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Levels in algebra and topology

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§1. Introduction

By definition, the level $s(A)$ of a commutative ring A is the smallest integer n such that -1 is the sum of n squares in A. (If -1 is not a sum of squares in A, we define $s(A)$ to be ∞.) By a well-known theorem of A. Pfister, if A is a field and if $s(A) < \infty$, then $s(A)$ must be a power of 2 (and any power of 2 is possible). This result, however, does not extend to rings: in [DLP], it was shown that there exist commutative $\mathbb{R}$-algebras of any prescribed level, or, equivalently, for any integer n, the "generic" algebra $A_n = \mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)$ has level exactly equal to n.

The proof that $s(A_n) = n$ in [DLP] was based on a topological fact: the Borsuk–Ulam Theorem. The idea of this proof suggested that there is a natural and interesting relationship between the topology of spheres and the arithmetic of sums of squares in rings. To study this relationship more formally, we defined in

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[DLP], for any topological space X with involution ε, the following two invariants

$$s(X) = \inf\{n\colon \text{there exists an equivariant map } X \to S^{n-1}\},$$
$$s'(X) = \sup\{m\colon \text{there exists an equivariant map } S^{m-1} \to X\},$$

called, respectively, the *level* and *colevel* of X. After [DLP] appeared in print, we realized that these invariants had been introduced much earlier by topologists: up to a constant 1, $s'(X)$ and $s(X)$ are the *index* and *coindex* (= B-index) of the space (X, ε) in the sense of Conner–Floyd [CF_1, CF_2] and C. T. Yang [Y_1, Y_2]. However, much of the past work on the index and coindex was focused on the computation of these invariants for specific spaces and their applications in topology; the potential applications of these invariants in algebra were not explored. In [DLP], we found that there is a close relationship between the level in topology and the level in algebra: for any space (X, ε), the toplogical level $s(X)$ is always equal to the algebraic level $s(A_X)$, where A_X denotes the $\mathbb{R}$-algebra of complex-valued functions $f\colon X \to \mathbb{C}$ such that $f(\varepsilon x) = \overline{f(x)}$ for every $x \in X$. In particular, taking $X = S^{n-1}$, one gets immediately an $\mathbb{R}$-algebra $A_{S^{n-1}}$ of level n.

The discovery that $s(X) = s(A_X)$ provided the basis of the present work, in which we try to probe more deeply into the process of applying known results in topology to prove new results in algebra. For instance, instead of using the Borsuk–Ulam Theorem, one can try to use other homotopy properties of the spheres. Thus, the property that odd (resp. even) mappings of S^{n-1} to itself have odd (resp. even) degrees can be used to show that, over the generic ring A_n defined above, not only is the level equal to n, but in fact the quadratic form $(n+1)\langle 1\rangle$ $(= t_0^2 + t_1^2 + \cdots + t_n^2)$ has no unimodular zero. Generalizing this idea further, instead of working with spheres, one can work with the Stiefel manifolds $V_{n,m}$. On $V_{n,m}$, consider the involution

$$(v_1, \ldots, v_m) \to (v_1, \ldots, v_r, -v_{r+1}, \ldots, -v_{r+s})$$

where $s > 0$ and $r + s = m$; the resulting space with involution is denoted by $V_{n,m}^{r,s}$. By an argument inspired by a communication of M. Kerviare and W. Scharlau, we show that, *for any space with involution* (X, ε) *the form* $n\langle 1\rangle$ *over* A_X *has a subform isometric to* $r\langle 1\rangle \perp s\langle -1\rangle$ *iff* (X, ε) *admits an equivariant map into* $V_{n,m}^{r,s}$ (*where, again,* $m = r + s$). This result enables us to study decompositions of the type $n\langle 1\rangle \cong r\langle 1\rangle \perp s\langle -1\rangle \perp \phi$ over $\mathbb{R}$-algebras by using equivariant properties of the Stiefel manifolds $V_{n,m}^{r,s}$. About the latter, of course, quite a bit is known in the topology literature. By our general machinery, many of the known results in topology about $V_{n,m}^{r,s}$ can thus be utilized to yield parallel results in algebra concerning the structure of the forms $n\langle 1\rangle$ over rings. To illustrate this point, let us mention some of the most interesting applications:

(a) Adams' Theorem on vector fields on spheres implies that there is no equivariant map from S^{n-1} to $V_{n,m}^{0,m}$ for $m > \rho(n)$ where ρ denotes the Radon function. This, combined with our results, shows that over any ring A, if $n\langle 1\rangle$ contains a subform $\langle -1\rangle$, then it contains a subform $\rho(n)\langle -1\rangle$, *but it need not contain a subform* $(\rho(n)+1)\langle -1\rangle$.

(b) Assuming a forthcoming result in [LL], we show in (10.2) that $V_{n,q+1}^{q,1}$ has colevel $s'(V_{n,q+1}^{q,1}) = n-q$. This, combined with our results, shows that over a ring A, if $n\langle 1\rangle$ contains $q\langle 1\rangle \perp \langle -1\rangle$, it need not contain $(q+1)\langle 1\rangle \perp \langle -1\rangle$. This implies, in particular, that over a ring A, if an (r-fold) Pfister form ϕ is isotropic (having a unimodular zero vector), it need not be hyperbolic (i.e. not $\cong 2^{r-1}\langle 1,-1\rangle$), contrary to the well-known behavior of Pfister forms over fields.

(c) For $n = 2^i - 1$, one can also compute the colevel of $V_{n,q+1}^{0,q+1}$; again, $s'(V_{n,q+1}^{0,q+1}) = n-q$. This computation implies that, over a ring A, if $n\langle 1\rangle$ contains $q\langle -1\rangle$, it need not contain $(q+1)\langle -1\rangle$.

To study the level and colevel more systematically, we define in §10 the notion of σ-levels and σ-colevels. Thus, for any space (X, ε) with an involution ε, we have two sequences of invariants $\{\sigma_k(X)\}, \{\sigma'_k(X)\}$ $(k \geq 0)$, with $\sigma_0(X) = s(X)$, $\sigma'_0(X) = s'(X)$ and

$$\cdots \leq \sigma'_{k+1}(X) \leq \sigma'_k(X) \leq \cdots \leq \sigma'_0(X) \leq \cdots \leq \sigma_{k+1}(X) \leq \sigma_k(X) \leq \cdots \leq \sigma_0(X).$$

For commutative $\mathbb{R}$-algebras A, we can define similar invariants $\{\sigma_k(A)\}$, $\{\sigma'_k(A)\}$ $(k \geq 0)$ satisfying the same chain of inequalities, with $\sigma_0(A) = s(A)$. Again, we have the relation $\sigma_k(X) = \sigma_k(A_X)$ for all X, and $\sigma'_k(X) = \sigma'_k(A_X)$ holds for a large class of spaces X with involution.

Several possible directions for future work seem to suggest themselves. One direction would be to develop more topological machinery to help compute the invariants $s(X)$, $s'(X)$ (and their higher analogues $\sigma_k(X)$, $\sigma'_k(X)$). Some of these invariants have been computed for certain types of Stiefel manifolds, but computations for the general type $V_{n,m}^{r,s}$ seem to be very difficult. In fact, even for the special type $V_{n,m}^{0,m}$, a full computation of the colevel would amount (essentially) to solving the skew-linear version of the Hopf Problem [H], the immersion problem of projective spaces into euclidean spaces, and the Generalized Vector Field Problem of Atiyah–Bott–Shapiro [ABS, §15]. Some explicit computations of $s(V_{n,m}^{r,s})$ and $s'(V_{n,m}^{r,s})$ will appear in [LL]. A second direction of work would be to develop more purely algebraic techniques to attack quadratic form problems over finitely generated k-algebras. When k is a formally real field, a natural idea would be to go to a real closure $\bar{k}$ of k; over $\bar{k}$, one can usually hope to get the same results as in the case over the real numbers (say by Artin-Lang, or by Tarski's Principle). However, in case k is non-real, this method will no longer work and a completely different approach would be needed. It is quite remarkable, therefore, that Arason and Pfister [AP] have succeeded, by using purely field-theoretic techniques, to solve the "level problem" algebraically: if k is any (possibly nonreal) field, then the level of the generic ring $k[x_1, \ldots, x_n]/(1+x_1^2+\cdots+x_n^2)$ is given by $\min\{s(k), n\}$. It seems to us that this statement ought to be true for any commutative ring k, so we raise it as the "Level Conjecture" in §11. It is hoped that this challenging problem will stimulate the development of further purely algebraic techniques, in complement to the technique of using topological results to solve algebraic problems over affine algebras over the real numbers.

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§2. Spaces with involution

In this section, we set the stage for the application of spaces with involution to quadratic forms. We shall write $(X, -)$ to denote a topological space X with an involution "bar" which is a homeomorphism from X to itself. If "bar" is given and fixed, we shall often write X for $(X, -)$. Whenever confusion is unlikely, involutions in different spaces will all be denoted by "bars". A continuous map $f:(X,-)\to(Y,-)$ will be called *equivariant* if f commutes with the involutions, i.e. if $f(\bar{x})=\overline{f(x)}$ for all $x\in X$. As a notational device, we shall write $f:X \twoheadrightarrow Y$ to denote (continuous) equivariant maps.

Throughout this paper, we shall write $\mathscr{C}$ for the category whose objects are $(X,-)$ as above, and whose morphisms are continuous equivariant maps $f:X\twoheadrightarrow Y$. We shall often write $X\in \mathrm{Obj}\,\mathscr{C}$ to indicate that $(X,-)$ is a space with a given involution "bar." A distinguished family of objects in $\mathscr{C}$ is given by the spheres S^n $(n\geq 0)$. Throughout this paper, whenever we talk about S^n, it will always be assumed that it is given the antipodal involution: $\bar{x}=-x$ for $x\in S^n$. Some other interesting objects of the category $\mathscr{C}$ are as follows:

(2.1) The space $\mathbb{R}^n$ with the involution $x\to -x$. This contains $(S^{n-1},-)$ as a subobject.

(2.2) The space $\mathbb{C}^n$ with the involution $(x_1,\ldots,x_n)\to(\overline{x_1},\ldots,\overline{x_n})$ given by complex conjugation of the coordinates.

(2.3) An affine variety $X = V_{\mathbb{C}}(\mathfrak{A}) \subseteq \mathbb{C}^n$ defined over $\mathbb{R}$ by an ideal $\mathfrak{A} \subseteq \mathbb{R}[x_1, \ldots, x_n]$. (We give X the strong topology, not the Zariski topology.) This variety is stable under complex conjugation, and is thus a subobject of the object $\mathbb{C}^n$ in (2.2).

(2.4) The Stiefel manifold $V_{n,m}$ with the involution

$$\varepsilon_{r,s}(v_1, \ldots, v_r, v_{r+1}, \ldots, v_{r+s}) = (v_1, \ldots, v_r, -v_{r+1}, \ldots, -v_{r+s})$$

where $m = r + s$. The objects $(V_{n,m}, \varepsilon_{r,s})$ will play an important role in later sections, and will be abbreviated by $V_{n,m}^{r,s}$. (Of course $V_{n,1}^{0,1} = S^{n-1}$.)

For an object $X \in \operatorname{Obj} \mathscr{C}$, we attach the following two invariants, called, respectively, its *level* and *colevel:*

(2.5) $s(X) = \inf\{n : \exists X \twoheadrightarrow S^{n-1}\}$ (level of X),

(2.6) $s'(X) = \sup\{m : \exists S^{m-1} \twoheadrightarrow X\}$ (colevel of X).[2]

In the former, if X does not map equivariantly into any sphere, we take $s(X) = \infty$ by convention. On the other hand, if X is non-empty, we can always find $S^0 \twoheadrightarrow X$, so we have $1 \le s'(X) \le \infty$.

The invariants $s(X)$ and $s'(X)$ coincide essentially with the co-index (= B-index) and index defined by Yang [$\mathrm{Y}_1, \mathrm{Y}_2$] and Conner–Floyd [$\mathrm{CF}_1$]; in fact $s(X) = \operatorname{coind} X + 1$ and $s'(X) = \operatorname{ind} X + 1$. For the purposes of the present work, it turns out to be more natural and more convenient to work with $s(X)$ and $s'(X)$ as defined in (2.5) and (2.6).

We have the following two lemmas: the first is clear, and the second is the Borsuk–Ulam Theorem, in a notational disguise:

LEMMA 2.7. *If there exists a morphism $X \twoheadrightarrow Y$, then $s(X) \le s(Y)$ and $s'(X) \le s'(Y)$.*

LEMMA 2.8. *For any object X in $\mathscr{C}$, we have $s'(X) \le s(X)$. Moreover, $s'(S^{n-1}) = s(S^{n-1}) = n$ for any $n \ge 1$.*

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[2] We called this invariant the "sublevel" in [DLP] but it should really be called the colevel. "Sublevel" shall mean a different invariant in this work: see §5.

Note that the two invariants s and s' are of interest only for objects $(X, -)$ of $\mathscr{C}$ whose involution "bar" is fixed-point-free. In fact, if the involution has a fixed point $x \in X$, then for any n we have $f: S^{n-1} \twoheadrightarrow X$ by $f(S^{n-1}) = x$, so $s'(X) = \infty$. On the other hand, we cannot have any $X \twoheadrightarrow S^{n-1}$ so $s(X) = \infty$ also. Even if the involution is fixed-point-free, we may still have $s(X) = s'(X) = \infty$. Such an example is provided by $X = \bigcup_{n=1}^{\infty} S^{n-1}$, where the spheres are imbedded into each other by equator maps, and the involution is again the antipodal map. However, if X is a finite dimensional separable metric space, then indeed $s'(X) \leq s(X) < \infty$ assuming that the involution is fixed-point-free: see, e.g. [CF_1]. The following easy Proposition gives an obvious upper bound for $s(X)$ for a large class of spaces of interest:

PROPOSITION 2.9. *Let $(X, -)$ be a space with a fixed-point-free involution.*
(1) *Suppose X is a topological subspace of $\mathbb{R}^n$. Then $s(X) \leq n$.*
(2) *Suppose X is a topological subspace of $\mathbb{C}^n$ and the involution on X is induced by the complex conjugation on $\mathbb{C}^n$. Then $s(X) \leq n$.*

Proof. (1) Define $f: X \twoheadrightarrow S^{n-1}$ be $f(x) = (x - \bar{x})/\|x - \bar{x}\|$. Equivariance is clear, and so is continuity.

(2) Define $g: X \twoheadrightarrow S^{n-1}$ by $f(z) = (y_1/\delta, \ldots, y_n/\delta)$, where $z_j = x_j + iy_j$ and $\delta = \sqrt{y_1^2 + \cdots + y_n^2}$. Again, equivariance and continuity are both clear. Q.E.D.

For later reference, we shall collect here some more elementary facts about s and s'.

PROPOSITION 2.10. *If $X \in \operatorname{Obj} \mathscr{C}$ is m-connected, then $s'(X) \geq m + 2$.*

Proof. This is a tautology when $m = -1$. For $m \geq 0$, assume, inductively, that $s'(X) \geq m + 1$, i.e. there exists $f: S^m \twoheadrightarrow X$. By the m-connectedness of X, f can be extended continuously to S_+^{m+1}, the upper $(m+1)$-sphere. Now extend f to $\hat{f}: S^{m+1} \twoheadrightarrow X$ by $\hat{f}(x) = -f(-x)$ for $x \in S_-^{m+1}$, the lower $(m+1)$-sphere. (The continuity of $\hat{f}$ is easy to check.) This shows that $s'(X) \geq m + 2$. Q.E.D.

In general, $s'(X)$ may be strictly less than $s(X)$.[(3)] The following Proposition gives a necessary condition for the equality of these two invariants:

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[3] It is not difficult to exhibit spaces X with involution for which $s(X) - s'(X)$ is arbitrarily large. In fact, as pointed out to us by Professor P. Conner, for any given natural number n, there exist spaces X with involution for which $s(X) - s'(X) = n$.

PROPOSITION 2.11. *Let* $X \in \operatorname{Obj} \mathscr{C}$. *If* $s'(X) = s(X) = k < \infty$, *then the homotopy group* $\pi_{k-1}(X)$ *cannot be a torsion group. In fact,* $\pi_{k-1}(X)$ *has a quotient group which is infinite cyclic.*

Proof. By hypothesis there exist $S^{k-1} \xrightarrow{g} X \xrightarrow{f} S^{k-1}$. This induces group homomorphisms

$$\pi_{k-1}(S^{k-1}) \xrightarrow[g_*]{} \pi_{k-1}(X) \xrightarrow[f_*]{} \pi_{k-1}(S^{k-1}).$$

The composition $f \circ g$ is an antipodal-preserving self-map of S^{k-1}, so by the theorem of Borsuk, it has odd degree, and in particular not null-homotopic. Thus, $(f \circ g)_* = f_* \circ g_*$ is a nontrivial endomorphism of $\pi_{k-1}(S^{k-1}) \cong \mathbb{Z}$, and so $\operatorname{im}(f_*)$ is infinite cyclic. Q.E.D.

COROLLARY 2.12. *Let* $X \in \operatorname{Obj} \mathscr{C}$. *If* X *is* m*-connected and* $\pi_{m+1}(X)$ *does not have an infinite cyclic quotient group (e.g.* $\pi_{m+1}(X)$ *is torsion), then* $s(X) \geq m+3$.

Proof. By (2.10), we have $s'(X) \geq m+2$. If this is a strict inequality, then $s(X) \geq s'(X) \geq m+3$, as desired. Thus, we may assume that $s'(X) = m+2$. Applying (2.11) for $k = m+2$, we see that $s(X) \neq s'(X)$, and so $s(X) \geq 1 + s'(X) = m+3$. Q.E.D.

§3. The Level Theorem

The goal of this section is to show that the level $s(X)$ of an object $X \in \operatorname{Obj} \mathscr{C}$ can be computed in purely algebraic terms; in fact, it is given by the (algebraic) level of a certain function ring canonically associated with X. We shall begin by introducing this important function ring.

For $X = (X, -) \in \operatorname{Obj} \mathscr{C}$, we define A_X to be the ring of continuous functions $f: X \to \mathbb{C}$ with the property that $f(\bar{x}) = \overline{f(x)}$ for any $x \in X$. Thus, A_X is the set of all $\mathscr{C}$-morphisms of X into $\mathbb{C}$ (with the complex conjugation as involution); it is a ring under the usual addition and multiplication of functions. Constant maps of X into $\mathbb{R}$ are in A_X, so A_X is an $\mathbb{R}$-algebra. (In general, A_X is not a $\mathbb{C}$-algebra). Clearly, any equivariant map $f: X \twoheadrightarrow Y$ induces an $\mathbb{R}$-algebra homomorphism $f^*: A_Y \to A_X$. Thus, the association $X \mapsto A_X$ gives a contravariant functor from $\mathscr{C}$ to the category of commutative $\mathbb{R}$-algebras. In the following, we shall call A_X the function ring of X.

