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Comment. Math. Helvetici 59 (1984) 592–599 0010-2571/84/040592-08$01.50 + 0.20/0

Some topologically locally-flat surfaces in the complex projective plane

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§ 1. Introduction; statement of results

THEOREM 1. *For every integer* $n \geq 6$, *there exists in the homology class* $n[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$ *a topologically locally-flatly embedded surface of genus strictly less than that of a nonsingular complex algebraic curve of degree n.*

THEOREM 2. *For every pair* (m, n) *of integers greater than or equal to* 5, (*except possibly* $(5, 5)$) *there is a topologically locally-flatly embedded surface in the 4-disk with boundary a torus link* $O\{m, n\}$ *of type* (m, n) *and genus strictly less than the* (*classical*) *genus of* $O\{m, n\}$.

Here, a surface S topologically embedded in a 4-manifold M will be called "topologically locally-flatly embedded" if S has a neighborhood N in M which is homeomorphic to an open 2-disk bundle over S by a homeomorphism carrying S to a section. This is evidently some kind of local homogeneity assumption on the embedding of S in M. (For instance, if S is smoothly, or P.L. locally-flatly, embedded in M then it is *a fortiori* topologically locally-flatly embedded. After preparing this paper, the author learned of a new theorem of Akbulut – showing that certain "topologically slice" knots very similar to $\hat{\beta}_6$ in §3, below, definitely are not smoothly slice – which implies that not every topologically locally-flat surface is just a smooth or P.L. locally-flat surface up to a global topological change of coordinates.)

One construction will be used to prove both theorems. It is an instance of a general construction discussed in earlier papers by the author [7, 8, 9]; it now proves the theorems because of a recent result of M. Freedman. The specific construction is given below, following some motivating remarks and a short new (and, I believe, improved) exposition of the general construction.

* Research partially supported by the Fonds National Suisse at the Mathematics Institute of the University of Geneva.

Remark 1. A conjecture frequently attributed to R. Thom[1] is that no *smoothly* embedded surface in $\mathbb{CP}^2$ can have genus strictly smaller than that of a homologous (smooth) complex algebraic curve. It is well-known [5] that, by being willing to sacrifice local flatness, one can represent every homology class by a piecewise-linearly embedded 2-sphere – for instance, up to orientation, by the complex algebraic curve with affine equation $w = z^n$. But this sphere need not be piecewise-linearly locally-flat – in the example, for $n \geq 3$, there is a singular point at infinity.

The point of Theorem 1 is that by making a global (or at least regional) sacrifice of smoothness, one can salvage a weaker sort of homogeneity of normal structure while "chopping off handles."

Remark 2. Theorem 2 is vaguely related to the "problem of Milnor" on the Gordian number (Überschneidungszahl, or unknotting number) of the link of a singularity (cf. [6], [2]). Indeed, $O\{m, n\}$ is such a link, and the problem in this case asks whether the Gordian number $\ddot{u}(O\{m, n\})$ equals $(m-1)(n-1)/2$, which is the classical genus of $O\{m, n\}$ (i.e. the least genus of a surface smoothly embedded in S^3 with boundary $O\{m, n\}$). If the answer is affirmative, then any smoothly embedded surface in the 4-disk with boundary $O\{m, n\}$ has genus at least $(m-1)(n-1)/2$. However, even if a smooth surface existed with boundary $O\{m, n\}$ and small genus, no conclusion could be necessarily drawn about $\ddot{u}(O\{m, n\})$; much less for the topologically locally-flat surface of Theorem 2.

Remark 3. Here is a sketch of the strategy used to prove both theorems. "By hand" we construct a smooth complex algebraic curve Γ of degree 6 in $\mathbb{CP}^2$, and a piecewise-smooth 4-ball D in $\mathbb{C}^2 \subset \mathbb{CP}^2$, such that (i) the transverse intersection $\Gamma \cap \partial D$ is a "topologically slice" knot, i.e., bounds a topologically locally-flatly embedded disk in D, while (ii) the smooth surface $\Gamma \cap D$, with the same boundary, has genus 1. Then replacing the surface of genus 1 by the disk, we produce a topologically locally-flatly embedded surface homologous to Γ in $\mathbb{CP}^2$, of genus 1 smaller.

