

COROLLARY 7.9. $X(V, S)$ *is contractible.*

*Proof.* We use 7.8 with $W^\vee$ chosen to be a maximal isotropic subspace of $V$. Since $S'$ is definite, $X(W/W^\vee, S')$ is a point, so the result follows from 7.7 and 4.1.  Q.E.D.

DISCUSSION 7.10. Suppose from now on that $P$ is a finitely generated free abelian group of rank $n$, $V = P \otimes \mathbb{R}$, and $S : P \to P^*$ is a nondegenerate alternating or symmetric bilinear form – this means that $S$ is injective, but not necessarily surjective. Let $G = O(S) = \{A \in Gl(V) : {}^t ASA = S\}$, and let $\Gamma = O(S) \cap Gl(P)$; $G$ is a real Lie group and an algebraic group, and $\Gamma$ is an arithmetic subgroup of $G$. Let $X = X(P, S) = X(V, S)$ be the space of all lattices $L$ with $P$ as underlying abelian group, and for which $S : L \to L^*$ is compatible with the inner product $H$.

The group $G$ acts on $X$ (on the left) via $A*H = {}^tA^{-1}HA^{-1}$; this agrees with the action previously defined on $\tilde{X}$.

LEMMA 7.11. *X is homeomorphic to the homogeneous space $G/K$, where $K$ is a maximal compact subgroup of $G$, and is a contractible manifold.*

*Proof.* We adapt a proof Siegel [1957, Chapter III] used when $S$ is symmetric. First we show that $X$ is nonempty (which we also did in 7.9). To do this, choose a basis for $V$ which puts $S$ in normal form $S_0$, i.e. block diagonal form where the blocks are all

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

if $S$ is alternating, and are all

$$(+1) \quad \text{or} \quad (-1)$$

if $S$ is symmetric. In the symmetric case we agree to put the $-1$'s after the $+1$'s: this ensures that $S_0$ depends only on $S$, because the signature is well-defined. With respect to this basis, we may set $H_0 = 1$; it clearly is in $X$, because $S_0$ is orthogonal.

Now we must see that $G$ acts transitively on $X$. Suppose $H \in X$; choose an orthonormal basis for it: this makes $H = 1$, and our equation becomes ${}^tSS = 1$, i.e. $S$ is orthogonal. According to the orthogonal form of the spectral theorem, we may find a new orthonormal basis for $H$ which puts $S$ in block diagonal form, with blocks

$$(+1) \quad \text{or} \quad (-1) \quad \text{or} \quad \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Taking into account that $S = \pm{}^tS$, we may assume that the matrix of $S$ in this new basis is $S_0$. Let $A$ be the change of basis matrix connecting the two bases we have found: then since $S$ has the same matrix in both bases, we see that $A \in G$. Since the matrix for $H_0$ with respect to the first basis is 1, and the matrix for $H$ with respect to the second basis is 1, too, we see that ${}^tA^{-1}H_0A^{-1} = H$, which proves transitivity.

Let $K$ be the stabilizer in $G$ of the inner product $H_0$, and consider the bijection $G/K \xrightarrow{\sim} X$. We apply a standard category argument [Helgason, Theorem, p. 121] to conclude that this map is a homeomorphism. The hypotheses required

for that theorem are: $G$ is a locally compact group with countable base, acting as a transitive topological transformation group on a locally compact Hausdorff space $X$; these hypotheses are fulfilled because $X$ is a closed subset of a Euclidean space. Homogeneous spaces are always manifolds, so now we know that $X$ is a manifold.

The orthogonal group of $H_0$ is a compact group and contains $K$ as a closed subset, so $K$ is compact; it remains to show that $K$ is a maximal compact subgroup of $G$. It is enough to show that any compact subgroup of $G$ is conjugate to a subgroup of $K$, and Siegel did this for $S$ symmetric [1957, Chapter III, Section 2]. To make his proof work when $S$ is alternating, we only need to know the following. Fix an inner product $P$ on $V$; it allows us to regard $S$ as an endomorphism of $V$. The complexification of $S$ is skew-hermitian, diagonalizable by an unitary change of basis, and has purely imaginary eigenvectors. It follows that we can find a basis for $V$ which is orthonormal for $P$ and which splits $S$ up into blocks of the form

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.$$

By rescaling the basis vectors we make the matrix of $S$ be $S_0$, and the matrix $D$ of $P$ be diagonal, with $DS_0 = S_0D$.   Q.E.D.