Note that A_X admits other algebraic structures as well. For instance, it carries a natural involution: for $f \in A_X$, we can define $\bar{f} \in A_X$ by $\bar{f}(x) = \overline{f(x)}$ (for every

$x \in X$). Also, in the case when X is compact, we can equip functions in A_X with a "sup-norm," thereby making A_X into a topological algebra. This kind of topological algebras with involution had been studied many years ago by Kaplansky and Arens [AK]. For the purposes of the present work, we shall be interested in A_X mainly as an $\mathbb{R}$-algebra.

For any function $f \in A_X$, write $f(x)=p(x)+iq(x)$, where p, q are real-valued functions on X. The equation $f(\bar{x})=\overline{f(x)}$ gives $p(\bar{x})+iq(\bar{x})=p(x)-iq(x)$, so we have $p(\bar{x})=p(x)$ and $q(\bar{x})=-q(\bar{x})$ for every $x \in X$. Thus, each $f \in A_X$ may be thought of as a pair of real-valued (continuous) functions (p, q) where p is "even" and q is "odd."

Recall that $s(A)$ denotes the level of a ring A. The following result computes the (topological) level of a space X with involution in terms of the (algebraic) level of its function ring:

LEVEL THEOREM 3.1. *For any $X \in \operatorname{Obj} \mathscr{C}$, $s(X)=s(A_X)$. In other words, the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{A} & \{\mathbb{R}\text{-algebras}\} \\ & {}_{s_{top}}\searrow \quad \swarrow_{s_{alg}} & \\ & \mathbb{N} \cup \{\infty\} & \end{array}$$

Proof. Step 1. First, we note that $s(A_{S^{n-1}}) \le n$. In fact, define $f_j : S^{n-1} \to \mathbb{C}$ by $f_j(x)=ix_j$, where $x=(x_1, \ldots, x_n) \in S^{n-1}$, and $i=\sqrt{-1}$. Clearly $f_j \in A_X$ and

$$(f_1^2+\cdots+f_n^2)(x)=(ix_1)^2+\cdots+(ix_n)^2=-1,$$

so $-1=f_1^2+\cdots+f_n^2$ in A_X, i.e. $s(A_{S^{n-1}}) \le n$.

Step 2. Let m be any integer $<s(X)$, and let $h(x_1, \ldots, x_m)$ be any real polynomial which does not represent -1 over $\mathbb{R}$. Then h does not represent -1 over A_X. In fact, assume there exist $f_1, \ldots, f_m \in A_X$ such that $-1=h(f_1, \ldots, f_m) \in A_X$. Write $f_j=p_j+iq_j$ $(1 \le j \le m)$. Then the $\{q_j\}$ do not have a common zero on X. For if $x \in X$ is such a common zero, then evaluation of $h(f_1, \ldots, f_m)$ at x gives $-1=h(p_1(x), \ldots, p_m(x))$, a contradiction. Thus, we can define a continuous map $q : X \to S^{m-1}$ by

$$q(x)=(q_1(x)/\delta(x), \ldots, q_m(x)/\delta(x)),$$

where $\delta(x)=\sqrt{q_1(x)^2+\cdots+q_m(x)^2} \neq 0$. This is an equivariant map since the q_j's are *odd* functions. This shows that $s(X) \le m$, a contradiction.

Step 3. Applying Step 2 to $h=x_1^2+\cdots+x_m^2$ where $m<s(X)$, we see that -1 is not a sum of fewer than $s(X)$ squares in A_X. Thus $s(A_X) \ge s(X)$. To show the

reversed inequality, we may assume that $n := s(X) < \infty$. Take an equivariant map $X \twoheadrightarrow S^{n-1}$. This induces a ring homomorphism $A_{S^{n-1}} \to A_X$. Therefore $s(A_X) \leq s(A_{S^{n-1}}) \leq n$, by Step 1. Q.E.D.

From (2.8) and (3.1), we have $s(A_{S^{n-1}}) = s(S^{n-1}) = n$, so there exist $\mathbb{R}$-algebras of any prescribed level n. In particular, we have

COROLLARY 3.2. *The generic algebra* $A_n := \mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)$ *has level* n (cf [DLP, Theorem 1]).

We close this section with some refinements and extensions of (3.2).

PROPOSITION 3.3. *The level of an integral domain A and the level of its quotient field F can differ by an arbitrary amount.*

Proof. The example $A = \mathbb{R}[t_1, \ldots, t_m]/(t_1^2 + \cdots + t_m^2)$ shows that we can have $s(A) = \infty$, and $s(F) =$ any prescribed 2-power (see [La$_1$: p. 306]). Next, let $m = 2^i \leq n < \infty$. Changing notations, let $A = A_n[t_1, \ldots, t_{m+1}]/(t_1^2 + \cdots + t_{m+1}^2)$ (where A_n is as defined in (3.2)), and F be its quotient field. Since there exist homomorphism $A \to A_n \to A$, we have $s(A) = s(A_n) = n$ by (3.2). Let F_n be the quotient field of A_n. By Pfister's Theorem [La$_1$: p. 306], $s(F_n) \geq m$. Since F is the quotient field of $F_n[t_1, \ldots, t_{m+1}]/(t_1^2 + \cdots + t_{m+1}^2)$, the same theorem of Pfister, applied once more, shows that $s(F) = m$. Q.E.D.

PROPOSITION 3.4. *Let k be a (commutative) semireal ring in the sense of* [La$_2$] *(i.e. with* $s(k) = \infty$*). Then the ring* $A = k[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots x_n^2)$ *has level n.*

Proof. As is well-known (e.g. [La$_2$, §2]), k has a real prime ideal, so k admits a homomorphism into a formally real field, and therefore into a real-closed field. Thus, we may as well assume that k *is* itself a real-closed field. In this case we can deduce $s(A) = n$ from (3.2) by Tarski's Principle. Alternatively, following a suggestion of M. Knebusch, we can proceed as follows: Assume that $s(A) < n$. Then there exists an equation

(3.5) $$-1 = f_1(x)^2 + \cdots + f_{n-1}(x)^2 + f_0(x)(1 + x_1^2 + \cdots + x_n^2)$$

in $k[x] = k[x_1, \ldots, x_n]$. Pick a finitely generated $\mathbb{Q}$-algebra $R \subset k$ which contains all the coefficients of $f_0, f_1, \ldots, f_{n-1}$. By Lang's Homomorphism Theorem [La$_2$, §5], there exists a ring homomorphism of R into $\mathbb{R}$ (in fact even into the field of real algebraic numbers). Thus, an equation similar to (3.5) exists in $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$, contradicting (3.2). Q.E.D.

§4. The Colevel Theorem

In view of the Level Theorem (3.1), it seems natural to ask if one could also find a suitable definition for the "colevel" of a ring such that the (topological) colevel of a space X with involution is given by the (algebraic) colevel of its function ring A_X. In this direction, we have only partial success. In the following, we shall offer a definition for the colevel $s'(A)$ of an $\mathbb{R}$-algebra A: for any space X with involution, we have an inequality $s'(X) \leq s'(A_X)$; we can establish the equality only for those X's which arises as affine varieties defined over $\mathbb{R}$ (with complex conjugation as involution).

DEFINITION 4.1. For any $\mathbb{R}$-algebra A, the *colevel* $s'(A)$ is defined by

$$s'(A) = \sup\{m \geq 1 : \exists\ \mathbb{R}\text{-algebra homomorphism } A \to A_{S^{m-1}}\}.$$

If there is no such $m \geq 1$, we defined $s'(A) = 0$.

Examples and Remarks (4.2)

(a) For any $\mathbb{R}$-algebra A, we have $s'(A) \leq s(A)$. this follows from the observation that, if there is a homomorphism $A \to A_{S^{m-1}}$, then $s(A) \geq s(A_{S^{m-1}}) = m$.

(b) If we have an $\mathbb{R}$-algebra homomorphism $B \to A$, then $s'(B) \geq s'(A)$.

(c) Since $s(A_{S^{n-1}}) = n$, $A_{S^{n-1}}$ has no homomorphism into A_{S^n}. Therefore $s'(A_{S^{n-1}}) = n$. By a similar argument, we have also $s'(A_n) = n$ where $A_n = \mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)$.

COLEVEL THEOREM 4.3. *For any space with involution* $(X, -)$, *we have* $s'(X) \leq s'(A_X)$. *Equality holds if X is an affine variety defined over $\mathbb{R}$, with involution given by complex conjugation.*

Proof. To prove the claimed inequality, we may assume that $n = s'(A_X) < \infty$. If $s'(X) \geq n+1$, then by definition there is an equivariant map $S^n \twoheadrightarrow X$. But then we have an induced homomorphism $A_X \to A_{S^n}$; from (4.2)(b, c), we get $s'(A_X) \geq s'(A_{S^n}) = n+1$, a contradiction. Therefore $s'(X) \leq n = s'(A_X)$.

Next, we shall deal with affine varieties defined over $\mathbb{R}$. Let $\mathfrak{A}$ be an ideal in $\mathbb{R}[x_1, \ldots, x_n]$, and $X = V_{\mathbb{C}}(\mathfrak{A})$ be the affine variety in $\mathbb{C}^n$ it defines. As observed in (2.3), we have $(X, -) \in \mathscr{C}$ where "bar" is induced by complex conjugation. We shall denote the real coordinate ring $\mathbb{R}[x_1, \ldots, x_n]/\mathfrak{A}$ by $\mathbb{R}[X]$. (Actually this depends not only on X but also on the choice of $\mathfrak{A}$. However we shall always work with a fixed $\mathfrak{A}$ so the notation is not likely to cause confusion.) Note that each $f \in \mathbb{R}[X]$ induces an equivariant function from $(X, -)$ to $(\mathbb{C}, \bar{})$, so there is a natural $\mathbb{R}$-algebra homomorphism from $\mathbb{R}[X]$ to A_X. Note that the following statements are equivalent:

(a) $s(\mathbb{R}[X]) < \infty$,

(b) $s(X) = s(A_X) < \infty$,

(c) $s'(X) < \infty$,

(d) X has no real point.

In fact, (a)$\Rightarrow$(b) is obvious since there is a homomorphism from $\mathbb{R}[X]$ to A_X; (b)$\Rightarrow$(c) follows from $s'(X) \leq s(X)$; (c)$\Rightarrow$(d) is obvious; finally (d)$\Rightarrow$(a) follows from the Real Nullstellensatz of Dubois and Risler (see e.g. [La_2, §2]).

Now let $(Y, -)$ be *any* space with involution, and consider any $\mathbb{R}$-algebra homomorphism $g: \mathbb{R}[X] \to A_Y$. We shall show that such a g must "arise" from some equivariant map $\gamma: Y \twoheadrightarrow X$. In fact, let $\xi_i \in \mathbb{R}[X]$ be the coordinate functions on X. For $y \in Y$, we can define

$$\gamma(y) = (g(\xi_1)(y), \ldots, g(\xi_n)(y)) \in \mathbb{C}^n.$$

This point lies in the affine variety X, since, for any polynomial $a(x_1, \ldots, x_n) \in \mathfrak{A}$,

$$\begin{aligned} a(\xi_1, \ldots, \xi_n) = 0 &\Rightarrow a(g(\xi_1), \ldots, g(\xi_n)) = 0 \\ &\Rightarrow a(g(\xi_1)(y), \ldots, g(\xi_n)(y)) = 0. \end{aligned}$$

It is routine to check that $\gamma: Y \to X$ is continuous and equivariant, and that the induced map $\gamma^*: A_X \to A_Y$ "extends" the given homomorphism $g: \mathbb{R}[X] \to A_Y$. In particular, we conclude that, for X (affine) and Y (arbitrary) as above,

$$\begin{aligned} &\exists\, Y \twoheadrightarrow X \\ \Leftrightarrow &\exists\, \mathbb{R}\text{-algebra homomorphism } A_X \to A_Y \\ \Leftrightarrow &\exists\, \mathbb{R}\text{-algebra homomorphism } \mathbb{R}[X] \to A_Y. \end{aligned}$$

Taking Y to be the unit spheres S^{m-1}, we see now that

(4.4) $s'(X) = s'(A_X) = s'(\mathbb{R}[X])$.

Moreover, if X has no real points, then these numbers are $\leq s(X) \leq n$, by (2.9)(2). (Otherwise, they are ∞.)

For the level, we have $s(X) = s(A_X) \leq s(\mathbb{R}[X])$. *Is the inequality actually an equality*? It turns out that equality does hold if $n = 1$ (cf. (4.8) below), but may no longer hold if $n \geq 2$. In the latter case, we have $s(X) = s(A_X) \leq n$ (assuming X has no real points), but we may have $s(\mathbb{R}[X]) > n$, as the following Proposition shows.

PROPOSITION 4.5. *Let $\gamma(x)$ be a real nonconstant polynomial such that $\gamma(a) \geq 1$ for all $a \in \mathbb{R}$. Let $\mathfrak{A}$ be the principal ideal generated by $\gamma(x)^2 + y^2$ in $\mathbb{R}[x, y]$, and let $X = V_{\mathbb{C}}(\mathfrak{A})$. Then $s(X) = s(A_X) = 2$ but $s(\mathbb{R}[X]) = 3$.*

Proof. Clearly X has no real points, so $s(X) \leq 2$. Geometrically X is the union of the two curves $y = \pm i\gamma(x)$ which intersect at the points $\{(c, 0) \in \mathbb{C}^2 : \gamma(c) = 0\}$, so X is connected. From (2.10) it follows that $s(X) = 2$. Let $A = \mathbb{R}[X]$ be the real coordinate ring of X. The quotient field of A has level 1 but we claim that A *has level* 3. Let $\theta(x) = \gamma(x)^2 - 1$; by hypothesis $\theta(a) \geq 0$ for all $a \in \mathbb{R}$ so we can write $\theta(x) = \theta_1(x)^2 + \theta_2(x)^2$ for suitable $\theta_i \in \mathbb{R}[x]$. Then in A we have $-1 = \theta_1(\bar{x})^2 + \theta_2(\bar{x})^2 + \bar{y}^2$, so $s(A) \leq 3$. Assume $s(A) \leq 2$: then we would have an equation

$$(4.6) \quad -1 = f_1(x, y)^2 + f_2(x, y)^2 + h(x, y)(\gamma(x)^2 + y^2).$$

We may assume that f_1 and f_2 are at most of degree 1 in y. Then clearly $h := h(x, y)$ cannot involve y. Write $f_j = p_j + yq_j$ $(p_j, q_j \in \mathbb{R}[x])$. Plugging these expressions into (4.6), we obtain the following three equations:

$$(4.7) \quad \begin{cases} -1 = p_1^2 + p_2^2 + h \cdot \gamma^2 \\ 0 = p_1 q_1 + p_2 q_2 \\ 0 = q_1^2 + q_2^2 + h. \end{cases}$$

Writing $\alpha = p_1^2 + p_2^2$, we have, by the 2-square identity

$$\begin{aligned} -\alpha \cdot h &= (p_1^2 + p_2^2)(q_1^2 + q_2^2) \\ &= (p_1 q_1 + p_2 q_2)^2 + (p_1 q_2 - p_2 q_1)^2 \\ &= (p_1 q_2 - p_2 q_1)^2. \end{aligned}$$

But from (4.7), $1 + \alpha = -h\gamma^2$ so

$$\alpha(1 + \alpha) = -\alpha h \gamma^2 = (p_1 q_2 - p_2 q_1)^2 \gamma^2.$$

Since α and $1 + \alpha$ are relatively prime, *each* must be a perfect square, say $\alpha = \phi^2$, $1 + \alpha = \psi^2$. But then $1 = \psi^2 - \phi^2 = (\psi + \phi)(\psi - \phi)$ implies that $\psi \pm \phi \in \mathbb{R}$ and so $\psi, \phi, \alpha \in \mathbb{R}$. This clearly contradicts $1 + \alpha = -h\gamma^2$ since γ is a nonconstant polynomial. Thus $s(A) = 3$. Q.E.D.

Finally, for the case $n = 1$, we prove:

PROPOSITION 4.8. *Let $\mathfrak{A}=(f(x))\subset\mathbb{R}[x]$ where f is a nonconstant polynomial without real root. Then the monogenic $\mathbb{R}$-algebra $A=\mathbb{R}[x]/\mathfrak{A}$ has level* 1.

Proof. Write $f=\varepsilon f_1^{r_1}\cdots f_k^{r_k}$ where $\varepsilon\in\mathbb{R}$ and f_i are distinct monic irreducible quadratic polynomials. By the Chinese Remainder Theorem, $A\cong\prod\mathbb{R}[x]/(f_i^{r_i})$, so it suffices to show that each $A_j=\mathbb{R}[x]/(f_j^{r_j})$ has level 1. Now A_j is a local algebra whose maximal ideal $\mathfrak{M}=(f_j)/(f_j^{r_j})$ is nilpotent, and $A_j/\mathfrak{M}\cong\mathbb{R}[t]/(f_j)\cong\mathbb{C}$, in which -1 is a square. By Hensel's Lemma,[4] it follows that -1 is a square in A_j. (For instance, if $A_j=\mathbb{R}[t]/(t^2+1)^2$, the usual proof of Hensel's Lemma using Newton's Method gives $\overline{(t^3+3t)/2}$ as a square root of -1 in A_j.) Q.E.D.

§5. The sublevel of a ring

In this section, we shall define the sublevel of a ring. For any $X\in\operatorname{Obj}\mathscr{C}$ (a space with involution), we shall establish a useful inequality ((5.11)) between the colevel of X and the sublevel of its function ring A_X. In general, however, these two numbers need not be equal.

DEFINITION 5.1. Let A be a commutative ring, and $f\in A[x_1,\ldots,x_m]$ be a form (i.e. a homogeneous polynomial) of degree d over A. We say that f is *isotropic over* A if f has a unimodular zero vector, i.e. if there exist $a_1,\ldots,a_m\in A$ generating A as an ideal, such that $f(a_1,\ldots,a_m)=0\in A$. If f is not isotropic, we shall say that f is *anisotropic* over A. For $b\in A\setminus\{0\}$, we shall say that f *represents* b *over* A if $b=f(b_1,\ldots,b_m)$ for some $b_1,\ldots,b_m\in A$.