It is clear that by various expedients (most naively, doing essentially the same surgery in k disjoint balls, on a curve of degree $6k$; or using a more complicated topologically slice knot, which bounds a piece of a curve of degree $5k+1$ that has genus k) one can produce as large a gap as desired between the genus of a smooth algebraic curve and that of a homologous topologically locally-flat surface. However, I know of no construction which makes a *proportional* gap bigger than

[1] Professor Thom has remarked (personal communication, November 19, 1982) that the conjecture perhaps more properly belongs to folklore.

10 per cent, which is already achieved by the example of degree 6 (where the genus of the algebraic curve is $\frac{1}{2}(6-1)(6-2)=10$ and one handle is chopped off). In any case, the proportional gap can't be *too* big (whether the topologically locally-flat surface is produced, as here, by "surgery" – rather, amputation – or not); for, as Shmuel Weinberger has kindly pointed out to me, Wall's topological version [10] of the G-signature theorem fits into the proof of Hsiang and Szczarba [4] to yield, for topologically locally-flat surfaces in 4-manifolds, exactly the estimates given in [4] for smooth surfaces.

In particular, topologically locally-flat 2-spheres in $\mathbb{CP}^2$ occur in degrees 0, ±1, ±2 only (where there are smooth examples).

§2. A construction of closed braids

Fix an integer $n \geq 2$. For $k=1,\ldots,n-1$, let $\eta_k = \exp(2\pi(k-1)i/(n-1))$ (so $\eta_1 = 1$), and let $J_k = \eta_k[0,1]$ be the line segment in $\mathbb{C}$ from 0 to η_k. Write $Q_{n-1} = \{\eta_k : k=1,\ldots,n-1\}$. The fundamental group $\pi_i(\mathbb{C}\backslash Q_{n-1}, 0)$ is free of rank $n-1$, with free basis $x_1,\ldots,x_{n-1}$, where x_k is represented by a loop based at 0 and running once counter-clockwise around the boundary of a convex region containing η_k and no η_j, $j \neq k$. This group is, of course, identical to $\pi_1((\mathbb{C}\cup\{\infty\})\backslash(Q_{n-1}\{\infty\}), 0)$. Represent it in the symmetric group on $\{1,\ldots,n\}$ by sending x_k to the transposition $(k \quad k+1)$. Let X be the corresponding n-sheeted branched covering space of $\mathbb{C}\cup\{\infty\}$, branched over $Q_{n-1}\cup\{\infty\}$. One readily verifies that X is a 2-sphere, with a single point over ∞. Thus the covering map "is" a polynomial of degree n, with $n-1$ critical points, and critical values η_k $(k=1,\ldots,n-1)$; further requiring the polynomial to be monic and have constant term 0 will specify it completely. We assume this is done, and call the result $p(w) = w^n + \alpha_{n-1}w^{n-1} + \cdots + \alpha_1 w$. Write $\mathbb{C}_w$ for $X - \{\infty\}$, $\mathbb{C}_z$ for the base space $\mathbb{C}$, so $p : \mathbb{C}_w \to \mathbb{C}_z$.

Remark 4. Except for $n=2,3$, I have been unable to find $p(w)$ explicitly. It is *not* in general the elegant $w^n - \alpha w$, where $\alpha = n(1-n)^{1-1/n}$; this (like the even simpler $w^n - nw$, which only differs by rotation and homothety in the base space) corresponds, apparently, to the representation $x_k \to (1k+1)$. (Of course the construction could be adapted to these polynomials, at the expense of complicating the braid theory a bit.) For $n=2,3$, the two representations are equivalent.