DISCUSSION 7.12. We are now ready to bring the canonical filtrations into the study of $X$. Each lattice of $X$ certainly has a canonical filtration, as defined before; moreover, for each $\mathbb{Q}$-subspace $W$ we have the restriction of the continuous function $d_W$ to $X$ available. It turns out that we won't need all of these functions: only those for which $W$ is totally isotropic or is coisotropic will be needed, for the others will remain bounded on $X$. This is fortunate, for it is only for such $W$ that we will be able to define a "geodesic action", to be used for retracting the cusps.

LEMMA 7.13. *Suppose $L$ is a lattice, and $M$ is a subgroup of finite index less than or equal to $k$. If $W$ is a $\mathbb{Q}$-subspace of $V = L \otimes R$, then $k^2 \cdot d(W, M) \geq d(W, L) \geq k^{-2} \cdot d(W, M)$.*

*Proof.* We clearly have

$$\text{vol } L \leq \text{vol } M \leq k \text{ vol } L$$

If $W_1 \subseteq W_2$ are $\mathbb{Q}$-subspaces of $V$, then the index $[L \cap W_2/L \cap W_1 : M \cap W_2/M \cap$

$W_1$] is bounded by $k$, too, so we get similar inequalities for these lattices. Apply them to the definition

$$d(W, L) = \exp \inf (\text{slope} \ (L \cap W_2/L \cap W) - \text{slope} \ (L \cap W/L \cap W_1))$$

where $W_2$ (resp. $W_1$) runs over all $\mathbb{Q}$-subspaces of $V$ containing (resp. contained in) $W$.   Q.E.D.

COROLLARY 7.14.  *There is a constant $C \geqslant 1$ such that for any lattice $L$ in $X$ and any $\mathbb{Q}$-subspace $W$ of $V$, if $d(W, L) > C$, then $W$ and its orthogonal complement for $S$ are both contained in the canonical filtration of $L$, $W$ is either totally isotropic or coisotropic, and $Cd(W, L) \geqslant d(W^\vee, L) \geqslant C^{-1}d(W, L)$.*

*Proof.* Apply 7.1 and 7.13 to the inclusion $L \subseteq L^*$, taking $C = k^2$. See figure 7.15.   Q.E.D.

DISCUSSION 7.16.  Lemma 7.14 tells us which cusps of $\tilde{X}$ may be inhabited by cusps of $X$, and they come in pairs, corresponding to a $\mathbb{Q}$-subspace $W$ and its orthogonal $W^\vee$, with $W^\vee \subseteq W \subseteq V$. According to 7.13, if a point of $X$ is far out along one of the cusps, it is just as far out along the other, roughly. For each such pair, we will use one distance function, say $d_W$, where $W$ is the coisotropic member of the pair, and we will have one type of geodesic action for retracting the cusp, which we now describe.