DEFINITION 5.2. The *sublevel* $\sigma(A)$ of a ring A is defined by

$$1\leq\sigma(A):=\min\{n:(n+1)\langle 1\rangle \text{ is isotropic over } A\}.$$

Here, $r\langle 1\rangle$ denotes the r-dimensional quadratic form $x_1^2+\cdots+x_r^2$ over A. (More generally, $\langle b_1,\ldots,b_r\rangle$ denotes the quadratic form $b_1x_1^2+\cdots+b_rx_r^2$.)

DEFINITION 5.3. The *pythagoras number* $P(A)$ of a ring A is the smallest integer n such that any sum of squares in A can be written as a sum of n squares. If there is no such integer n, we define $P(A)=\infty$.

Remarks 5.4. (a) If we have a homomorphism $A\to B$, then $\sigma(A)\geq\sigma(B)$.
(b) For any ring A, we have $\sigma(A)\leqslant s(A)$. In fact, assume that $s=s(A)<\infty$.

[4] We thank George Bergman who suggested the use of Hensel's Lemma here.

Then there is an equation $1+a_1^2+\cdots+a_s^2=0$, so $(1, a_1, \ldots, a_s)$ is a unimodular zero vector for $(s+1)\langle 1\rangle$. This gives $\sigma(A)\leq s=s(A)$ (hence the term "sublevel"). If A is an $\mathbb{R}$-algebra, we shall show later in this section that $s'(A)\leqslant\sigma(A)$.

(c) If A is an integral domain, with quotient field F, then $s(F)\leq\sigma(A)\leq s(A)$.

The level and the sublevel of a ring A are both related to the pythagoras number of A, as the following Proposition shows.

PROPOSITION 5.5. *Let A be a ring in which 2 is a unit, and $\sigma(A)<\infty$. Then we have*

$$s(A)\leq P(A)\leq 1+\sigma(A)=P(A[t]).$$

Proof. The first inequality is obvious. For the second inequality, let $n=\sigma(A)$. It is well-known that (under $2\in U(A)$)

(5.6) A (regular) quadratic form over A is isotropic iff it contains the hyperbolic plane $\langle 1, -1\rangle$ as an orthogonal direct summand.

Since $\langle 1, -1\rangle$ represents all elements of A, the same holds for $(n+1)\langle 1\rangle$, and so $P(A)\leq n+1$. It remains only to prove the last equality in (5.5). First note that $\sigma(A[t])=\sigma(A)$ (e.g. by (5.4)(a)). Therefore, by the second inequality in (5.5) (applied to $A[t]$), we have

$$P(A[t])\leq 1+\sigma(A[t])=1+\sigma(A).$$

Next, suppose $P(A[t])=m$. Since $s(A)<\infty$, any element $f(t)\in A[t]$ is a sum of squares, and therefore a sum of m squares. Write

$$t=\sum_{i=1}^{m}(a_0^{(i)}+a_1^{(i)}t+\cdots+a_d^{(i)}t^d)^2, \qquad a_j^{(i)}\in A.$$

Then we have

$$\sum_{i=1}^{m}a_0^{(i)2}=0 \quad \text{and} \quad 2\sum_{i=1}^{m}a_0^{(i)}a_1^{(i)}=1.$$

Since $2\in U(A)$, $(a_0^{(1)}, \ldots, a_0^{(m)})$ is a unimodular zero vector for $m\langle 1\rangle$ over A. By definition, we have $\sigma(A)\leq m-1$ so $1+\sigma(A)\leq P(A[t])$. Q.E.D.

COROLLARY 5.6. *Let A be a PID with $2\in U(A)$. Let n be the level of its* *quotient field F. Then* $\sigma(A)=n$ *and* $s(A)\in\{n, n+1\}$. (The latter is a special case of a result of Baeza [Ba] for Dedekind rings.)

Proof. We may assume that $n<\infty$. From $s(F)=n$, we have an equation $a_0^2+\cdots+a_n^2=0$ with $a_i\in A\setminus\{0\}$. After knocking out common factors, we may assume that $\{a_0,\ldots,a_n\}$ have no common factor. Since A is a PID, this means that $(a_0,\ldots,a_n)$ is unimodular, so $\sigma(A)=n$. Given this, the Proposition implies that $s(A)=n$ or $n+1$. Q.E.D.

EXAMPLE 5.7. *There do exist rings with* $\sigma(A)\neq s(A)$. For instance, $A=\mathbb{Q}[x,y]/(1+x^2+2y^2)$ is a PID with $s(A)=3$ (see [CLRR, (3.8)]). However, by (5.6), $\sigma(A)=s(F)=2$ (and therefore, by (5.5), $P(A)=3$).

This example raises the following interesting

QUESTION 5.8. *For* $n\geq 1$ *what pairs* (n,n), $(n,n+1)$ *can be realized as* $(\sigma(A), s(A))$ *for some commutative ring* A?

The following Proposition shows that not all pairs $(n, n+1)$ can be so realized:

PROPOSITION 5.9. *If* $s(A)=1$, 2, 4 *or* 8, *then* $\sigma(A)=s(A)$.

Proof. Let us explain the proof first in the case when $s(A)=4$. Assume that $\sigma(A)=3$. Then there exist two equations: $a_1^2+\cdots+a_4^2=0$, $a_1b_1+\cdots+a_4b_4=1$ in A. Consider the classical 4-square identity

$$(x_1^2+\cdots+x_4^2)(y_1^2+\cdots+y_4^2)=(x_1y_1+\cdots+x_4y_4)^2+f_2^2+f_3^2+f_4^2,$$

where f_2, f_3, f_4 are bilinear forms over $\mathbb{Z}$. Plugging in $x_i=a_i$, $y_i=b_i$ and transposing, we see that -1 is a sum of three squares in A, contradicting $s(A)=4$. The cases $s(A)=2$, $s(A)=8$ follow similarly from the 2-square identity and the 8-square identity. (The case $s(A)=1$ is trivial since, by definition, $\sigma(A)\geq 1$.) Q.E.D.

The Proposition above shows that the four pairs (0, 1), (1, 2), (3, 4), (7, 8) *cannot* be realized as $(\sigma(A), s(A))$ for any ring A. Later, we shall show, however, that, with these four exceptions, *all* pairs $(n, n+1)$ and (n,n) $(n\geq 1)$ can be realized as $(\sigma(A), s(A))$ for some ring A. One of the key results needed for this is the following:

THEOREM 5.10. *Let* $(X,-)$ *be a space with involution, and* $n\leq s'(X)$. *Let*

$f(t_1, \ldots, t_n)$ be a real homogeneous polynomial and $g(t_1, \ldots, t_{n-1})$ be any real polynomial.

(1) *If f is anisotropic over $\mathbb{R}$, then f remains anisotropic over A_X.*

(2) *If $g(a_1, \ldots, a_{n-1}) \neq 0$ for all $a_i \in \mathbb{R}$, then $g(d_1, \ldots, d_{n-1}) \neq 0$ in A_X for all $d_i \in A_X$.*

Before we prove this important principle, let us first state some of its main consequences. Applying (5.10)(1) to quadratic forms, we see that, for $n \leq s'(X)$, the forms $n\langle 1\rangle$ are anisotropic over A_X. Therefore, from the definition of the sublevel, we deduce the following inequality:

COROLLARY 5.11. *For any space with involution $(X, -)$, we have $s'(X) \leq \sigma(A_X)$.*

We note, however, that $s'(X)$ and $\sigma(A_X)$ may not be equal in general. In fact, by (4.5), $\sigma(A_X)$ is at most one less than $s(A_X) = s(X)$; since we have pointed out (in an earlier footnote) that $s(X)$ and $s'(X)$ can differ by an arbitrary amount, we see that $\sigma(A_X)$ and $s'(X)$ may also differ by a large integer.

COROLLARY 5.12. $\sigma(A_{S^{n-1}}) = \sigma(A_n) = n$ *(where* $A_n = \mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)$.

Proof. By (5.11) (plus (3.1) and (5.4)(b)), we have $s'(X) \leq \sigma(A_X) \leq s(X)$. Applying this to S^{n-1} yields $\sigma(A_{S^{n-1}}) = n$. To get the similar equation for A_n, note that there is a homomorphism from A_n to $A_{S^{n-1}}$ by mapping the $\overline{x}_j$'s in A_n to the functions $f_j \in A_{S^{n-1}}$ defined in the proof of (3.1). Therefore, by (5.4)(a), $\sigma(A_n) \geq \sigma(A_{S^{n-1}}) = n$. This must be an equality since we also have $\sigma(A_n) \leq s(A_n) = n$. Q.E.D.

Note that this Corollary already settles half of Question (5.8), since for $n \geq 1$, the pair (n, n) is realized as $(\sigma(A), s(A))$ by taking $A = A_n$ (or $A_{S^{n-1}}$). The realizability of $(n, n+1)$ for $n \neq 1, 3, 7$ depends on deeper topological facts, so we shall postpone it to a later section.

COROLLARY 5.13. *For any $\mathbb{R}$-algebra A, we have $s'(A) \leq \sigma(A)$ $(\leq s(A))$.*

Proof. We may assume that $n = \sigma(A) < \infty$. If $s'(A) \geq n + 1$, then by definition of the colevel there is a homomorphism $A \to A_{S^n}$. But then by (5.4)(a) and (5.12), $\sigma(A) \geq \sigma(A_{S^n}) = n + 1$, a contradiction. Thus, $s'(A) \leq n = \sigma(A)$. Q.E.D.

Remark 5.14. In a later section, we shall give a purely topological definition for the sublevel $\sigma(X)$ of any space X with involution, and shall prove a "sublevel theorem" to the effect that, for any X, $\sigma(X)=\sigma(A_X)$. Therefore, in view of (5.11), we also have $s'(X)\le\sigma(X)\le s(X)$ for any space X with involution, in parallel to (5.13).

With Choi and Reznick, we have shown earlier [CDLR] that there exist rings with arbitrarily prescribed pythagoras numbers. However, the rings constructed in [CDLR] are not integral domains. The last Corollary 5.13 enables us to show:

COROLLARY 5.15. *There exist integral domains with any prescribed pythagoras number* $n+1$.

Proof. We may assume $n>0$. By (5.5) and (5.12) we have $P(A_n[t])=1+\sigma(A_n)=n+1$. Q.E.D.

We shall now try to give the proof for Theorem 5.10. The key to the proof is the following geometric fact:

COLLINEARITY LEMMA 5.16. *Let* $(X,-)$ *be a space with involution, and let* $d_1,\dots,d_n\in A_X$, *where* $n\le s'(X)$. *Then there exists a point* $z\in X$ *such that* $d_1(z),\dots,d_n(z)\in\mathbb{C}$ *are collinear on a line in the Gaussian plane passing through the origin.*

Proof. By definition there exists an equivariant map $\lambda: S^{n-1}\twoheadrightarrow X$. We can compose the functions d_j with λ to get n functions in $A_{S^{n-1}}$. Therefore, we may assume that $X=S^{n-1}$ in the following. Write, as usual, $d_j(z)=p_j(z)+iq_j(z)$ where p_j, q_j are real functions on S^{n-1}.

CASE 1. $\{q_j\}$ have a common zero $z\in S^{n-1}$. Then $\{d_j(z)\}$ all lie on the real axis of $\mathbb{C}$ and we are done.

CASE 2. $\{p_j\}$ have a common zero $z\in S^{n-1}$. Then $\{d_j(z)\}$ all lie on the imaginary axis of $\mathbb{C}$ and we are done as before.

CASE 3. We may now assume that $\{q_j\}$ have no common zero on S^{n-1}, and also that $\{p_j\}$ have no common zero on S^{n-1}. After a normalization, each of these defines a continuous mapping, say q, respectively p, from S^{n-1} to S^{n-1}. Since q_j are odd functions, q is an odd mapping and hence has an odd (topological) degree. Similarly, p_j are even functions, so p is an even mapping, and hence has an even degree. In particular, q and p cannot be homotopic, so there exists a point $z \in S^{n-1}$ at which q and p are antipodal, i.e. $q(z) = -p(z)$. This means that $q_j(z) = -\delta \cdot p_j(z)$ $(1 \le j \le m)$ for some nonzero real number δ independent of j. Therefore, the complex numbers $\{d_j(z)\}$ all lie on the line with equation $y + \delta x = 0$ in the Gaussian plane, as claimed.

COROLLARY 5.17. *In the notation of the Lemma, there exists a point* $z' \in X$ *such that* $d_1(z'), \ldots, d_{n-1}(z')$ *are all real.*

Proof. This Corollary follows by applying the Lemma to the functions $\{d_1, \ldots, d_{n-1}, 1\}$. Alternatively, it may also be proved by replacing X by S^{n-1} as before and applying the Borsuk–Ulam Theorem to the odd mapping $(q_1, \ldots, q_{n-1}): S^{n-1} \to \mathbb{R}^{n-1}$, where q_i are the imaginary parts of d_i. Q.E.D.

Finally, we proceed to the

Proof of (5.10). Note that $(d_1, \ldots, d_n) \in A_X^n$ is unimodular over A_X iff $\{d_1, \ldots, d_n\}$ have a common zero on X. (If there is no common zero, $d_1\bar{d}_1 + \cdots + d_n\bar{d}_n$ will be invertible in A_X.) Now assume $f(d_1, \ldots, d_n) = 0$ where the notation is as in (5.10)(1). By the Lemma, there exists a point $z \in X$ such that $\{d_j(z)\}$ are collinear on a line through the origin in $\mathbb{C}$. Thus, there is an angle θ such that $d_j(z) = r_j e^{i\theta}$ $(1 \le j \le n)$, where $r_j \in \mathbb{R}$. Now we have

$$\begin{aligned} 0 &= f(d_1(z), \ldots, d_n(z)) \\ &= f(r_1 e^{i\theta}, \ldots, r_n e^{i\theta}) \\ &= (e^{i\theta})^k f(r_1, \ldots, r_n), \end{aligned}$$

where $k = \deg(f)$. (The homogeneity of f plays an important role here!) This implies that $f(r_1, \ldots, r_n) = 0$ and hence $r_1 = \cdots = r_n = 0$ since f is anisotropic over $\mathbb{R}$. Therefore $d_j(z) = 0$ for all j, and the n-tuple $(d_1, \ldots, d_n)$ cannot be unimodular over A_X. This proves (5.10)(1).

For (5.10)(2), assume that $g(d_1, \ldots, d_{n-1}) = 0 \in A_X$, where $d_1, \ldots, d_{n-1} \in A_X$. By (5.17), there exists a point $z' \in X$ such that $a_i = d_i(z') \in \mathbb{R}$ for $1 \le i \le n-1$. Evaluating at z', we get $g(a_1, \ldots, a_{n-1}) = 0$ in $\mathbb{R}$. Q.E.D.

Let A be any $\mathbb{R}$-algebra which is contained in some formally real field K. Then, for any anisotropic form $f(t_1, \ldots, t_n) \in \mathbb{R}[t_1, \ldots, t_n]$, we can conclude from Tarski's Principle that f has no nontrivial zero over K, and hence also no nontrivial zero over A. However, the function rings A_X are a very different kind of rings. For reasonable spaces X, they have finite level, and so cannot be mapped into formally real fields. It is somewhat surprising, therefore, that certain forms of the "Transfer Principle" (namely (5.10)(1), (2)) survive in this context. Note, however, that we cannot hope for the same "strong" transfer as in the formally real case, i.e. even though the form $f \in \mathbb{R}[t_1, ..., t_n]$ has no nontrivial zero in $\mathbb{R}^n$, it may well have a nontrivial zero in A_X^n. (Our Theorem (5.10)(1) guarantees only the nonexistence of a *unimodular* zero.) To give an example of this, take an integer n such that $2^i < n < 2^{i+1}$, and take $X = S^{n-1}$. The homomorphism $A_n \to A_{S^{n-1}}$ constructed in the proof of (5.12) can be easily checked to be an injection. Since the level of the quotient field of A_n is 2^i by Pfister's theorem, the form $n\langle 1 \rangle$ does have a nontrivial zero over A_n and hence also over $A_{S^{n-1}}$ (although it does not have a *unimodular* zero over either ring).

§6. Affine varieties $X \subset \mathbb{C}^n$ with level n

In this section, we continue to consider affine varieties $X \subset \mathbb{C}^n$ defined over $\mathbb{R}$, with involution given by complex conjugation. If X no real points, we have, from (2.9)(2), $s(A_X) = s(X) \leq n$. In this section, we shall construct large classes of examples of X for which this inequality becomes an equality. (This will enable us to construct many quotient rings of $\mathbb{R}[x_1, \ldots, x_n]$ with level exactly equal to n.) In view of the inequality $s'(X) \leq s(X)$, the natural way to get such examples is to look for varieties X for which $s'(X) \geq n$.

EXAMPLE 6.1. Let $f_1(t), \ldots f_n(t)$ be nonzero real polynomials each of which has at least one real root, and let $X = V_{\mathbb{C}}(g) \subset \mathbb{C}^n$ where $g = 1 + f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)$. Then $s'(X) \geq n$.

To show this, we must construct an equivariant map $H: S^{n-1} \twoheadrightarrow X$. We shall construct maps $h_j: [-1, 1] \to \mathbb{C}$ and obtain H by the formula

(6.2) $H(a_1, \ldots, a_n) = (h_1(a_1), \ldots, h_n(a_n))$

for $(a_1, \ldots, a_n) \in S^{n-1}$. To define the h_j's, fix a real root δ_j for the polynomial f_j. We shall first define h_j on $[0, 1]$ and then extend h_j to $[-1, 0]$ by "reflection." In detail, we define $h_j(0) = \delta_j$, and for $a \in [0, 1]$, we define $h_j(a) = b$ where b is a complex root of $f_j(t) = -a^2$ depending continuously on a. For $a' \in [-1, 0]$, let $a = -a' \in [0, 1]$, and $b = h_j(a)$; then define $h_j(a') = \bar{b}$. (This is also a root of $\overline{f_j(t) = -a'^2}$ since a is a root of $f_j(t) = -a^2$.) Our definition ensures that $h_j(-a) = \overline{h_j(a)}$ for *all* $a \in [-1, 1]$. Moreover, for $(a_1, \ldots, a_n) \in [-1, 1]^n$, we have

$$f_1(h_1(a_1)) + \cdots + f_n(h_n(a_n)) = -(a_1^2 + \cdots + a_n^2).$$

Thus, if $(a_1, \ldots, a_n) \in S^{n-1}$, the pont $H(a_1, \ldots, a_n)$ in (6.2) lies on the hypersurface $X = V_{\mathbb{C}}(g)$. By construction we have $H(-a_1, \ldots, -a_n) = \overline{H(a_1, \ldots, a_n)}$, so $H: S^{n-1} \twoheadrightarrow X$ is the desired equivariant map.