Now consider $p^{-1}(J_k)$. This has $n-1$ components, each a simple arc; let I_k be the one containing the critical point with critical value η_k. Then the endpoints of I_k are two of the preimages of 0, call them w_k and w_{k+1}; it is easy to see that they may be numbered so that $I_k \cap I_{k+1} = \{w_{k+1}\}$ for $k = 1, \ldots, n-2$, while w_1 belongs only to I_1, w_n only to I_{n-1}, and $I_k \cap I_l = \emptyset$ if $|k-l| > 1$. Let $I = \bigcup_{k=1}^{n-1} I_k$. Then I is a simple arc in $\mathbb{C}_w$.

Next consider the *configuration space* $E_n \backslash \Delta$ of unordered n-tuples of distinct points of $\mathbb{C}_w$; that is, form the symmetric product $E_n = \mathbb{C}_w^n/\mathcal{S}_n$, and delete from it the multidiagonal Δ of n-tuples with at least one pair of equal elements. The *n-string braid group* is by definition the fundamental group of the configuration space.

Specifically, we will take $p^{-1}(0) \in E_n \backslash \Delta$ as our basepoint. In the usual description of B_n, the basepoint is taken to be $\{1, \ldots, n\}$, and for $k = 1, \ldots, n-1$, the loop $l_k : S^1 \to E_n - \Delta : z \to \{1, \ldots, k-1, k+2, \ldots, n\} \cup \{k + \frac{1}{2}(1 \pm z^{\frac{1}{2}})\}$ (where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$) represents an element of $\pi_1(E_n \backslash \Delta, \{1, \ldots, n\})$ called the *standard generator* σ_k. Here, let $h : \mathbb{C}_w \to \mathbb{C}_w$ be an orientation-preserving homeomorphism with $h(I) = [1, n]$ and $h(w_k) = k$, $k = 1, \ldots, n$. Then h enforces an identification of $\pi_1(E_n \backslash \Delta, \{1, \ldots, n\})$ with $B_n = \pi_1(E_n \backslash \Delta, p^{-1}(0))$, giving a meaning to the standard generators $\sigma_1, \ldots, \sigma_{n-1} \in B_n$.

Finally, note that p^{-1} is well-defined as a continuous map $\mathbb{C}_z \to E_n$, and that by construction $p^{-1} \mid (\mathbb{C}_z - Q_{n-1})$ has image in $E_n \backslash \Delta$.

PROPOSITION. *The induced homomorphism $p^{-1} \mid (\mathbb{C}_z - Q_{n-1})_*$ from the free group $\pi_1(\mathbb{C}_z - Q_{n-1}, 0)$ to B_n carries the free generator x_k to the standard generator σ_k, for $k = 1, \ldots, n-1$.*

Proof. Recall that x_k is represented by a loop which traverses (counterclockwise) the boundary of a convex region – call it D_k – in $\mathbb{C}_z$, and that $\eta_k \in \operatorname{Int} D_k$, $\eta_j \notin D_k$ $(j \neq k)$, and $0 \in \partial D_k$ $(k = 1, \ldots, n-1)$. As with $I_k \subset D_k$, the preimage $p^{-1}(D_k)$ has $n-1$ components; $n-2$ of them are carried to D_k homeomorphically by p, and one – call it D_k' – is a 2-sheeted branched cover of D_k *via* p, branched at $w_k \in \operatorname{Int} D_k'$; so D_k' is again homeomorphic to a 2-disk. No component of $p^{-1}(D_k)$ other than D_k' contains any critical point w_j of p. The loop in E_n, with domain the simple closed curve ∂D_k, which takes $z \in \partial D_k$ to $p^{-1}(z) \in E_n$, clearly has image in $E_n \backslash \Delta$. It can easily be homotoped (respecting its basepoint $p^{-1}(0)$), in $E_n \backslash \Delta$, to a path of n-tuples each containing the $n-2$ points of $p^{-1}(0)$ not in D_k', together with two points on $\partial D_k'$ which exchange positions (by a counterclockwise "rotation") as the loop is traversed; but such a path clearly represents σ_k. $\square$

Recall that an oriented (closed) 1-manifold L in the open solid torus $S^1 \times \mathbb{C}$ is a closed braid (on n strings) if $\mathrm{pr}_1 \mid L : L \to S^1$ is an oriented covering projection (of degree n). A braid $\beta \in B_n$ yields a closed braid $\hat{\beta} \subset S^1 \times \mathbb{C}$ (unique up to isotopy respecting pr_1) by taking a loop $l: S^1 \to E_n \backslash \Delta$ representing β and considering its "graph" (as an n-valued complex function) $\operatorname{gr} l = \{(z, w) \in S^1 \times \mathbb{C} : w \in l(z)\}$.