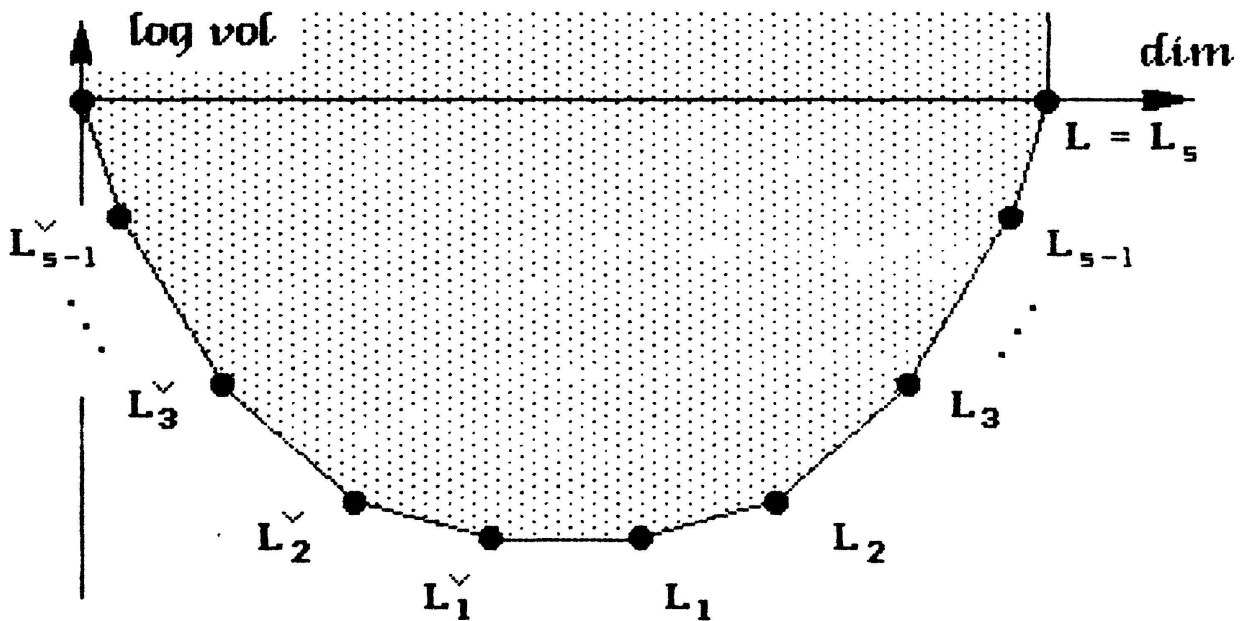Let $r$ be a positive real number. According to 7.6 and 7.8, we may multiply



Figure 7.15

the inner product on $V/W$ by $r$, leave the inner product on $W/W^\vee$ unchanged, and reassemble the pieces to get a new inner product on $V$ compatible with $S$. (It follows from the proof of 7.8 that the norm on $W^\vee$ gets divided by $r$.) For the new lattice which results from replacing the norm on $L$ with this new one we use the notation $L\{W; r\}$. We have

$$L\{W; r\} = L[r^{-1}][W; r][W^\vee; r]$$

$$\text{vol } L\{W; r\} = \text{vol } L$$

DEFINITION 7.17. Let $X_{ss}(t)$, for any real number $t \geq C$, be the space $X \cap \tilde{X}_{ss}(t)$. Define $X(W, t) = X \cap \tilde{X}(W, t)$, for any coisotropic $\mathbb{Q}$-subspace $W$ of $V$. Suppose $q = (W_1 \subseteq \cdots \subseteq W_k)$ is a strictly increasing chain of coisotropic $\mathbb{Q}$-subspaces of $V$: call $q$ a coisotropic flag on $V$, as before, and define $X_q = X \cap \tilde{X}_q$.

THEOREM 7.18. *Suppose $t$ is a real number, and $t > C$.*

(a) *The space $X_{ss}(t)$ is a manifold with boundary, and the boundary consists of those points $x$ with $d(W, x) = d$ for some $W$.*

(b) *The space $X_{ss}(t)$ is contractible.*

(c) *For any coisotropic flag $q$ on $V$, the space $X_q$ is contractible.*

(d) *The open set $X - X_{ss}(t)$ has the homotopy type of the Tits building of $G$, as does the boundary of the manifold $X_{ss}(t)$.*

(e) *The space $X_{ss}(t)$ is compact modulo $\Gamma$.*

*Proof.* (a) Just as in the proof of 3.4, making the obvious adjustments.

(b) The proof here can be done just as in 4.2. Continuity of the deformation retraction is automatic because $X_{ss}(t)$ is a subspace of $\tilde{X}$.

(c) This goes as in 4.6, but we use 7.8 instead of the decomposition presented there.