COROLLARY 6.3. *Let f_j $(1 \leq j \leq n)$ be as above. Then the ring* $A = \mathbb{R}[x_1, \ldots, x_n]/(1 + f_1(x_1)^2 + \cdots + f_n(x_n)^2)$ *has level n.*

Proof. The affine variety Y defined by $1 + f_1(x_1)^2 + \cdots + f_n(x_n)^2 = 0$ over $\mathbb{C}$ has no real points, so, by (6.1), $s(Y) = s'(Y) = n$. Therefore $s(A) = s(\mathbb{R}[Y]) \geq s(A_Y) = s(Y) = n$. On the other hand we have clearly $s(A) \leq n$, so equality follows.

To better understand the construction of the equivariant map H in (6.2), let us examine it more closely in the special case $f_j(t) = t^{r_j}$, where all $r_j > 0$. Here, the δ_j's are all zero, and the ring in question is $A = \mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^{r_1} + \cdots + x_n^{r_n})$. For $a \in [-1, 1]$, $b = h_j(a)$ is supposed to be a root of $t^{r_j} = -a^2$. Fixing a primitive $2r_j$-th root of unity, say ζ_j, we can define b explicitly as follows:

$$(6.4) \quad b = \begin{cases} \zeta_j (a^2)^{1/r_j} & \text{if } \ a \geq 0, \\ \bar{\zeta}_j (a^2)^{1/r_j} & \text{if } \ a \leq 0. \end{cases}$$

This, of course, depends continuously on a. Note that, in the special case when all $r_j = 2$, the definition above boils down simply to $b = i \cdot a$ $(i = \sqrt{-1})$, irrespective of the sign of a. Note that $a \to i \cdot a$ was exactly the map exploited in the proof of $s(\mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)) = n$ given in [DLP], though this crucial equivariant map was disguised there as a "substitution of variables" $x_j \to ix_j$.

EXAMPLE 6.5. Let $q \in \mathbb{R}[x_1, \ldots, x_n]$ be a nonconstant, absolutely irreducible polynomial, $Y = V_{\mathbb{C}}(q) \subseteq \mathbb{C}^n$ and $A = \mathbb{R}[Y] = \mathbb{R}[x_1, \ldots, x_n]/(q)$. If the projective closure of Y is nonsingular, then $s'(Y) \geq n$. In particular, the conclusions of (5.10) are applicable to A and to A_Y. If Y has no real points, then $n = s'(Y) = s(Y) = s(A_Y) \leq s(A) < \infty$.

To show this, use the fact that, under the stated hypotheses, the affine hypersurface Y has the homotopy type of a bouquet of $(n-1)$-spheres (see [M], [GH, p. 486]). In particular, Y is $(n-2)$-connected. By (2.10), we have therefore $s'(Y) \geq n$.

To construct the next family of examples, we shall need the following result on the function ring of a space X which is in some sense "dual" to (5.10). We thank M. Knebusch who suggested to us the statement of this result as well as the key ideas for its proof.

THEOREM 6.6. *Let $(X,-)$ be a space with involution, with $s(X)\leq n$. Let $p(t_1,\ldots,t_n)$ be a nonconstant, absolutely irreducible real homogeneous polynomial. If p is anisotropic over $\mathbb{R}$, then p represents all (nonzero) real numbers over A_X.*

(For the terminology used here, see (5.1).)

Proof. Since an equivariant map $X \twoheadrightarrow S^{n-1}$ induces an $\mathbb{R}$-algebra homomorphism $A_{S^{n-1}}\to A_X$, we may assume in the following that $X=S^{n-1}$. Note that the degree of p must be even, and p cannot be indefinite on $\mathbb{R}^n$. Therefore, we may assume that p is positive definite (i.e. $p(\mathbb{R}^n\setminus\{0\})>0$); in this case it will be sufficient to prove that p represents -1 over A_X. The proof will be carried out in two steps.

Step 1. We assume here that p is a *regular* form, in the sense that the partial derivatives $\partial p/\partial x_1,\ldots,\partial p/\partial x_n$ do not have a nontrivial common zero in $\mathbb{C}^n$. Since p is absolutely irreducible, clearly so is

$$q(x_1,\ldots,x_n):=1+p(x_1,\ldots,x_n).$$

Consider the affine hypersurface $Y:=V_{\mathbb{C}}(q)\subseteq\mathbb{C}^n$. We claim that

(6.7) *The projective closure $\bar{Y}$ of Y is nonsingular.*

If this is the case, then by (6.5) there exists an equivariant map $X=S^{n-1}\twoheadrightarrow(Y,-)$ ("bar" = complex conjugation), and we have $\mathbb{R}$-algebra homomorphisms

$$\mathbb{R}[x_1,\ldots,x_n]/(q)\to A_Y\to A_X.$$

Since p represents -1 in $\mathbb{R}[x_1,\ldots,x_n]/(q)$, it follows that p also represents -1 in A_X, as desired. To prove the claim (6.7), let us assume, instead, that $\bar{Y}$ does have a singular point $(a_0:a_1:\cdots:a_n)$ in $\mathbb{CP}^n$.

CASE 1. $a_0\neq 0$, say $a_0=1$. Then $(a_1,\ldots,a_n)$ kills all the partial derivatives $\partial q/\partial x_i=\partial p/\partial x_i$. By Euler's formula it follows that $p(a_1,\ldots,a_n)=0$. But then $q(a_1,\ldots,a_n)=1+p(a_1,\ldots,a_n)=1$, a contradiction.

CASE 2. $a_0=0$. Since $\bar{Y}$ is given by the homogeneous equation $x_0^d+p(x_1,\ldots,x_n)=0$ $(d=\deg p)$, we must have $p(a_1,\ldots,a_n)=0$. Meanwhile, the singular point $(0:a_1:\cdots:a_n)$ kills all partial derivatives

$$\frac{\partial}{\partial x_i}(x_0^d+p(x_1,\ldots,x_n))=\frac{\partial p}{\partial x_i}\quad(1\leq i\leq n),$$

so $(a_1, \ldots, a_n)$ is a nontrivial common zero of $\partial p/\partial x_1, \ldots, \partial p/\partial x_n$, a contradiction to the regularity of p.

Step 2. In the general case, we know by algebraic geometry that p can be approximated arbitrarily well (coefficient-wise) by a regular, absolutely irreducible (n-ary d-ic) form $\tilde{p}$. Moreover, if the approximation is good enough, the form $\tilde{p}$ will also be positive definite. By Step 1, we know that $\tilde{p}$ represents -1 over A_X, say $-1 = \tilde{p}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in A_X$. Let $\phi := -p(\lambda_1, \ldots, \lambda_n) \in A_X$. Since $X = S^{n-1}$ is compact, by choosing $\tilde{p}$ sufficiently close to p (coefficient-wise), we can insure that the function $\phi: X \to \mathbb{C}$ takes only values near 1, say within an open ball of radius $\frac{1}{2}$ around 1. *Then, ϕ is a d-th power in A_X.* In fact, consider the binomial expansion

$$(1+z)^{1/d} = \sum_{j=0}^{\infty} \binom{1/d}{j} z^j \quad \text{for} \quad |z| < \tfrac{1}{2}.$$

Taking $z = \phi(x) - 1$ $(x \in X)$, we see that $\sum_{j=0}^{\infty} \binom{1/d}{j} (\phi(x)-1)^j$ is a d-th root of $\phi(x)$. The resulting function $\psi := \sum_{j=0}^{\infty} \binom{1/d}{j} (\phi - 1)^j$ is clearly in A_X since all coefficients in the summation are real, and we have

$$\psi^d = \phi = -p(\lambda_1, \ldots, \lambda_n).$$

Since ψ is nowhere zero on X, we get $-1 = p(\lambda_1/\psi, \ldots, \lambda_n/\psi)$, so -1 is represented by p over A_X. Q.E.D.

The result above leads to one more large family of affine varieties $Y \subset \mathbb{C}^n$ with $s'(Y) = s(Y) = n$.

EXAMPLE 6.8. Let $p(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n] \setminus \mathbb{R}$ be a form which is positive definite and absolutely irreducible. Let $Y = V_{\mathbb{C}}(1+p)$ and $A = \mathbb{R}[Y] = \mathbb{R}[x_1, \ldots, x_n]/(1+p)$. Then the conclusions of (5.10) are applicable to A and to A_Y, and we have

$$n = s'(Y) = s(Y) = s(A_Y) \le s(A) < \infty.$$

In particular, p is not a sum of $n-1$ squares of real polynomials.

In fact, since p represents -1 over $A_{S^{n-1}}$, we can find an $\mathbb{R}$-algebra homomorphism from A to $A_{S^{n-1}}$. As we have seen before in the proof of (4.3), this gives rise to an equivariant map $S^{n-1} \twoheadrightarrow Y$, so $s'(Y) \geq n$. Since Y has no real points, we also have $s(Y) \leq n$ by (2.9)(2). Now the rest follows as usual.

Note that the assumption that p be absolutely irreducible is essential for both (6.6) and (6.8). This can be seen by considering the form

$$p(x_1, \ldots, x_n) = x_1^4 + (x_2^2 + \cdots + x_n^2)^2.$$

This form is positive definite, irreducible but not absolutely irreducible. For $n \geq 3$ the conclusion in (6.8) is clearly false since p is a sum of two squares in $\mathbb{R}[x_1, \ldots, x_n]$.

§7. Level and colevel of $V^{\varepsilon}_{n,2}$, $V^{\delta}_{n,2}$

Let $V_{n,m}$ denote the Stiefel manifold of orthonormal m-frames in the real euclidean space $\mathbb{R}^n$. In this and the next section, we shall be interested in two basic fixed-point-free involutions of $V_{n,m}$, denoted by ε and δ. These are defined as follows:

$$(7.1) \quad \begin{cases} \varepsilon\{v_1, \ldots, v_m\} = \{v_1, \ldots, v_{m-1}, -v_m\}, \\ \delta\{v_1, \ldots, v_m\} = \{-v_1, \ldots, -v_m\}. \end{cases}$$

The resulting spaces with involutions $(V_{n,m}, \varepsilon)$, $(V_{n,m}, \delta)$ will be denoted in the sequel by $V^{\varepsilon}_{n,m}$ and $V^{\delta}_{n,m}$. *In this section, we shall focus our attention on the case* $m = 2$, *and regard n as fixed.* Therefore, to simplify the notations, we shall write (throughout this section) V^{ε} for $V^{\varepsilon}_{n,2}$, and V^{δ} for $V^{\delta}_{n,2}$. In the first half of the section, we shall compute the level and colevel of V^{ε} and V^{δ}; in the second half, we shall then study the algebraic implications of these computations.

We note, in passing, that there is actually a third natural involution ε' on $V_{n,2}$, defined by $\varepsilon'\{v_1, v_2\} = \{v_2, v_1\}$. However, it is easy to see that, as a space with involution, $(V_{n,2}, \varepsilon')$ is isomorphic to V^{ε}. In fact, the homeomorphism $h: V^{\varepsilon} \to (V_{n,2}, \varepsilon')$ defined by

$$h\{v_1, v_2\} = \{(v_1 - v_2)/\sqrt{2}, (v_1 + v_2)/\sqrt{2}\}$$

is easily checked to be equivariant with respect to the two specified involutions. Therefore, there is no need to consider ε'.

To begin our computations, note that the "projection map" $V_{n,2} \twoheadrightarrow S^{n-1}$ given by $\{v_1, v_2\} \mapsto v_2$ is equivariant with respect to both ε and δ on $V_{n,2}$. This map will be used freely in the following computations. It shows that $s(V^{\varepsilon})$ and $s(V^{\delta})$ are both $\leq n$.

THEOREM 7.2.

$$(1)\ s'(V^{\delta}_{n,2}) = \begin{cases} n & \textit{if } n \textit{ is even.} \\ n-1 & \textit{if } n \textit{ is odd.} \end{cases}$$

$$(2)\ s(V^{\delta}_{n,2}) = \quad n \qquad \textit{for all } n.$$

Proof. First assume n is even. We can construct a map $f: S^{n-1} \rightarrow V^{\delta}$ by

$$f(v) = \{(v_1, \ldots, v_n), (v_2, -v_1, \ldots, v_n, -v_{n-1})\}$$

for $v = (v_1, \ldots, v_n) \in S^{n-1}$. It is easy to check that f is an equivariant map, so we get $s(V^{\delta}) = s'(V^{\delta}) = n$. In the following, we may, therefore, assume that n is odd. In this case, the idea used above gives an equivariant map $g: S^{n-2} \twoheadrightarrow V$, namely

$$g(t_1, \ldots, t_{n-1}) = \{(t_1, \ldots, t_{n-1}, 0), (t_2, -t_1, \ldots, t_{n-1}, -t_{n-2}, 0)\}$$

for $(t_1, \ldots, t_{n-1}) \in S^{n-2}$. We claim that

(7.3) *There does not exist an equivariant map* $V^{\delta} \twoheadrightarrow S^{n-2}$.

For, if such a map exists, then we would have $s'(V^{\delta}) = s(V^{\delta}) = n-1$ and so by (2.11) $\pi_{n-2}(V_{n,2})$ would have an infinite cyclic quotient group, contradicting Stiefel's Theorem that (for n odd) $\pi_{n-2}(V_{n,2}) \cong \mathbb{Z}_2$ [St: p. 132]. This proves (7.3), and therefore $s(V^{\delta}) = n$. For the colevel, the existence of the map g already shows that $s'(V^{\delta}) \geq n-1$. To show that this is an equality, we need to show that

(7.4) *There does not exist an equivariant map* $S^{n-1} \twoheadrightarrow V^{\delta}$.

Indeed, if such a map exists, then we would have $s'(V^{\delta}) = s(V^{\delta}) = n$ and so, by (2.11) again, $\pi_{n-1}(V_{n,2})$ would have an infinite cyclic quotient group, contradicting Whitehead's Theorem [Wh][5] that, for n odd:

$$\pi_{n-1}(V_{n,2}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n > 3 \\ 0 & \text{if } n = 3. \end{cases} \qquad \text{Q.E.D.}$$

Next we shall compute the level and colevel for the space $V^{\varepsilon} = V^{\varepsilon}_{n,2}$.

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[5] Another argument showing $s'(V^{\delta}_{n,2}) = n-1$ for n odd, without using homotopy, will be given in (8.3)(1).

THEOREM 7.5.

(1) $s'(V^\varepsilon_{n,2})=n-1$.

(2) (Conner–Floyd) $s(V^\varepsilon_{n,2})=\begin{cases} n & \text{if } n\neq 2,4,8,\\ n-1 & \text{if } n=2,4,8.\end{cases}$

Proof. First, note that there is an equivariant map $S^{n-2} \twoheadrightarrow V^\varepsilon$ defined by $v \to \{e, v\}$, where e is a fixed vector, and v ranges over the unit sphere in the orthogonal complement of e. Therefore, we have $s'(V^\varepsilon)\geq n-1$. To show that this is an equality, we need to show that there does not exist $f: S^{n-1} \twoheadrightarrow V^\varepsilon$. Indeed, assume such an equivariant map exists, say $f(v)=\{p(v), q(v)\}$. Then p is an even map and q is an odd map from S^{n-1} to itself. Since $p(v)\perp q(v)$, these two mappings must be homotopic, and hence have the same degree. However, deg (q) is odd and deg (p) is even, a contradiction.

To compute the level $s(V^\varepsilon)$, first let $n=2, 4, 8$. Using the properties of the complex numbers, quaternons and Cayley numbers, it is easy to construct an equivariant map $V^\varepsilon \twoheadrightarrow S^{n-2}$, so $s(V^\varepsilon)=n-1$ in these cases. *Now assume* $n\neq 2, 4, 8$. If n is odd, we can get $s(V^\varepsilon)=n$ by the same argument used before to prove (7.3). If n is not necessarily odd, the same conclusion is considerably deeper: the proof given in $[CF_2]$[6] uses Adams' Theorem on the nonexistence of Hopf invariant one, plus a certain construction of Milnor and Spanier. In the following, we shall present a more "elementary" proof sketched to us by I. M. James which uses only Adams' Theorem but not the Milnor–Spanier construction.

Assume there exists an equivariant map $f: V^\varepsilon \twoheadrightarrow S^{n-2}$. Let e be the "north pole" of S^{n-1} and S^{n-2} be the "equator" of S^{n-1}. By identifying $z\in S^{n-2}$ with $\{e, z\}$, S^{n-2} is equivariantly imbedded in S^{n-1}. By restriction, f induces $f_0: S^{n-2} \twoheadrightarrow S^{n-2}$. We now define a map $\phi: S^{n-1}\times S^{n-1}\to S^{n-1}$ as follows. If $\{x, y\}\in V_{n,2}$, we set $\phi(x, y)=f\{x, y\}\in S^{n-2}\subset S^{n-1}$. Next consider $(x, z)\in S^{n-1}\times S^{n-1}$. Choose y coplanar with the vectors x, z such that z makes an acute angle ($\leq\pi/2$) with y, and write $z=x\cos\theta+y\sin\theta$, where θ is the angle between z and x $(0\leq\theta\leq\pi)$. One then sets

$$\phi(x, z)=\phi(x, x\cos\theta+y\sin\theta)=e\cos\theta+f\{x, y\}\sin\theta.$$

It is easy to check that ϕ is well-defined and has the following properties:

(a) $\phi(x, x)=e$;

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[6] Conner and Floyd used the involution $\varepsilon': \{v_1, v_2\}\to\{v_2, v_1\}$. But, as pointed out before, $(V_{n,2}, \varepsilon')$ is isomorphic as a space with involution to our V^ε.

(b) Let $\phi_e : S^{n-1} \to S^{n-1}$ be defined by $\phi_e(z) = \phi(e, z)$. Then $\phi_e \mid S^{n-2} = f_0$, and ϕ_e maps the upper (resp. lower) hemisphere of S^{n-1} into the upper (resp. lower) hemisphere of S^{n-1}.

Now let $d = \deg(f_0) = \deg(\phi_e \mid S^{n-2})$ which is an odd integer since f_0 is equivariant. Property (b) above implies that $\deg(\phi_e) = \deg(\phi_e \mid S^{n-2}) = d$, and Property (a) implies that $\phi'_e : S^{n-1} \to S^{n-1}$ defined by $\phi'_e(x) = \phi(x, e)$ has degree $-d$. Thus ϕ has bidegree $(d, -d)$. Since d is odd, Adams' Hopf Invariant One Theorem implies that this is possible only for $n = 2, 4, 8$. Q.E.D.