COROLLARY. *If $x_{i(1)}^{\varepsilon(1)} \cdots x_{i(s)}^{\varepsilon(s)}$ is any word in the free group $\pi_1(\mathbb{C}_z - Q_{n-1}, 0)$, and $\gamma: S^1 \to \mathbb{C}_z - Q_{n-1}, \gamma(1) = 0$, is a loop representing it, then the set $\{(z, w) : \gamma(z) = p(w)\}$ is a closed braid $\hat{\beta}$ on n strings in $S^1 \times \mathbb{C}_w$, where $\beta = \sigma_{i(1)}^{\varepsilon(1)} \cdots \sigma_{i(s)}^{\varepsilon(s)} \in B_n$.* □

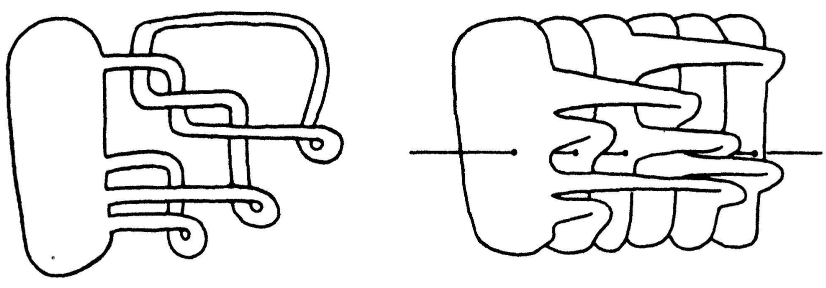
§3. Freedman's theorem; proofs of theorems 1 & 2

The profound researches of Michael Freedman into the topology of 4-manifolds have recently led him to the following improvement [3a] of a theorem published in [3] (the original theorem applied only to a knot K which was an untwisted double of a knot with Alexander polynomial 1).

FREEDMAN'S THEOREM. *Let $K \subset S^3 = \partial D^4$ be a (smooth) knot with Alexander polynomial $\Delta_K(t)$ identically 1. Then K bounds a topologically locally-flat disk $S \subset D^4$.* □

It is not important for the following proofs to know what an Alexander polynomial is; it is enough to believe that the knot K pictured in Figure 1A, where it is shown as the boundary of a punctured torus in $\mathbb{R}^3$, has $\Delta_K(t) = 1$. (This K is in fact an untwisted double of a trefoil knot; from that fact, or calculating directly from the obvious Seifert matrix of the visible surface, those in the know will see that $\Delta_K(t) = 1$. As readers of [8] will have guessed, this particular K was chosen simply as being about the easiest "quasipositive" knot with corresponding "braided surface" of genus 1 and Alexander polynomial 1.)

Figure 1B shows an isotopic surface, punctured by a line in $\mathbb{R}^3$; the boundary knot is a closed braid in the open solid torus complementary to the line, and is the