(d) This follows from (c) just as in the proof of 4.7. In order to apply the definition of the Tits building [Tits, 1974, Section 5], which is phrased in terms of parabolic subgroups, we need the additional information that a one-to-one, order-reversing correspondence exists between the $\mathbb{Q}$-parabolics of $G$ and the coisotropic $\mathbb{Q}$-flags of $V$: each parabolic is the stabilizer of a unique flag.

(e) As in 5.2; according to 7.14, it is okay to ignore the functions $d(W, x)$ for those $\mathbb{Q}$-subspaces $W$ of $V$ which are not coisotropic. The needed fact that $G$ is reductive follows from the fact that for a basis of $V$ which makes $S$ into an orthogonal matrix, $g \in G$ implies $^t g \in G$.   Q.E.D.

CONCLUSION 7.19. Now all the remarks of 6 apply to the situation introduced in 7.10, because of 7.18.

## REFERENCES

A. ASH, D. MUMFORD, M. RAPAPORT and Y. TAI, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, Brookline, Mass. 1975.

M. F. ATIYAH and R. BOTT, *The Yang-Mills Equations over Riemann Surfaces*, Phil. Trans R. Soc., London A 308 (1982) 523.

R. BIERI and B. ECKMANN, *Groups with homological duality generalizing Poincaré duality*, Inventiones Mathematicae 20 (1973) 103.

A. BOREL, *Reduction Theory for Arithmetic Groups*, Proc. Sympos. Pure Math. 9 (1966) 20, AMS, Providence, RI.

A. BOREL and J.-P. SERRE, *Corners and Arithmetic Groups*, Comment. Math. Helv. 48 (1973) 436.

D. GRAYSON, *Finite Generation of K-groups of a curve over a finite field* [after Daniel Quillen], Lecture Notes in Mathematics no. 966, Algebraic K-theory, Proceedings, Oberwolfach, 1980, Part I, Springer, Berlin, 1982.

G. HARDER and M. NARASIMHAN, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. 212 (1975) 215.

C. HERMITE, *Oeuvres*, I, Paris (1905) 94.

R. KIRBY and L. SIEBENMANN, *Foundational Essays on Topological Manifolds, Smoothing, and Triangulations*, Annals of Math. Study 88, Princeton University Press, 1977.

S. LANG, *Algebraic Number Theory*, Addison-Wesley, Reading, Massachusetts, 1970.

E. MENDOZA, *Cohomology of $PGl_2$ over Imaginary Quadratic Integers*, Bonner Mathematische Schriftstellen 128 (1980).

H. MINKOWSKI, *Geometrie der Zahlen*, Leipzig (1896).

H. MINKOWSKI, *Discontinuitätsberiech fur arithmetische Äquivalenz*, Ges. Werke II (1911) 53–100.

D. QUILLEN, *Finite generation of the groups $K_i$ of rings of algebraic integers*, in Algebraic K-theory I, Battelle Institute Conference 1972, Lecture Notes in Math. no. 341, Springer, Berling, Heidelberg, New York, 1973.

H. RADEMACHER, *Topics in Analytic Number Theory*, Grundlehren Math. Wiss. 169, Springer, 1973.

J-P. SERRE, *Arbres, Amalgames, $SL_2$*, Astérisque no. 46, Soc. Math. France, 1977.

——, *Arithmetic Groups*, in "Homological Group Theory", Cambridge University Press, 1979.

C. SIEGEL, *Lectures on quadratic Forms*, TATA, Bombay, 1957.

U. STUHLER, *Eine Bemerkung zur Reduktionstheorie quadratischen Formen*, Archiv der Math. 27 (1976) 604.

——, *Zur Reduktionstheorie der positiven quadratischen Formen II*, Archiv der Math., 28 (1977) 611.

J. TITS, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics 386, Springer, Berlin, Heidelberg, New York, 1974.

A. WEIL, *Sur l'analogie entre les corps de nombres algébriques et les corps de fonctions algébrique* [1939a], Collected Papers, vol I, Springer, 1980.

*University of Illinois*
*Dept of Mathematics*
*273 Altgeld Hall*
*1409 West Green Street*
*Urbana, Ill. 61801/USA*