(Professor James has further pointed out to us that the construction of ϕ from f above is made possible by the fact that $S^{n-1} \times S^{n-1}$ may be viewed as the "fibre-suspension" of $V_{n,2}$ with respect to the natural fibration $V_{n,2} \to S^{n-1}$.)

Next we shall study the applications of the topological results above to algebra. Look at $\mathbb{R}[x, y]$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$; let $\mathfrak{A}^\varepsilon$ and $\mathfrak{A}^\delta$ be the following ideals:

$$(7.6) \quad \mathfrak{A}^\varepsilon = \left(1 - \sum x_j^2, 1 + \sum y_j^2, \sum x_j y_j\right),$$

$$(7.7) \quad \mathfrak{A}^\delta = \left(1 + \sum x_j^2, 1 + \sum y_j^2, \sum x_j y_j\right),$$

and let $B^\varepsilon_{n,2} = \mathbb{R}[x, y]/\mathfrak{A}^\varepsilon$, $B^\delta_{n,2} = \mathbb{R}[x, y]/\mathfrak{A}^\delta$. Note that $B^\varepsilon_{n,2}$ is a "generic" $\mathbb{R}$-algebra over which $n\langle 1\rangle$ contains an orthogonal direct summand $\langle 1, -1\rangle$ (i.e. $n\langle 1\rangle$ is isotropic), and $B^\delta_{n,2}$ is a generic $\mathbb{R}$-algebra over which $n\langle 1\rangle$ contains an orthogonal direct summand $\langle -1, -1\rangle$. In particular, we have $s(B^\varepsilon_{n,2}) \le n$, $\sigma(B^\varepsilon_{n,2}) \le n - 1$ and $s(B^\delta_{n,2}) \le n$. In the following we propose to compute the invariants s (level), s' (colevel) and σ (sublevel) for the rings $B^\varepsilon_{n,2}$ and $B^\delta_{n,2}$. Since n will be held fixed, we shall henceforth write $B^\varepsilon = B^\varepsilon_{n,2}$ and $B^\delta = B^\delta_{n,2}$.

Remark 7.8. We can also look at a third algebra $B = \mathbb{R}[x, y]/\mathfrak{A}$ where $\mathfrak{A} = (\sum x_j^2, \sum x_j y_j - 1)$. But since this is also a generic $\mathbb{R}$-algebra over which $n\langle 1\rangle$ is isotropic, there exist algebra homomorphism $B^\varepsilon \to B \to B^\varepsilon$. Therefore, the results obtained below for the ring B^ε will hold equally for the ring B.

To relate the rings B^ε, B^δ to the Stiefel manifolds with involutions V^ε and V^δ, let Y^ε and Y^δ be the affine varieties defined, respectively, by $\mathfrak{A}^\varepsilon$ and $\mathfrak{A}^\delta$ in $\mathbb{C}^{2n}$. These have no real points and are, as usual, given the involution defined by complex conjugation. We can construct an equivariant map $V^\varepsilon \to Y^\varepsilon$ by $\{u, v\} \mapsto (u, iv)$, and an equivariant map $V^\delta \to Y^\delta$ by $\{u, v\} \mapsto (iu, iv)$, where $i = \sqrt{-1}$.

Therefore, we have the following $\mathbb{R}$-algebra homomorphisms:

(7.9) $B^{\varepsilon} \to A_{Y^{\varepsilon}} \to A_{V^{\varepsilon}}$,

(7.10) $B^{\delta} \to A_{Y^{\delta}} \to A_{V^{\delta}}$.

THEOREM 7.11.

(1) $s'(B^{\varepsilon}_{n,2}) = \sigma(B^{\varepsilon}_{n,2}) = n-1$.

(2) $s(B^{\varepsilon}_{n,2}) = \begin{cases} n & \textit{if } n \neq 2, 4, 8 \\ n-1 & \textit{if } n = 2, 4, 8. \end{cases}$

(Note that this computation, in particular, completely settles Question (5.8), since for $n \neq 2$, 4, 8, $(\sigma(B^{\varepsilon}), s(B^{\varepsilon}))$ realizes the pair $(n-1, n)$.)

Proof. From (7.9) and (5.11), we have $\sigma(B^{\varepsilon}) \geq \sigma(A_{V^{\varepsilon}}) \geq s'(V^{\varepsilon})$. Using (7.5)(1), this gives $\sigma(B^{\varepsilon}) \geq n-1$ and hence $\sigma(B^{\varepsilon}) = n-1$. For the colevel, we have from (7.9) and (4.3):

$$s'(B^{\varepsilon}) \geq s'(A_{V^{\varepsilon}}) \geq s'(V^{\varepsilon}) = n-1.$$

On the other hand, (5.13) gives $s'(B^{\varepsilon}) \leq \sigma(B^{\varepsilon}) = n-1$, so we also have $s'(B^{\varepsilon}) = n-1$. for the level, (7.9) and (7.5)(2) give

$$s(B^{\varepsilon}) \geq s(A_{V^{\varepsilon}}) = s(V^{\varepsilon}) = \begin{cases} n & \text{if } n \neq 2, 4, 8, \\ n-1 & \text{if } n = 2, 4, 8. \end{cases}$$

Therefore $s(B^{\varepsilon}) = n$ if $n \neq 2$, 4, 8. On the other hand, if $n = 2$, 4, 8, we must have $s(B^{\varepsilon}) = n-1$ for otherwise $s(B^{\varepsilon})$ would be n and (5.9) would give $\sigma(B^{\varepsilon}) = s(B^{\varepsilon}) = n$, contradicting the conclusion in part (1). Q.E.D.

Next we proceed to the computation of the invariants for the ring B^{δ}. Here we can also completely determine the level and colevel. However, the sublevel turns out to be more difficult: we can determine $\sigma(B^{\delta})$ only for n even (and later for $n = 3$, 7).

THEOREM 7.12.

(1) $s'(B^{\delta}_{n,2}) = \begin{cases} n & \textit{if n is even,} \\ n-1 & \textit{if n is odd.} \end{cases}$

(2) $s(B^{\delta}_{n,2}) = n$ *for all n.*

(3) $\sigma(B^{\delta}_{n,2}) = \begin{cases} n & \textit{if } n \textit{ is even} \\ n-1 & \textit{if } n = 3, 7. \end{cases}$

Proof. From (7.10) and (7.2)(2), we have $s(B^{\delta}) \geq s(A_{V^{\delta}}) = s(V^{\delta}) = n$. Therefore $s(B^{\delta}) = n$, proving (2). Similarly

$$\sigma(B^{\delta}) \geq \sigma(A_{V^{\delta}}) \geq s'(V^{\delta}) = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd,} \end{cases}$$

and the same inequalities hold with σ replaced by s'. Therefore, if n is even, we clearly have $\sigma(B^{\delta}) = s'(B^{\delta}) = n$. *Next, assume n is odd*; then $s'(B^{\delta})$ is either n or $n-1$. If it is n, there would be an $\mathbb{R}$-algebra homomorphism from B^{δ} to $A_{S^{n-1}}$. Hence, over $A_{S^{n-1}}$, the form $n\langle 1 \rangle$ admits an orthogonal direct summand $\langle -1, -1 \rangle$. By a later result (cf. (9.6)), this implies that there is an equivariant map $S^{n-1} \twoheadrightarrow V^{\delta}$. But this is impossible since $s'(V^{\delta}) = n-1$ by (7.2)(1); therefore $s'(B^{\delta}) = n-1$ for n odd. For $n = 3$ or 7, we shall show (cf. (9.17)) that $n\langle 1 \rangle$ is isotropic over $B^{\delta}_{n,2}$, so in these two cases $\sigma(B^{\delta}_{n,2}) = n-1$. Q.E.D.

For n odd $\neq 3, 7$, we conjuecture that $\sigma(B^{\delta}_{n,2}) = n$, i.e. $n\langle 1 \rangle$ is anisotropic over $B^{\delta}_{n,2}$. This will be proved later modulo a certain conjecture on equivariant maps (cf. end of Section 10).

§8. Colevel of $V^{\delta}_{n,q}$ and the Hopf Problem

In this section, we shall consider the problem of computing the colevel s' of $V^{\delta}_{n,q}$. It turns out that this problem amounts precisely to the "skew-linear" version of the Hopf Problem on the existence of nonsingular maps from $\mathbb{R}^p \times \mathbb{R}^q$ to $\mathbb{R}^n$ ([H]). The following lemma is due to K. Y. Lam [L3: (3.1)]; we include its statement and proof here for the sake of completeness.

LEMMA 8.1. *A (continuous) equivariant map $f: S^{p-1} \twoheadrightarrow V^{\delta}_{n,q}$ gives rise to a (continuous) nonsingular skew-linear map $\phi: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$, and conversely.* ("Nonsingular" means that $\phi(x, y) = 0 \Rightarrow x = 0$ or $y = 0$. "Skew-linear" means that $\phi(-x, y) = -\phi(x, y)$ and $\phi(x, \alpha y + \alpha' y') = \alpha\phi(x, y) + \alpha'\phi(x, y')$.)

Proof. For $x \in S^{p-1}$, we think of the q (column) vectors of $f(x)$ as forming an $n \times q$ matrix, again denoted by $f(x)$. For $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$, we can then define ϕ

(the "adjoint" of f) by

$$\phi(x, y) = \begin{cases} f(x/\|x\|) \cdot y & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Here, y is written as a column vector, and the dot denotes matrix multiplication. Clearly ϕ is nonsingular and skew-linear. Conversely, if such a ϕ is given, and $x \in S^{p-1}$, consider $\{\phi(x, e_1), \ldots, \phi(x, e_q)\}$, where $\{e_j\}$ are the unit vectors in $\mathbb{R}^q$. By the linearity of ϕ in the second variable, and the nonsingularity of ϕ, the $\phi(x, e_j)$'s are linearly independent in $\mathbb{R}^n$. Therefore, we can define $f(x)$ to be the Gram–Schmidt Orthonormalization of $\{\phi(x, e_1), \ldots, \phi(x, e_q)\}$. From the orthonormalization formulas, it is easy to check that the skewness of ϕ in the first variable implies the equivariance of $f: S^{p-1} \to V^{\delta}_{n,q}$. Q.E.D.

For given $p, q \geq 1$, let $p \# q$ be the least integer n for which there exists a nonsingular skew-linear map $\phi: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$. The lemma above says that $s'(V^{\delta}_{n,q}) \geq p$ iff $p \# q \leq n$; from this, we conclude that

COROLLARY 8.2. *$s'(V^{\delta}_{n,q})$ is the largest integer p such that $p\#q \leq n$.*

In the notation of [L_4], we have therefore $s'(V^{\delta}_{n,q}) = s(n, q-1)$, where the latter is the largest number of independent sections for the n-fold Whitney sum of the Hopf line bundle on $\mathbb{RP}^{q-1}$.

From the known results on $p \# q$ in the literature, we can record the following consequences on the computation of $s'(V^{\delta}_{n,q})$:

COROLLARY 8.3. (1) *$s'(V^{\delta}_{n,q}) \leq n$, with equality iff $q \leq \rho(n)$ (the Radon function). (This subsumes, in particular,* (7.2)(1).)

(2) *Let* "neg" *denote the involution on the orthogonal group $O(n)$ defined by $M \to -M$. Then $s'(O(n), \text{neg}) = \rho(n)$. If n is even, then $s'(SO(n), \text{neg}) = \rho(n)$.*

Proof. The equivariant map $V^{\delta}_{n,q} \to S^{n-1}$ obtained by projection to the first vector shows that $s'(V^{\delta}_{n,q}) \leq s(V^{\delta}_{n,q}) \leq n$. The rest follows from (8.2) and Adams' solution to the Vector Field Problem [Ad].

Note that if $q \leq 8$, $p \# q$ has been completely determined by Behrend [B]. In fact, in this case, $p \# q$ just coincides with the $p \circ q$ defined in Pfister's paper [Pf] in connection to the composition of a sum of p squares with a sum of q squares in fields. For p', $q \leq 8$, the computation for $p' \# q = p' \circ q$ is easy, and for $p = 8m + p'$ ($m \geq 0, 1 \leq p' \leq 8$), we simply have $p \# q = p \circ q = 8m + p' \circ q$. Similarly, using the fact that $\rho(16) = 9$, one can show that $p \# q = p \circ q$ also holds for $q = 9$. Therefore, (8.2) leads to a complete determination of $s'(V^{\delta}_{n,q})$ for $q \leq 9$.

For the reader's convenience, we compile in the following table the known values of $p \# q$ for $q \le p \le 17$:

p \ q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1																
2	2	2															
3	3	4	4														
4	4	4	4	4													
5	5	6	7	8	8												
6	6	6	8	8	8	8											
7	7	8	8	8	8	8	8										
8	8	8	8	8	8	8	8	8									
9	9	10	11	12	13	14	15	16	16								
10	10	10	12	12	14	14	16	16	16	16							
11	11	12	12	12	15	16	16	16	16	17	17						
12	12	12	12	12	16	16	16	16	16	17	17	17					
13	13	14	15	16	16	16	16	16	16	19	19	19	19				
14	14	14	16	16	16	16	16	16	16	20	20	20	23	23			
15	15	16	16	16	16	16	16	16	16	20	20	20	23	23	23		
16	16	16	16	16	16	16	16	16	16	22	23	23	23	23	23	23	
17	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	32

Here, the first eight columns follow from Behrend's computation, and the ninth (up to $p = 16$) follows from $\rho(16) = 9$. The rest follows from the work of K. Y. Lam [L_1–L_4] and J. Adem [A_1–A_3]. Note that in this table, *if* $p = 2^i + 1$, *then* $p \# q = p + q - 1$ *for all* $q < p$. This follows easily from the work of Hopf [H].

From this table and from (8.2), we can easily read off the $s'(V^s_{n,q})$ table on the next page.

Finally, we make an observation on a lower bound for $s'(V^\delta_{n,q})$. Translating Hopf's upper bound $p \# q \le p + q - 1$ [H], our (8.2) implies that, for all n, q,

(8.4) $$s'(V^\delta_{n,q}) \ge n - q + 1.$$

In fact, the map obtained by sending $v = (v_1, \ldots, v_{n-q+1}) \in S^{n-q}$ to the Gram–Schmidt Orthonormalization of the q (linearly independent) row vectors of

$$\begin{pmatrix} v_1, \ldots\ldots, v_{n-q+1}, & & & \\ & v_1, \ldots\ldots, v_{n-q+1}, & & \\ & & \ldots\ldots\ldots & \\ & & & \ldots\ldots\ldots \\ & & & v_1, \ldots\ldots, v_{n-q+1} \end{pmatrix}$$

Table of Values for $s'(V^{\delta}_{n,q})$* $(n \leq 19)$

$n \backslash q$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																		
2	2	2																	
3	3	2	1																
4	4	4	4	4															
5	5	4	4	4	1														
6	6	6	4	4	2	2													
7	7	6	5	4	3	2	1												
8	8	8	8	8	8	8	8	8											
9	9	8	8	8	8	8	8	8	1										
10	10	10	8	8	8	8	8	8	2	2									
11	11	10	9	8	8	8	8	8	3	2	1								
12	12	12	12	12	8	8	8	8	4	4	4	4							
13	13	12	12	12	9	8	8	8	5	4	4	4	1						
14	14	14	12	12	10	10	8	8	6	6	4	4	2	2					
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1				
16	16	16	16	16	16	16	16	16	16	10	9	9	9	9	9	9			
17	17	16	16	16	16	16	16	16	16	12	12	9	9	9	9	9	1		
18	18	18	16	16	16	16	16	16	16	12	12	12	9	9	9	9	2	2	
19	19	18	17	16	16	16	16	16	16	13	13	13	13	9	9	9	3	2	1

* From the definition of $s'(V^{\delta}_{n,q})$, it is easy to check that, in this table, the rows must be nonincreasing, and the columns must be nondecreasing.

(undesignated entries are zero; see [L_5]) is evidently an equivariant map from S^{n-q} to $V^{\delta}_{n,q}$. From our table of values for $s'(V^{\delta}_{n,q})$, one finds that there exist various pairs (n, q) for which (8.4) is actually an equality: this is the case, for instance, when $n=7$ or $n=15$. More generally, we have the following

PROPOSITION 8.5. *Let* $n=2^i-1$. *Then, for all* q, $s'(V^{\delta}_{n,q})=n-q+1$.

Proof. In view of (8.4) and (8.2), it is enough to show that $p \# q = p+q-1$ if $p+q=2^i+1$. If this is not the case, there would exist a nonsingular skew-linear map $\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$. Since $p+q-n=2$, there is exactly one integer k strictly between $n-p$ and q. By the Hopf Condition on the existence of nonsingular (biskew) maps [H], the binomial coefficient $\binom{n}{k}$ must be even. However, since $n=2^i-1$ has only 1's in its dyadic expansion, Lucas' Lemma implies that all binomial coefficients $\binom{n}{j}$ are odd (see [B]), a contradiction. Q.E.D.

There are also values (n, q) with $n \neq 2^i - 1$ for which $s'(V^{\delta}_{n,q}) = n - q + 1$; for instance, for $q = 5$, this equality holds not only for $n = 7, 15, \ldots$, but also for $n = 5, 6, 12, 13, 14, 15$, etc. In general, it can be shown that $s'(V^{\delta}_{n,q}) = n - q + 1$ *iff the binomial coefficient* $\binom{n}{q-1}$ *is odd.* The proof of this criterion will appear in [LL].

For the level $s(V^{\delta}_{n,q})$, computations seem to be more difficult (for $q \geqslant 3$). Some partial results are given in [LL].

§9. Equivariant maps into Stiefel manifolds

In the remaining sections of this paper, we shall study certain generic rings which are generalizations of the rings A_n, $B^{\varepsilon}_{n,2}$ and $B^{\delta}_{n,2}$ defined before in §§3, 7. These generic rings, denoted by $B^{r,s}_{n,m}$ (where $m = r + s$), are defined as follows: they are generated over $\mathbb{R}$ by commuting variables (x_{jk}) $(1 \leq j \leq m, 1 \leq k \leq n)$ subject to the relations dictated by the matrix equation:

$$(9.1) \quad (x_{jk})(x_{jk})^t = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}.$$

Let $B = B^{r,s}_{n,m}$ and let x_j $(1 \leq j \leq m)$ be the jth row of (x_{jk}), viewed as a vector in $B^{(n)}$, the free B-module of rank n. With respect to the quadratic form $n\langle 1 \rangle$ $(:= t_1^2 + \cdots + t_n^2)$ on $B^{(n)}$, we have the inner-product relations:

$$(9.2) \quad x_j \cdot x_{j'} = \begin{cases} 0 & \text{if } j \neq j', \\ 1 & \text{if } j = j' \leq r, \\ -1 & \text{if } j = j' > r. \end{cases}$$

Therefore, the vectors $x_1, \ldots, x_m$ are linearly independent, and span an inner product subspace $r\langle 1 \rangle \perp s\langle -1 \rangle$ in $n\langle 1 \rangle$. This leads to an orthogonal decomposition

$$(9.3) \quad n\langle 1 \rangle \cong r\langle 1 \rangle \perp s\langle -1 \rangle \perp \phi$$

over B, where ϕ is the inner product space given by the orthogonal complement of $\sum_{j=1}^{m} B \cdot x_j$. If $m (= r + s) > n$, the decomposition above implies that $B = 0$, but if $m \leq n$, it will be clear that B is nonzero. In the following, we shall always assume that $m \leq n$ and $s \geq 1$.