A B

Fig. 1

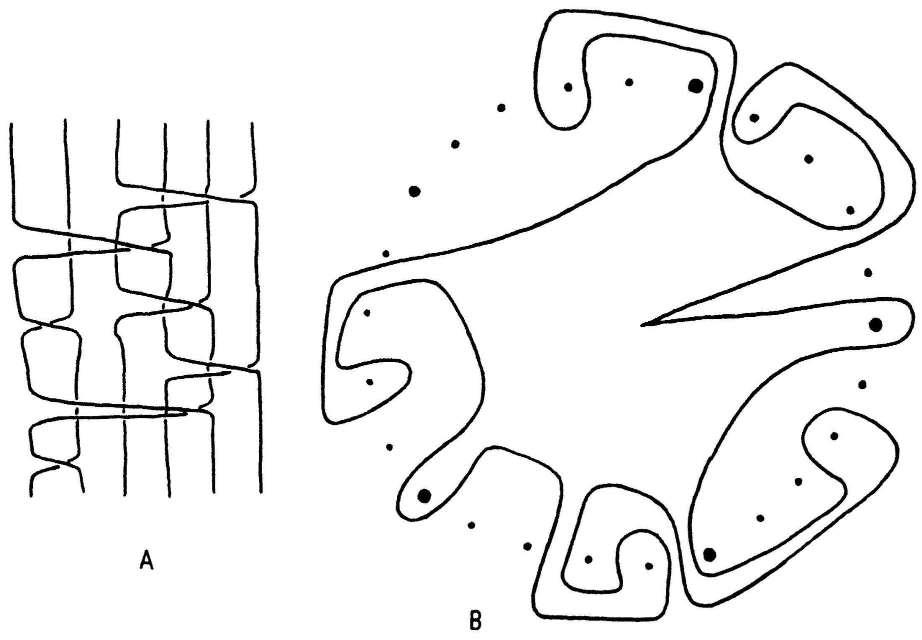


Fig. 2

same type of closed braid as in Figure 2A. (The surface is less explicit but still visible.) Call the pictured braid $\beta_6 \in B_6$. If we abbreviate aba^{-1} by ${}^ab$, and σ_k by k $(k=1,\dots,5)$, then we may write $\beta_6 = {}^{34}5 \cdot {}^{12}3 \cdot {}^34 \cdot 1 \cdot {}^45 \cdot {}^{123}4 \cdot 1$. (The raised dots are for clarity only.)

Fix integers $n \geq 2$, $m \geq 1$. Let $p: \mathbb{C}_w \to \mathbb{C}_z$ be the n^{th} degree polynomial of §2; let $f(z, w) = p(w) - z^m$; and let $\Gamma_\varepsilon(m, n) = \{(z, w) \in \mathbb{C}^2 : f(z, w) = \varepsilon\}$. Then $\mathrm{pr}_1 \mid \Gamma_0(m, n): \Gamma_0(m, n) \to \mathbb{C}_z$ is an n-sheeted branched covering branched over $Q_{n-1}^{1/m} = \{\xi : \xi^m \in Q_{n-1}\} = \{\exp[2\pi i(k-1)/m(n-1)] : k = 1, \dots, m(n-1)\}$.

Let $\gamma: S^1 \to \mathbb{C}_z - Q_{n-1}^{1/m}$ be a loop with $\gamma(1) = 0$. Then in $S^1 \times \mathbb{C}_w$ the set $\{(z, w) : (\gamma(z), w) \in \Gamma_0(m, n)\}$ is a closed n-string braid, and it is easy to see which one it is: compose γ with $z \to z^m$ to obtain $\gamma^m : S^1 \to \mathbb{C}_z - Q_{n-1}$, $\gamma^m(1) = 0$; then look at the element of B_n corresponding to the class of γ^m *via* the proposition of §2 and its corollary.

In particular, if R is a closed region homeomorphic to a disk in $\mathbb{C}_z$, with $0 \in \partial R$, $Q_{n-1}^{1/m} \cap \partial R = \varnothing$, then we may take γ to be a (counterclockwise) parametrization of ∂R; we find that $L = \{(z, w) : z \in \partial R,\ (z, w) \in \Gamma_0(m, n)\}$ is a closed braid in $\partial R \times \mathbb{C}_w$. Being compact, L lies in some closed solid torus $\partial R \times D$, $D \subset \mathbb{C}_w$ a closed disk; finally, then, L lies in the 3-sphere (with corners) $\partial(R \times D)$. In fact (by, say, the maximum principle), $L = \Gamma_0(m, n) \cap \partial(R \times D)$, that is, L is the complete boundary of $\Gamma_0(m, n) \cap R \times D$. Also, it is easy to calculate the Euler characteristic of the surface $\Gamma_0(m, n) \cap R \times D$, for it is the branched cover of R branched over $Q_{n-1}^{1/m} \cap R$.