Let C be any commutative $\mathbb{R}$-algebra. If there is an orthogonal decomposition of the type (9.3) over C (for some C-inner-product space ϕ), we shall write

$$n\langle 1\rangle \geq r\langle 1\rangle \perp s\langle -1\rangle \text{ over } C.$$

Clearly this is the case iff there is an $\mathbb{R}$-algebra homomorphism of $B = B^{r,s}_{m,n}$ into C. To refer to this property, we shall say that B is the generic ring for which $n\langle 1\rangle \geq r\langle 1\rangle \perp s\langle -1\rangle$. Note that $B^{0,1}_{n,1} = A_n$ is the generic ring of level n in (3.2), and $B^{1,1}_{n,2}$, $B^{0,2}_{n,2}$ are respectively the rings denoted by $B^{\varepsilon}_{n,2}$ and $B^{\delta}_{n,2}$ in §7 (cf. (7.6), (7.7)).

Now let $Y = Y^{r,s}_{n,m}$ be the affine variety in $\mathbb{C}^{nm}$ defined by the polynomial equations given by (9.1). This variety is defined over $\mathbb{R}$, and $B = B^{r,s}_{n,m}$ is its real coordinate ring. Since $s \geq 1$, Y has no real points; as usual, we equip Y with the (fixed-point-free) involution given by complex conjugation.

To study $B^{r,s}_{n,m}$ and $Y^{r,s}_{n,m}$, we shall use the Stiefel manifolds $V^{r,s}_{n,m}$ defined in (2.4), with the involution

(9.4) $\quad (v_1, \ldots, v_m) \mapsto (v_1, \ldots, v_r, -v_{r+1}, \ldots, -v_{r+s}).$

Note that $V^{m-1,1}_{n,m} = V^{\varepsilon}_{n,m}$, $V^{0,m}_{n,m} = V^{\delta}_{n,m}$, and $V^{0,1}_{n,1}$ is just the unit sphere S^{n-1} with the antipodal involution.

LEMMA 9.5. *There exists an equivariant map* $f: V^{r,s}_{n,m} \to Y^{r,s}_{n,m}$.

Proof. We define f by sending an orthonormal m-frame $(v_1, \ldots, v_m) \in V^{r,s}_{n,m}$ to the m-tuple of n-vectors $x_1, \ldots, x_m$ where

$$x_j = \begin{cases} v_j & \text{if } 1 \leq j \leq r, \\ iv_j & \text{if } r < j \leq m, \end{cases}$$

and $i = \sqrt{-1}$. Clearly the vectors x_j $(1 \leq j \leq m)$ satisfy the inner product equations (9.2), so their coordinates (x_{jk}) define a point in $Y^{r,s}_{n,m}$. Clearly f is an equivariant map from $V^{r,s}_{n,m}$ to $Y^{r,s}_{n,m}$. Q.E.D.

(This Lemma, incidentally, shows that $Y^{r,s}_{n,m} \neq \varnothing$, so, in particular, $B^{r,s}_{n,m} \neq 0$ for $m \leq n$.)

We now come to the basic result of this section, which relates the behavior of the form $n\langle 1\rangle$ over the function ring A_X of a space X with involution to the existence of equivariant maps of X into the Stiefel manifolds $V^{r,s}_{n,m}$. We are greatly indebted to M. Kervaire and W. Scharlau for a valuable communication which was instrumental to the inception and proof of the following result.

THEOREM 9.6. *For any space X with involution, we have $n\langle 1\rangle \geq r\langle 1\rangle \perp s\langle -1\rangle$ over A_X iff there exists an equivariant map from X to $V^{r,s}_{n,m}$ where $m = r + s$.*

Proof. First assume there is an equivariant map $X \twoheadrightarrow V^{r,s}_{n,m}$. Composing this with the map f constructed in the proof of (9.5), we get an equivariant map $X \twoheadrightarrow Y := Y^{r,s}_{n,m}$. This induces an $\mathbb{R}$-algebra homomorphism $A_Y \to A_X$. Composing this with the standard map $B^{r,s}_{n,m} \to A_Y$, we get a homomorphism $B^{r,s}_{n,m} \to A_X$. Therefore, we have $n\langle 1\rangle \geq r\langle 1\rangle \perp s\langle -1\rangle$ over A_X.

Conversely, assume that $n\langle 1\rangle \geq r\langle 1\rangle \perp s\langle -1\rangle$ over A_X. Let $\langle ,\rangle$ denote the inner product given by $n\langle 1\rangle$ over any ring, and let F_j $(1 \leq j \leq m)$ be vectors in A_X^n giving an orthogonal basis (w.r.t. $\langle ,\rangle$) for the orthogonal summand $r\langle 1\rangle \perp s\langle -1\rangle$. We think of each F_j as a vector function, and decompose it into its real and imaginary parts, say $F_j = G_j + iH_j$. Then the coordinates of G_j are "even" functions and those of H_j are "odd" functions (from X to $\mathbb{R}$). Our next step is to express the inner product properties of the F_j's in terms of the G_j's and H_j's.

(1) Let $j \neq k$. Then

$$\begin{aligned} 0 &= \langle F_j, F_k\rangle = \langle G_j + iH_j, G_k + iH_k\rangle \\ &= \langle G_j, G_k\rangle - \langle H_j, H_k\rangle + i(\langle G_j, H_k\rangle + \langle G_k, H_j\rangle). \end{aligned}$$

Therefore, for $j \neq k$, we have

(9.7) $\langle G_j, G_k\rangle = \langle H_j, H_k\rangle$,

(9.8) $\langle G_j, H_k\rangle = -\langle G_k, H_j\rangle$.

(2) For $j = k$, we get instead

$$\langle F_j, F_j\rangle = \langle G_j, G_j\rangle - \langle H_j, H_j\rangle + 2i\langle G_j, H_j\rangle.$$

Therefore, we have

(9.9) $\langle G_j, G_j\rangle - \langle H_j, H_j\rangle = \begin{cases} 1 & \text{if } 1 \leq j \leq r, \\ -1 & \text{if } r < j \leq m, \end{cases}$

(9.10) $\langle G_j, H_j\rangle = 0$.

Now consider the map

(9.11) $x \to (G_1(x), \ldots, G_r(x), H_{r+1}(x), \ldots, H_{r+s}(x))$

$$\underset{X}{\cap} \qquad \underset{\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{r\text{-copies}}}{\cap} \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{s\text{-copies}}$$

We claim that, *for any* $x \in X$, *the* m *vectors listed above are linearly independent in* $\mathbb{R}^n$. In fact, fix a point $x \in X$ and assume that

$$\sum_{1 \le j \le r} \mu_j G_j(x) + \sum_{r < k \le m} \lambda_k H_k(x) = 0.$$

where μ_j, $\lambda_k \in \mathbb{R}$. Let

$$\text{(9.12)} \quad \rho := \sum_{1 \le j \le r} \mu_j H_j(x) - \sum_{r < k \le m} \lambda_k G_k(x).$$

To simplify the notations, we shall suppress the (fixed) point x in the following computations, and always assume that the indices j, j' range from 1 to r, while k, k' range from $r+1$ to m. For any j', we have

$$\begin{aligned} 0 &= \left\langle \sum_j \mu_j G_j + \sum_k \lambda_k H_k, G_{j'} \right\rangle \\ &= \sum_{j \neq j'} \mu_j \langle G_j, G_{j'} \rangle + \mu_{j'} \langle G_{j'}, G_{j'} \rangle + \sum_k \lambda_k \langle H_k, G_{j'} \rangle \end{aligned}$$

Using (9.7), (9.8) and (9.9), we get

$$\begin{aligned} 0 &= \sum_{j \neq j'} \mu_j \langle H_j, H_{j'} \rangle + \mu_{j'} (1 + \langle H_{j'}, H_{j'} \rangle) - \sum_k \lambda_k \langle G_k, H_{j'} \rangle \\ &= \mu_{j'} + \left\langle \sum_j \mu_j H_j - \sum_k \lambda_k G_k, H_{j'} \right\rangle \\ &= \mu_{j'} + \langle \rho, H_{j'} \rangle \end{aligned}$$

Therefore, $\mu_{j'} = -\langle \rho, H_{j'} \rangle$ (for $1 \le j' \le r$). Similarly, for $r < k' \le m$, we have

$$\begin{aligned} 0 &= \left\langle \sum_j \mu_j G_j + \sum_k \lambda_k H_k, H_{k'} \right\rangle \\ &= \sum_j \mu_j \langle G_j, H_{k'} \rangle + \sum_{k \neq k'} \lambda_k \langle H_k, H_{k'} \rangle + \lambda_{k'} \langle H_{k'}, H_{k'} \rangle \\ &= -\sum_j \mu_j \langle H_j, G_{k'} \rangle + \sum_{k \neq k'} \lambda_k \langle G_k, G_{k'} \rangle + \lambda_{k'} (1 + \langle G_{k'}, G_{k'} \rangle) \\ &= \lambda_{k'} - \langle \rho, G_{k'} \rangle. \end{aligned}$$

Therefore, $\lambda_{k'}=\langle \rho, G_{k'}\rangle$ (for $r<k'\le m$). We have then

$$\begin{aligned}\langle \rho, \rho\rangle &= \left\langle \rho, \sum_j \mu_j H_j - \sum_k \lambda_k G_k \right\rangle \\ &= \sum_j \mu_j \langle \rho, H_j\rangle - \sum_k \lambda_k \langle \rho, G_k\rangle \\ &= -\left(\sum_j \mu_j^2 + \sum_k \lambda_k^2\right) \le 0.\end{aligned}$$

Since ρ is a real vector, this implies that the μ_j's are all zero, and the λ_k's are all zero. Therefore, we have proved that, for any $x\in X$, the m vectors in (9.11) form an m-frame in $\mathbb{R}^n$. Let $\tilde{V}^{r,s}_{n,m}$ be the space of (not necessarily orthonormal) m-frames in $\mathbb{R}^n$, with involution given again by (9.4). Since the G_j's are even functions and the H_k's are odd functions, (9.11) defines an equivariant map $X \twoheadrightarrow \tilde{V}^{r,s}_{n,m}$. Therefore, we are done by the following

LEMMA 9.12. *There exists an equivariant map* $g: \tilde{V}^{r,s}_{n,m} \twoheadrightarrow V^{r,s}_{n,m}$.

Proof. We define g by sending an m-frame $\{u_1, \ldots, u_m\}\in \tilde{V}^{r,s}_{n,m}$ to its Gram–Schmidt Normalization $\{v_1, \ldots, v_m\}\in V^{r,s}_{n,m}$. A routine computation with the standard normalization formula shows that g is an equivariant map with respect to the involutions defined in (9.4). We suppress the details here. Q.E.D.

COROLLARY 9.13. *There exist equivariant maps* $V^{r,s}_{n,m} \twoheadrightarrow Y^{r,s}_{n,m} \twoheadrightarrow V^{r,s}_{n,m}$.

Proof. The first map has been constructed in (9.5). To show the existence of the second map, let $Y=Y^{r,s}_{n,m}$. The natural homomorphism $B^{r,s}_{n,m}\to A_Y$ shows that $n\langle 1\rangle \ge r\langle 1\rangle \perp s\langle -1\rangle$ over A_Y. Therefore, Theorem (9.6) implies the existence of $Y \twoheadrightarrow V^{r,s}_{n,m}$. Q.E.D.

COROLLARY 9.14. *For given integers n, m, r, s and n', m', r', s' with $m=r+s$ and $m'=r'+s'$, consider the following statements:*

(1) *Over $B^{r,s}_{n,m}$, one has $n'\langle 1\rangle \ge r'\langle 1\rangle \perp s'\langle -1\rangle$.*

(1′) *Over any $\mathbb{R}$-algebra A, $n\langle 1\rangle \ge r\langle 1\rangle \perp s\langle -1\rangle \Rightarrow n'\langle 1\rangle \ge r'\langle 1\rangle \perp s'\langle -1\rangle$.*

(2) *There exists an equivariant map $V^{r,s}_{n,m} \twoheadrightarrow V^{r',s'}_{n',m'}$.*

(2′) *Over any $\mathbb{R}$-algebra of the type A_X (X a space with involution), $n\langle 1\rangle \ge r\langle 1\rangle \perp s\langle -1\rangle \Rightarrow n'\langle 1\rangle \ge r'\langle 1\rangle \perp s'\langle -1\rangle$.*

We have $(1)\Leftrightarrow(1')\Rightarrow(2')\Leftrightarrow(2)$.

Proof. $(1)\Leftrightarrow(1')\Rightarrow(2')$ are obvious, and $(2')\Leftrightarrow(2)$ follows easily from (9.6). Q.E.D.

The implication (2)$\Rightarrow$(1) is probably not true, though we won't try to construct an example to show this. The point is that there may exist equivariant maps $V_{n,m}^{r,s} \twoheadrightarrow V_{n',m'}^{r',s'}$ which may not be expressible in algebraic terms. Whenever we can construct an equivariant map $V_{n,m}^{r,s} \twoheadrightarrow V_{n',m'}^{r',s'}$ "algebraically," we can usually use the construction to show (1). In the following, we shall give several examples to illustrate this.

(9.15) There is a well-known map $f: V_{7,2} \to V_{7,3}$ given by vector products in $\mathbb{R}^7$ (cf. [E: p. 339]). We think of $\mathbb{R}^7$ as the space of Cayley numbers without a real part and take

$$f\{u, v\} = \{u, v, u \cdot v\},$$

where $u \cdot v$ is the Cayley multiplication in $\mathbb{R}^8$. It is easy to see that $u, v \in \mathbb{R}^7$ and $u \perp v$ imply that $u \cdot v \in \mathbb{R}^7$ and that $u \cdot v$ is perpendicular to both u and v. This gives two equivariant maps:

(9.16) $V_{7,2}^{1,1} \twoheadrightarrow V_{7,3}^{1,2}$ and $V_{7,2}^{0,2} \twoheadrightarrow V_{7,3}^{1,2}$.

Since the Cayley multiplcation can be defined over any commutative ring A, the construction of f actually shows the following: If $7\langle 1\rangle$ over A contains a subform $\langle a, b\rangle$ where a, b are units, then the orthogonal complement of $\langle a, b\rangle$ contains $\langle ab\rangle$. In particular, we have the following algebraic analogues of (9.16):

(9.17) $$\begin{cases} 7\langle 1\rangle \geq \langle 1, -1\rangle \Rightarrow 7\langle 1\rangle \geq \langle 1, -1, -1\rangle. \\ 7\langle 1\rangle \geq \langle -1, -1\rangle \Rightarrow 7\langle 1\rangle \geq \langle 1, -1, -1\rangle \quad \text{(over any } A\text{)}. \end{cases}$$

(Similar conclusions can be drawn for $3\langle 1\rangle$, but for this form the conclusions are already clear by determinant considerations.)

(9.18) There is also a vector product for three vectors in $\mathbb{R}^8$ which has been explicitly determined by G. Whitehead and P. Zvengrowski ([W], [Z]). In the case of an orthonormal 3-frame $\{u, v, w\}$, the vector product turns out to be $-u(\bar{v}w)$, and this is perpendicular to each of u, v, w. This leads to three equivariant maps:

(9.19) $V_{8,3}^{2,1} \twoheadrightarrow V_{8,4}^{2,2}$, $V_{8,3}^{1,2} \twoheadrightarrow V_{8,4}^{2,2}$ and $V_{8,3}^{0,3} \twoheadrightarrow V_{8,4}^{0,4}$.

For inner product spaces, this construction implies that, over any commutative ring, if $8\langle 1\rangle$ contains $\langle a, b, c\rangle$ where a, b, c are units, then the orthogonal complement of $\langle a, b, c\rangle$ contains $\langle abc\rangle$. In particular,

$$(9.20)\quad \begin{cases} 8\langle 1\rangle \geq \langle 1, 1, -1\rangle \Rightarrow 8\langle 1\rangle \geq \langle 1, 1, -1, -1\rangle, \\ 8\langle 1\rangle \geq \langle 1, -1, -1\rangle \Rightarrow 8\langle 1\rangle \geq \langle 1, 1, -1, -1\rangle, \\ 8\langle 1\rangle \geq \langle -1, -1, -1\rangle \Rightarrow 8\langle 1\rangle \geq \langle -1, -1, -1, -1\rangle. \end{cases}$$

(9.21) By the Hurwitz–Radon Theorem, we have an equivariant map S^{n-1} $(= V^{\delta}_{n,1}) \twoheadrightarrow V^{\delta}_{n,\rho(n)}$. It is well-known that the Hurwitz–Radon equations can be solved over $\mathbb{Z}$ and hence over any commutative ring A. Therefore, if we apply the Hurwitz–Radon Theorem to $n\langle 1\rangle$ over A, we conclude that, for any unit $a \in A$,

(9.22) $n\langle 1\rangle \geq \langle a\rangle \Rightarrow n\langle 1\rangle \geq \rho(n)\langle a\rangle$.

The power of (9.14) lies in the fact that it enables us to show that, in general, the statement (9.22) is already the best possible. In fact, for $a = -1$ and $A = A_n = B^{0,1}_{n,1}$, we have $n\langle 1\rangle \geq \rho(n)\langle -1\rangle$, but $n\langle 1\rangle \ngeq (\rho(n)+1)\langle -1\rangle$. (If $n\langle 1\rangle \geq (\rho(n)+1)\langle -1\rangle$ over $B^{0,1}_{n,1}$, (9.14) would imply that there is an equivariant map $S^{n-1} \twoheadrightarrow V^{\delta}_{n,\rho(n)+1}$; this contradicts the fact (8.3)(1) that $s'(V^{\delta}_{n,\rho(n)+1}) < n$.)

(9.23) We can get similar negative results by using the values of $s'(V^{\delta}_{n,q})$ tabulated earlier for $n \leq 19$. For instance, take $n = 6$. There is a decrease in $s'(V^{\delta}_{6,q})$ when q goes from 2 to 3 and when q goes from 4 to 5. Therefore we cannot have equivariant maps $V^{\delta}_{6,2} \twoheadrightarrow V^{\delta}_{6,3}$ or $V^{\delta}_{6,4} \twoheadrightarrow V^{\delta}_{6,5}$. This implies that, for $\mathbb{R}$-algebras A, $6\langle 1\rangle \geq 2\langle -1\rangle$ need not imply $6\langle 1\rangle \geq 3\langle -1\rangle$ (take $A = B^{0,2}_{6,2}$), and $6\langle 1\rangle \geq 4\langle -1\rangle$ need not imply $6\langle 1\rangle \geq 5\langle -1\rangle$ (take $A = B^{0,4}_{6,4}$). If we take $n = 2^i - 1$, we can say a lot more since, by (8.5), $s'(V^{\delta}_{n,q})$ decreases at every step as q goes from 1 to n. This implies that there is no map $V^{\delta}_{n,q} \twoheadrightarrow V^{\delta}_{n,q+1}$. Therefore, for $n = 2^i - 1$ and for any q, $n\langle 1\rangle \geq q\langle -1\rangle$ does not imply $n\langle 1\rangle \geq (q+1)\langle -1\rangle$; in fact, over $B^{0,q}_{n,q}$, $n\langle 1\rangle$ contains q copies of $\langle -1\rangle$, but not $q+1$ copies.

(9.24) Let A be any ring with $\frac{1}{2} \in A$, and let $W(A)$ be its Witt ring. Then we have

$$s(A) \leq 2 \Rightarrow 4W(A) = 0,$$
$$s(A) \leq 4 \Rightarrow 8W(A) = 0,$$
$$s(A) \leq 8 \Rightarrow 16W(A) = 0.$$

In fact, assume $s(A)\leq 8$. Then, over A, we have $8\langle 1\rangle \geq \langle -1\rangle$. Since $\rho(8)=8$, we get $8\langle 1\rangle \geq 8\langle -1\rangle$ and hence $8\langle 1\rangle \cong 8\langle -1\rangle$. This gives $16\langle 1\rangle = 0\in W(A)$, so $16W(A)=0$. The other cases are dealt with similarly. Unfortunately, this argument does not extend to higher levels since $\rho(2^i)=2^i$ holds only for $i=1,2,3$. In the case when $s(A)<\infty$, it is known that $2^nW(A)=0$ for some n ([K_1: Chapter 3]); bounds on n in terms of $s(A)$ seem to depend on topological K-theory.

(9.25) Over a field, we know that if the n-fold Pfister form $2^n\langle 1\rangle$ is isotropic, then it is in fact hyperbolic. But over a commutative ring A, this is not the case. We shall deal with the case of general n in the next section after developing some more machinery; here, let us give counterexamples for $n=2$ and 3. For $n=2$, consider the generic ring $A=B^{1,1}_{4,2}$ over which $4\langle 1\rangle$ is isotropic; we claim that $4\langle 1\rangle \not\geq \langle 1,1,-1\rangle$ (à fortiori $4\langle 1\rangle \not\cong \langle 1,-1\rangle \perp \langle 1,-1\rangle$). In fact, if $4\langle 1\rangle \geq \langle 1,1,-1\rangle$, there would exist an equivariant map $V^{\varepsilon}_{4,2} \twoheadrightarrow V^{\varepsilon}_{4,3}$ (by (9.14)). Since there also exists a "forgetful" map $V^{\varepsilon}_{4,3} \twoheadrightarrow V^{\varepsilon}_{4,2}$, these two spaces would have the same level and colevel. Using (7.5), we have therefore $s(V^{\varepsilon}_{4,3})=s'(V^{\varepsilon}_{4,3})=3$ and so by (2.11) $\pi_2(V_{4,3})$ has a quotient group $\cong\mathbb{Z}$. This is a contradiction since $\pi_2(V_{4,3})=0$ (by [Wh]). Similarly, we can see that, for the generic ring $A=B^{1,1}_{8,2}$, though $8\langle 1\rangle$ is isotropic, $8\langle 1\rangle \not\geq \langle 1,1,-1\rangle$ (à fortiori $8\langle 1\rangle \not\cong 4\langle 1,-1\rangle$). In fact, if $8\langle 1\rangle \geq \langle 1,1,-1\rangle$, we would get, as before, that $s(V^{\varepsilon}_{8,3})=s'(V^{\varepsilon}_{8,3})=7$ and that $\pi_6(V_{8,3})$ has a quotient group $\cong\mathbb{Z}$. This is again a contradiction since $\pi_6(V_{8,3})\cong\mathbb{Z}/2\mathbb{Z}$ by [Wh].

§10. Colevel of $V^{\varepsilon}_{n,q}$; σ-levels and σ-colevels

In an earlier section, we have given some partial computations for the colevel of $V^{\delta}_{n,q}$. The full computation of this colevel will probably remain unknown for some time since it would amount to the solution of the "skew-linear" version of the Hopf Problem on nonsingular pairings (cf. (8.2)) which is well-known to be a tough problem. However, if we replace the involution δ by the involution ε, the computation of the colevel of $V^{\varepsilon}_{n,q}$ turns out to be completely feasible. The crucial fact is the following:

THEOREM 10.1. *For $n\geq q\geq 1$, there is no equivariant map from S^{n-q+1} to $V^{\varepsilon}_{n,q}$.*

The proof of this theorem (and other related results) will appear in [LL]. Note that, for $q=1$, (10.1) is just the Borsuk-Ulam Theorem. For $q=2$, (10.1) can be proved by the homotopy argument used in the proof of (7.5)(1). For $q\geq 3$, the proof of (10.1) proceeds by induction on q; therefore, the theorem may be regarded as a common generalization of the Borsuk-Ulam Theorem and the homotopy facts used in the proof of (7.5)(1).

As the main consequence of (10.1), we have

COROLLARY 10.2. $s'(V^{\varepsilon}_{n,q})=n-q+1$. *If* $q>1$ *and* $n-q$ *is odd, then* $s(V^{\varepsilon}_{n,q})>n-q+1$.

Proof. For a fixed orthonormal frame $\{v_1,\ldots,v_{q-1}\}$, let S^{n-q} be the unit sphere in the orthogonal complement of $\sum_{i=1}^{q-1}\mathbb{R}\cdot v_i$. We have an equivariant map $S^{n-q}\twoheadrightarrow V^{\varepsilon}_{n,q}$ by sending v to $\{v_1,\ldots,v_{q-1},v\}$, so $s'(V^{\varepsilon}_{n,q})\geq n-q+1$. Using (10.1), we have therefore $s'(V^{\varepsilon}_{n,q})=n-q+1$. Now assume $q>1$ and $n-q$ is odd. If $s(V^{\varepsilon}_{n,q})$ is also equal to $n-q+1$, it would follow from (2.11) that $\pi_{n-q}(V_{n,q})$ has a homomorphism onto $\mathbb{Z}$. This is impossible since $\pi_{n-q}(V_{n,q})\cong\mathbb{Z}_2$ [St: p. 132]. Therefore, $s(V^{\varepsilon}_{n,q})>n-q+1$. Q.E.D.

COROLLARY 10.3. *There exists an equivariant map* $V^{\varepsilon}_{m,p}\twoheadrightarrow V^{\varepsilon}_{n,q}$ *only if* $m-p\leq n-q$. *In particular, there exists an equivariant map* $V^{\varepsilon}_{n,p}\twoheadrightarrow V^{\varepsilon}_{n,q}$ *iff* $p\geq q$, *and there exists an equivariant map* $V^{\varepsilon}_{m,q}\twoheadrightarrow V^{\varepsilon}_{n,q}$ *iff* $m\leq n$.

Proof. This follows from (10.2) and (2.7).

We shall now record the algebraic consequences of the results obtained above.

COROLLARY 10.4. *In general, over a ring B,* $n\langle 1\rangle\geq q\langle 1\rangle\perp\langle -1\rangle$ *does not imply* $n\langle 1\rangle\geq (q+1)\langle 1\rangle\perp\langle -1\rangle$.

Proof. Consider the generic ring $B=B^{q,1}_{n,q+1}$ over which we have $n\langle 1\rangle\geq q\langle 1\rangle\perp\langle -1\rangle$. If $n\langle 1\rangle\geq (q+1)\langle 1\rangle\perp\langle -1\rangle$ over B, there would exist (by (9.14)) an equivariant map $V^{\varepsilon}_{n,q+1}\twoheadrightarrow V^{\varepsilon}_{n,q+2}$, contradicting the last Corollary. Q.E.D.

A special case is the following.

COROLLARY 10.5. *Over* $B^{1,1}_{n,2}$, *the form* $n\langle 1\rangle$ *has Witt index* 1. *In particular, over* $B^{1,1}_{2^r,2}$ $(r\geq 2)$, *the r-fold Pfister form* $2^r\langle 1\rangle$ *is isotropic, but not hyperbolic.*

(Here, we use the following definitions: The *Witt index* of a form ϕ over a ring B is the largest nonnegative integer i such that $\phi\geq i\langle 1,-1\rangle$ over B. A form ϕ is called *hyperbolic* if $\phi\cong r\langle 1,-1\rangle$ for some integer i.)

COROLLARY 10.6. *Over the ring* $A_n = \mathbb{R}[x_1, \ldots, x_n]/(1+x_1^2+\cdots+x_n^2)$, *we have* $m\langle 1\rangle \geq q\langle 1\rangle \perp \langle -1\rangle$ *iff* $m \geq q+n$. *Suppose* $m \geq n$. *Then over* A_n *the form* $m\langle 1\rangle$ *has Witt index* $\leq m-n$. *In particular, if* $2^{r-1} < n < 2^r$, *then* $2^r\langle 1\rangle$ *is isotropic but not hyperbolic.*

Proof. The "if" part is trivial. Conversely, assume $m\langle 1\rangle \geq q\langle 1\rangle \perp \langle -1\rangle$ over A_n. Let $X = V_{\mathbb{C}}(1+x_1^2+\cdots+x_n^2)$, with involution given by complex conjugation. By (9.6) we have an equivariant map $X \twoheadrightarrow V^{\varepsilon}_{m,q+1}$ so $s'(X) \leq s'(V^{\varepsilon}_{m,q+1}) = m-q$. Since $s'(X) = n$, we get $m \geq q+n$. The rest follows easily from this inequality. Q.E.D.

Motivated by the success in computing $s'(V^{\varepsilon}_{n,q})$, it seems useful to use the $V^{\varepsilon}_{n,q}$'s as "model" spaces, in generalization of the use of the spheres in the definition of levels and colevels. To formulate these generalizations, let $k \geq 0$ be any integer. For any space X with involution, we define

(10.7) $\sigma_k(X) = \inf\{n : \exists X \twoheadrightarrow V^{\varepsilon}_{n+k,k+1}\}$,

(10.7′) $\sigma'_k(X) = \sup\{n : \exists V^{\varepsilon}_{n+k,k+1} \twoheadrightarrow X\}$.

We call $\{\sigma_k(X) : k \geq 0\}$ the *σ-levels* and $\{\sigma'_k(X) : k \geq 0\}$ the *σ-colevels* of X. Since $V^{\varepsilon}_{n,1} = S^{n-1}$, we have $\sigma_0(X) = s(X)$ and $\sigma'_0(X) = s'(X)$, so the σ-levels and σ-colevels subsume the level and colevel discussed in the earlier sections.

THEOREM 10.8. *For any space X with involution, we have*

$$\cdots \leq \sigma'_{k+1}(X) \leq \sigma'_k(X) \leq \cdots \leq \sigma'_0(X) \leq \cdots \leq \sigma_{k+1}(X) \leq \sigma_k(X) \leq \cdots \leq \sigma_0(X).$$

Moreover, $\sigma'_k(X) \leq \sigma'_{k+1}(X)+1$ *and* $\sigma_k(X) \leq \sigma_{k+1}(X)+1$ *for every k.*

Proof. If there exists $X \twoheadrightarrow V^{\varepsilon}_{n+k,k+1}$, then by (10.1) there cannot exist $S^n \twoheadrightarrow X$ and so $s'(X) \leq n$. This shows that $\sigma'_0(X) = s'(X) \leq \sigma_k(X)$ for every k. Next consider the standard imbedding $V^{\varepsilon}_{n+k,k+1} \twoheadrightarrow V^{\varepsilon}_{n+k+1,k+2}$. Using definitions we get $\sigma'_{k+1}(X) \leq \sigma'_k(X)$ and $\sigma_{k+1}(X) \leq \sigma_k(X)$. Similarly, using the forgetful map $V^{\varepsilon}_{n+k+1,k+2} \twoheadrightarrow V^{\varepsilon}_{(n+1)+k,k+1}$, we see that $\sigma'_k(X) \leq \sigma'_{k+1}(X)+1$ and $\sigma_k(X) \leq \sigma_{k+1}(X)+1$. Q.E.D.

As algebraic analogues of (10.7) and (10.7′), we can define the σ-level of a (commutative) ring A and the σ-colevel of a (commutative) $\mathbb{R}$-algebra B as follows (where k is any nonnegative integer):

(10.9) $\sigma_k(A) = \inf\{n : (n+k)\langle 1\rangle \geq k\langle 1\rangle \perp \langle -1\rangle \text{ over } A\}$,

(10.9′) $\sigma'_k(B) = \sup\{n : \exists\, \mathbb{R}\text{-algebra homomorphism } B \to A_{V^{\varepsilon}_{n+k,k+1}}\}$.

As in the topological case, we have $\sigma_0(A)=s(A)$ and $\sigma_0'(B)=s'(B)$ (the latter was first introduced in §4). Moreover, $\sigma_1(A)$ is just the "sublevel" $\sigma(A)$ introduced in §5. We have the following algebraic analogue of (10.8):

THEOREM 10.10. *The inequalities in* (10.8) *remain true with the space X replaced by any commutative* $\mathbb{R}$*-algebra B.*

Proof. The inequalities $\sigma_{k+1}(B)\le\sigma_k(B)\le\sigma_{k+1}(B)+1$ and $\sigma'_{k+1}(B)\le\sigma'_k(B)\le\sigma'_{k+1}(B)+1$ follow as before. (The former, of course, holds for any commutative ring.) To show that $s'(B)=\sigma_0'(B)\le\sigma_k(B)$, we may assume that $n=\sigma_k(B)<\infty$ and so $(n+k)\langle 1\rangle\ge k\langle 1\rangle\perp\langle -1\rangle$ over B. If $s'(B)\ge n+1$, there would exist a ring homomorphism from B to A_{S^n} and hence $(n+k)\langle 1\rangle\ge k\langle 1\rangle\perp\langle -1\rangle$ also over the latter ring. But then by (9.6) there would exist an equivariant map from S^n to $V^{\varepsilon}_{n+k,k+1}$, contradicting (10.1). Therefore, $s'(B)\le n=\sigma_k(B)$. Q.E.D.

The algebraic σ-levels and σ-colevels are related to their topological counterparts by the following theorem:

THEOREM 10.11. *For any space X with involution, we have* $\sigma_k(X)=\sigma_k(A_X)$. *If X is an affine variety defined over* $\mathbb{R}$ *with involution given by complex conjugation, then* $\sigma'_k(X)=\sigma'_k(A_X)$.

Proof. The first statement follows from (9.6). With the additional assumption in the second statement, any $\mathbb{R}$-algebra homomorphism $A_X\to A_{V^{\varepsilon}_{n+k,k+1}}$ "arises" from an equivariant map $V^{\varepsilon}_{n+k,k+1}\twoheadrightarrow X$, as we have shown in the proof of (4.3). Therefore $\sigma'_k(X)=\sigma'_k(A_X)$. Q.E.D.

We shall now conclude this section by checking the following values of the invariants σ'_1, s', σ_k and s for the spaces S^{n-1}, $V^{\varepsilon}_{n,2}$ and $V^{\delta}_{n,2}$, where k is any integer ≥ 1.

		$\sigma'_1(X)$	$s'(X)$	$\sigma_k(X)$	$s(X)$
$X=S^{n-1}$	$n\ne 1,3,7$	$n-1$	n	n	n
	$n=1,3,7$	n	n	n	n
$X=V^{\varepsilon}_{n,2}$	$n\ne 2,4,8$	$n-1$	$n-1$	$n-1$	n
	$n=2,4,8$	$n-1$	$n-1$	$n-1$	$n-1$
$X=V^{\delta}_{n,2}$	$n=$ even	$n-1$	n	n	n
	$n=3,7$	$n-1$	$n-1$	$n-1$	n

Since $s'(X) \le \sigma_k(X) \le \sigma_1(X)$ for any space, it is sufficient to work with the case $k=1$ in the following.

(I) $X = S^{n-1}$. We have $s'(X) = s(X) = n$, so we need only compute $\sigma'_1(X)$. By (7.5)(2), we have $V^{\varepsilon}_{n+1,2} \twoheadrightarrow S^{n-1}$ iff $n = 1, 3, 7$. Therefore, $\sigma'_1(S^{n-1}) = n$ if $n = 1, 3, 7$, and $\sigma'_1(S^{n-1}) = n-1$ if $n \neq 1, 3, 7$.

(II) $X = V^{\varepsilon}_{n,2}$. The identity map $X \twoheadrightarrow X$ shows that $n-1 \le \sigma'_1(X) \le \sigma_1(X) \le n-1$, so $\sigma'_1(X) = s'(X) = \sigma_1(X) = n-1$. the computation of $s(X)$ is in (7.5)(2).

(III) $X = V^{\delta}_{n,2}$. In view of (7.2) we need only compute $\sigma'_1(X)$ and $\sigma_1(X)$. First assume $n = 7$. By (9.16), we have equivariant maps $V^{\varepsilon}_{7,2} \twoheadrightarrow V^{\delta}_{7,2} \twoheadrightarrow V^{\varepsilon}_{7,2}$. This implies that $6 \le \sigma'_1(X) \le \sigma_1(X) \le 6$ so $\sigma'_1(X) = \sigma_1(X) = 6 = n-1$. The case $n = 3$ is similar. Now assume n is even. In this case we have $s'(X) = s(X) = n$ so we are done if we can show that $\sigma'_1(X) < s'(X) = n$. Assume, instead, that $\sigma'_1(X) = n$. By definition, this means that there is an equivariant map $V^{\varepsilon}_{n+1,2} \twoheadrightarrow X = V^{\delta}_{n,2}$. By (2.7), we have $s(V^{\varepsilon}_{n+1,2}) \le s(V^{\delta}_{n,2})$; by (7.2) (2), this boils down to $n+1 \le n$, a contradiction.