THEOREM 1. *For every integer $n \geq 6$, there exists in the homology class $n[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$ a topologically locally-flatly embedded surface of genus strictly less than that of a nonsingular complex algebraic curve of degree n.*

Proof. In Figure 2B is sketched a simple closed curve in $\mathbb{C}_z \backslash Q_5^{1/5}$ which gives the braid β_6. (The 25[th] roots of 1 are indicated by dots, the 5[th] roots among them by larger dots; 0 is the basepoint.) Let R be the region it bounds. Then (for a suitably large disk $D \subset \mathbb{C}_w$) the surface $\Gamma_0(5,6) \cap R \times D$ has Euler characteristic -1 and a connected boundary (of type $\hat{\beta}_6$), so it is of genus 1. (It is essentially the surface of Figure 1A, "pushed in.") Now, $\Gamma_0(5,6)$ is nonsingular in $\mathbb{C}^2$, but has a singular point at infinity in $\mathbb{C}P^2$; but for sufficiently small $\varepsilon \neq 0$, $\Gamma_\varepsilon(5,6)$ will be nonsingular when completed in $\mathbb{C}P^2$, while $\Gamma_\varepsilon(5,6) \cap R \times D$ will still be a punctured torus with boundary in $\partial(R \times D)$ of type $\hat{\beta}_6$. The homology class of the completion of $\Gamma_\varepsilon(5,6)$ is of course $6[\mathbb{C}P^1]$.

By Freedman's Theorem, the smooth surface $S' = \Gamma_\varepsilon(5,6) \cap R \times D$, of genus 1, shares its boundary with a topologically locally-flatly embedded disk S in $R \times D$. Replace S' by S on the completion of $\Gamma_\varepsilon(5,6)$; the resulting surface is still in the homology class $6[\mathbb{C}P^1]$, is topologically locally flat, and has genus 1 smaller than the genus of $\Gamma_\varepsilon(5,6)$. The theorem is thus proved for $n = 6$.

For larger n, one may apply the same technique, starting with the braid $\beta_n = \beta_6 \sigma_6 \cdots \sigma_{n-1} \in B_n$ and taking the appropriate simple closed curve in $\mathbb{C} \backslash Q_{n-1}^{1/5}$; for $\hat{\beta}_n$ is of the same knot type as $\hat{\beta}_6$ (and 5 replications of Q_{n-1} still suffice to write the whole word properly). □

THEOREM 2. *For every pair* (m, n) *of integers greater than or equal to* 5, (*except possibly* (5, 5)) *there is a topologically locally-flatly embedded surface in the 4-disk with boundary a torus link* $O\{m, n\}$ *of type* (m, n) *and genus strictly less than the* (*classical*) *genus of* $O\{m, n\}$.

Proof. Follow the proof of Theorem 1 up to the final paragraph.

Without loss of generality, we may assume $n \geq m \geq 5$ and $n \geq 6$. Then we may apply the same technique as above, starting with β_n and taking the simple closed curve to lie in $\mathbb{C} \backslash Q_{n-1}^{1/m}$; again, $\hat{\beta}_n$ is the correct knot type, and extra replications of Q_{n-1} do no harm. So $\Gamma_0(m, n)$ can have a handle surgered away inside $\mathbb{C}^2$, in the topologically locally flat sense. But for r_1, r_2 sufficiently large, the intersection of $\Gamma_0(m, n)$ with the boundary of the bidisk $\{(z, w) : |z| \leq r_1, |w| \leq r_2\}$ is a link of type $O\{m, n\}$ (in fact it is the closure of the m^{th} power of the n-string braid $\sigma_1 \sigma_2 \cdots \sigma_{n-1}$), and the intersection of $\Gamma_0(m, n)$ with the whole bidisk has genus $(m-1)(n-1)/2$, the classical genus of $O\{m, n\}$ (by direct calculation). □

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Received January 25, 1984