In case (III), we have not been able to compute $\sigma'_1(X)$ and $\sigma_1(X)$ for odd integers $n \neq 3, 7$. We conjecture that they are given as follows:

	$\sigma'_1(X)$	$s'(X)$	$\sigma_1(X)$	$s(X)$
$X = V^{\delta}_{n,2}$ ($n = \text{odd} \neq 3, 7$)	$n-2$	$n-1$	n	n

Stated more explicitly in terms of equivariant maps, this conjecture says that

(10.12) *For n odd $\neq 3, 7$, there are no equivariant maps between $V^{\varepsilon}_{n,2}$ and $V^{\delta}_{n,2}$.*

For $n = 3, 7$, we have already pointed out that there exist $V^{\varepsilon}_{n,2} \twoheadrightarrow V^{\delta}_{n,2} \twoheadrightarrow V^{\varepsilon}_{n,2}$. For n even, there exists $V^{\varepsilon}_{n,2} \twoheadrightarrow V^{\delta}_{n,2}$ but not $V^{\delta}_{n,2} \twoheadrightarrow V^{\varepsilon}_{n,2}$. Therefore, only the case n odd $\neq 3, 7$ remains to be of interest. In this case $V^{\varepsilon}_{n,2}$ and $V^{\delta}_{n,2}$ both have level n and colevel $n-1$; therefore, in order to distinguish their "equivariant types," it is not enough to compare them with the spheres, but it will be necessary to delve more deeply into their equivariant properties.

Note that if the Conjecture (10.12) is true, we will be able to compute the sublevel of the $\mathbb{R}$-algebra $B^{\delta}_{n,2}$ (with generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ and relations $\sum x_i^2 = \sum y_i^2 = -1$ and $\sum x_i y_i = 0$) in the case n odd, $n \neq 3, 7$. (This is the missing case in (7.12).) In fact, if there is no equivariant map $V^{\delta}_{n,2} \twoheadrightarrow V^{\varepsilon}_{n,2}$, (9.14) will imply that $n\langle 1 \rangle$ is *anisotropic* over $B^{\delta}_{n,2}$, so $\sigma(B^{\delta}_{n,2}) = n$, for n odd $\neq 3, 7$.

§11. Open problems

While the topological methods developed in this paper have helped solve some of the basic problems concerning the level of rings, there still remain a number of other difficult problems which we are not able to solve. Aside from problems concerning the level, there are also problems concerning quadratic forms, orthogonal groups and equivariant maps between spaces with involution. In the following we shall state and comment on some of these open problems in the hope of stimulating future work.

The first problem concerns the level of the generic ring $A_n(k) = k[x_1, \ldots, x_n]/(1+x_1^2+\cdots+x_n^2)$ where k is an arbitrary commutative ring. We venture the following

(11.1) Level Conjecture. $s(A_n(k)) = \min\{s(k), n\}$.

To lend credence to this Conjecture, we note the truth of the Conjecture in the following important cases:

(A) By (3.4), the Conjecture is true for all semireal rings k.

(B) The Conjecture is clearly true if $s(k) \leq n$. In fact, in this case, we have homomorphisms $k \to A_n(k) \to k$, so $s(A) = s(k) = \min\{s(k), n\}$.

(C) The Conjecture is true in the important case when k is itself the generic ring $A_m(\mathbb{R})$. For this choice of k, we have

$$A_n(k) \cong \frac{\mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_m]}{(1+x_1^2+\cdots+x_n^2,\ 1+y_1^2+\cdots+y_m^2)}$$

By the symmetry of the x's and the y's, we see from (B) above that $s(A_n(k)) = \min\{m, n\} = \min\{s(k), n\}$.

(D) By an algebraization of the method used in [DLP] (and further field-theoretic techniques), Arason and Pfister [AP] have shown the Conjecture (11.1) to be true for all *fields* k (see also [K_2]). Unfortunately, this does not seem to imply the truth of the Conjecture for all rings k, since, when we map a ring into a field by a homomorphism, the level usually decreases.

Our second problem concerns the level of the tensor product of two commutative algebras. To be more specific, let A, B be commutative $\mathbb{R}$-affine algebras. Since A, B are both subalgebras of $A \otimes_{\mathbb{R}} B$, we have, of course, $s(A \otimes_{\mathbb{R}} B) \leq \min\{s(A), s(B)\}$. It seems natural to ask:

(11.2) Is it true that $s(A \otimes_{\mathbb{R}} B) = \min\{s(A), s(B)\}$?

This turns out to be true in all the cases in which we can make a determination of $s(A \otimes_{\mathbb{R}} B)$. We record some of these cases below.

(A) The formula in (11.2) is true if there exists an $\mathbb{R}$-algebra homomorphism $f: A \to B$. For, f induces homomorphisms $A \otimes_{\mathbb{R}} B \to B \otimes_{\mathbb{R}} B \to B$, from which we see that $s(A \otimes_{\mathbb{R}} B) \geq s(B) = \min\{s(A), s(B)\}$.

(B) The formula in (11.2) is true if the $\mathbb{R}$-affine algebra A is semireal. For, in this case, there exists an $\mathbb{R}$-algebra homomorphism $A \to \mathbb{R} \subset B$ [La$_2$: Th. 6.2], so we can use (A) above.

(C) Let $A = A_n(\mathbb{R})$. Then the formula in (11.2) is true for any $\mathbb{R}$-albegra B such that $n \notin (s'(B), s(B))$. For, if $n \geq s(B)$, then there exists an $\mathbb{R}$-algebra homomorphism $A \to B$ so we can use (A) above. On the other hand, if $n \leq s'(B)$, then there exists an $\mathbb{R}$-algebra homomorphism $B \to A_{S^{n-1}}$. Using this together with the homomorphism $A \to A_{S^{n-1}}$, we get a homomorphism $A \otimes_{\mathbb{R}} B \to A_{S^{n-1}}$, which implies that $s(A \otimes_{\mathbb{R}} B) \geq s(A_{S^{n-1}}) = n = \min\{s(A), s(B)\}$.

The next problem concerns the relationship between the level of an $\mathbb{R}$-affine algebra A and its number of generators. More specifically, we raise the following question:

(11.3) Suppose $A = \mathbb{R}[x_1, \ldots, x_n]/\mathfrak{A}$ has finite level. Does there exist a function $\alpha(n)$ of n such that $s(A) \leq \alpha(n)$ (independently of $\mathfrak{A}$)?

The answer to this question is "yes" in the case $n = 1$; in fact, we can choose $\alpha(1) = 1$ according to Proposition 4.8. However, the case $n = 2$ already seems to be open. We only know, from Proposition 4.5, that $\alpha(2) \geq 3$, if it exists.

The last problem on levels we want to mention is connected with Hurwitz' Problem of determining the least number of squares needed to express $(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2)$ as a sum of squares in $\mathbb{R}[x_1, \ldots, x_r, y_1, \ldots, y_s]$. Let this number be denoted by $r * s$. Then one can ask:

(11.4) Is the level of the ring $C = \mathbb{R}[x_1, \ldots, x_r, y_1, \ldots, y_s]/(1 + \sum_{i=1}^r x_i^2 \cdot \sum_{j=1}^s y_j^2)$ equal to $r * s$?

Since there exist surjections of C onto $A_r(\mathbb{R})$ and $A_s(\mathbb{R})$, it follows that

$$\max(r, s) \leq s(C) \leq r * s.$$

In particular, in the "classical" case when $r \leq \rho(s)$ (ρ the Radon function), we have $s(C) = r * s = s$. It seems natural to expect that $s(C)$ should still be equal to $r * s$ in the non-classical case, but we have not been able to give a proof.

Even for the ring $A_n = A_n(\mathbb{R})$, there remain difficult problems concerning, for instance, the behavior of quadratic forms and orthogonal groups. We shall mention two specific problems.

(11.5) Suppose we have an orthogonal decomposition $n\langle 1\rangle \cong \rho(n)\langle -1\rangle \perp \phi$ over A_n. Is the form ϕ uniquely determined (up to isometry)? Is the form ϕ orthogonally indecomposable?

(We only know, from the results in this paper, that ϕ cannot split off a one-dimensional subform $\langle a\rangle$ for any $a \in \dot{\mathbb{R}}$.)

(11.6) If $-1 = f_1(\bar{x})^2 + \cdots + f_n(\bar{x})^2$ in $A_n = \mathbb{R}[x_1, \ldots, x_n]/(1+\sum_{i=1}^n x_i^2)$, is $(f_1(\bar{x}), \ldots, f_n(\bar{x}))$ conjugate to $(\bar{x}_1, \ldots, \bar{x}_n)$ under the action of the orthogonal group of $n\langle 1\rangle$? What can we say about the structure of this orthogonal group?

This question was raised by W. Scharlau and M. Kervaire. The following observations were made by Kervaire in a letter to Scharlau in July, 1980.

(A) The first half of the question (11.6) has an affirmative answer for $n = 1, 2, 4, 8$. Consider, for instance, the case $n = 2$. For (f_1, f_2) as in (11.6), it is easy to check that the matrix

$$(11.7)\quad T = \begin{pmatrix} f_1 & f_2 \\ f_2 & -f_1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$$

is in the special orthogonal group $SO_2(A_2)$, and that $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. For $n = 4, 8$, we can construct T similarly by using the matrices arising from the multiplication law of quaternions and Cayley numbers.

(B) If we let $(f_1, f_2) = (x_1, -x_2)$ in (11.7), we obtain the matrix

$$(11.8)\quad J_0 = \begin{pmatrix} x_1^2 - x_1^2 & 2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} \in SO_2(A_2).$$

It has been shown by M. Kervaire that $SO_2(A_2)$ is the direct product of $SO_2(\mathbb{R})$ and the infinite cyclic group generated by the matrix J_0. Consider the matrix

$$(11.9)\quad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} J_0 = \begin{pmatrix} 1+2x_1^2 & 2x_1x_2 \\ 2x_1x_2 & 1+2x_2^2 \end{pmatrix} \in O_2(A_2).$$

We have $\det(J) = -\det(J_0) = -1$ and $J^2 = J^tJ = I$. Since $SO_2(A_2) = SO_2(\mathbb{R}) \times \langle J_0\rangle$, it follows immediately that $O_2(A_2)$ is generated by $O_2(\mathbb{R})$ and the (symmetric) matrix J.

For arbitrary n, we can generalize the definition of J by taking $J_n = (\delta_{ij} + 2x_i x_j)$, where (δ_{ij}) are the Kronecker deltas. It is easy to see that $J_n \in O_n(A_n)$ and that $\det(J_n) = -1$. To make the second part of Question (11.6) more specific (and in an attempt to generalize the case $n = 2$), one can ask: is $O_n(A_n)$ *generated by* $O_n(\mathbb{R})$ *and the (symmetric) matrix* J *for arbitrary* n?

Finally, there are various open problems concering equivariant maps between Stiefel manifolds whose solutions will be of importance in studying quadratic forms over $\mathbb{R}$-affine algebras. Stated in the most general form, the ultimate problem is that of determining all quadruples (n, q, r, s) and (n', q', r', s') for which there exists an equivariant map from $V_{n,q}^{r,s}$ to $V_{n',q'}^{r',s'}$. In this general form, however, the problem is perhaps too difficult. We shall state below two special cases of it which should be more tractable:

(11.10) If there exists an equivariant map $V_{n,q}^{r,s} \twoheadrightarrow V_{n',q}^{r,s}$, does it follow that $n \leq n'$?

An affirmative answer to this would represent an interesting generalization of the Borsuk-Ulam Theorem. By (10.5), we know that the answer is indeed affirmative in the special case when $s = 1$.

(11.11) Let n be a given integer. For what pairs $q < q'$ will there exist an equivariant map $V_{n,q}^{\delta} \twoheadrightarrow V_{n,q'}^{\delta}$?

For $q = 1$, we known from Adams' solution of the Vector Field Problem that such a map exists iff $q' \leq \rho(n)$. For $q > 1$, we also know from the work of G. W. Whitehead [W] that there exist *cross-sections* $V_{n,q} \to V_{n,q'}$ for the natural fibration $V_{n,q'}^{\delta} \to V_{n,q}^{\delta}$ only for a few specific values of n, q and q'. However, it is conceivable that there exist various equivariant maps $V_{n,q}^{\delta} \twoheadrightarrow V_{n,q'}^{\delta}$ which are *not* cross-sections. Solution of this problem (as well as (11.10)) will be of interest in the study of the decomposition of $n\langle 1\rangle$ into $r\langle 1\rangle \perp s\langle -1\rangle \perp \phi$, by the results of §9.

Note added in proof. K. Y. Lam informed us that the question (11.10) above has been answered affirmatively by Duane Randall in a recent preprint entitled "on equivariant maps of Stiefel manifolds." The algebraic implication of Randall's result is the following (cf. (9.14)): Over the ring $B_{n,q}^{\delta}$, one has $n\langle 1\rangle \geq q\langle -1\rangle$, but $n'\langle 1\rangle \not\geq q\langle -1\rangle$ for any $n' < n$.

REFERENCES

[A_1] J. ADEM, *On nonsingular bilinear maps*, in The Steenrod Algebra and its Applications, pp. 11–24, Lecture Notes in Math., Vol. 168, Springer Verlag, 1970.

[A_2] J. ADEM, *On nonsingular bilinear maps II*, Bol. Soc. Mat. Mexicana *16* (1971), 64–70.

[A_3] J. ADEM, *Construction of some normed maps*, Bol. Soc. Mat. Mexicana *20* (1975), 59–75.

[ABS] M. ATIYAH, R. BOTT and A. SHAPIRO, *Clifford modules*, Topology *3* (1964), 3–38.

[Ad] J. ADAMS, *Vector fields on spheres*, Ann. of Math. *75* (1962), 603–632.

[AK] R. ARENS and I. KAPLANSKY, *Topological representation of algebras*, Trans. Amer. Math. Soc. *62* (1948), 457–481.

[AP] J. K. ARASON and A. PFISTER, *Quadratische Formen über affinen Algebren und ein algebraischer Beweis des Satzes von Borsuk-Ulam*, J. reine angew. Math. *331* (1982), 181–184.

[B] F. BEHREND, *Über Systeme reeller algebraischer Gleichungen*, Composito Math. *7* (1939), 1–19.

[Ba] R. BAEZA, *Über die Stufe von Dedekind Ringen*, Arch. Math. *33*, (1979), 226–231.

[CDLR] M. D. CHOI, Z. D. DAI, T. Y. LAM and B. REZNICK, *The Pythagoras number of some affine algebras and local algebras*, J. reine angew. Math., *336* (1982), 45–82.

[CF_1] P. CONNER and E. FLOYD, *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. *66* (1960), 416–441.

[CF_2] P. CONNER and E. FLOYD, *Fixed point free involutions and equivariant maps II*, Trans. Amer. Math. Soc. *105* (1962), 222–228.

[CLRR] M. D. CHOI, T. Y. LAM, B. REZNICK and A. ROSENBERG, *Sums of squares in some integral domains*, J. Algebra *65* (1980), 234–256.

[DLP] Z. D. DAI, T. Y. LAM, and C. K. PENG, *Levels in algebra and topology*, Bull. Amer. Math. Soc. (New Series) *3* (1980), 845–848.

[E] B. ECKMANN, *Stetige Lösungen linearer Gleichungssysteme*, Comment. Math. Helv. *15* (1942/43), 318–339.

[GH] P. GRIFFITHS and J. HARRIS, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[H] H. HOPF, *Ein topologischer Beitrag zur reellen Algebra*, Comment. Math. Helv. *13* (1940/41), 219–239.

[K_1] M. KNEBUSCH, *Symmetric bilinear forms over algebraic varieties*, in Proc. Quadratic Form Conference (ed. G. Orzech), Queen's Papers in Pure and Applied Math. Vol. 46, pp. 103–283, Queen's University, Kingston, Ontario, 1977.

[K_2] M. KNEBUSCH, *An algebraic proof of the Borsuk-Ulam theorem for polynomial mappings*, Proc. Amer. Math. Soc. *84* (1982), 29–32.

[L_1] K. Y. LAM, *Construction of nonsingular bilinear maps*, Topology *6* (1967), 423–426.

[L_2] K. Y. LAM, *Construction of some nonsingular bilinear maps*, Bol. Soc. Mat. Mexicana *13* (1968), 88–94.

[L_3] K. Y. LAM, *On bilinear and skew-linear maps that are nonsingular*, Quart. J. Math. Oxford. (2), *19* (1968), 281–288.

[L_4] K. Y. LAM, *Sectioning vector bundles over real projective spaces*, Quart. J. Math. Oxford (2), *23* (1972), 97–106.

[L_5] K. Y. LAM, *A note on Stiefel manifolds and the generalised J-homomorphism*, Bol. Soc. Mat. Mexicana *21* (1976), 33–38.

[La_1] T. Y. LAM, *The Algebraic Theory of Quadratic Forms*, W. A. Benjamin, revised printing, 1980.

[La_2] T. Y. LAM, *An Introduction to Real Algebra*, Lecture Notes, Sexta Escuela Latinoamericana Matematicas, Oaxtepec, Mexico, 1982.

[LL] K. Y. LAM and T. Y. LAM, *Equivariant maps between Stiefel manifolds*, to appear.
[M] S. MORITA, *The Kervaire invariant of hypersurfaces in complex projective spaces*, Comment. Math. Helv. *50* (1975), 403–419.
[Pf] A. PFISTER, *Quadratische Formen in beliebigen Körpern*, Invent. Math. *1* (1966), 116–132.
[St] N. STEENROD, *The Topology of Fibre Bundles*, Princeton University Press, 1951, Princeton, N.J.
[W] G. W. WHITEHEAD, *Note on cross-sections in Stiefel manifolds*, Comment. Math. Helv. *37* (1962), 239–240.
[Wh] J. H. C. WHITEHEAD, *On the groups $\pi_r(V_{n,m})$ and sphere bundles*, Proc. London Math. Soc. *48* (1944), 243–291. Corrigendum, *49* (1947), 478–481.
[Y_1] C. T. YANG, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson, I*, Ann. of Math. *60* (1954), 262–282.
[Y_2] C. T. YANG, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson, II*, Ann. of Math. *62* (1955), 271–283.
[Z] P. ZVENGROWSKI, *A 3-fold vector product in $\mathbb{R}^8$*, Comment. Math. Helv. *40* (1965/66), 149–152.

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