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Satake compactifications

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Introduction

Let G be a semi-simple algebraic group defined over $\mathbf{Q}$, and let X denote the corresponding symmetric space, which we assume to be non-compact. Let Γ be an arithmetic subgroup of G. The quotients $\Gamma\backslash X$ are interesting spaces, in general non-compact. For example, when G is the standard form of $\mathrm{Sp}(2n, \mathbf{R})$, the group of $(2n)\times(2n)$ symplectic matrices, X is the Siegel upper half-space of genus n, and $\Gamma\backslash X$ is, for suitable Γ, the moduli space of n-dimensional principally polarized Abelian varieties with corresponding level structure. In order to discuss the geometry of these spaces, people have introduced various methods for compactifying them.

Though there are some ideas in the work of Siegel, the modern starting point for the theory of compactifications is the work of Satake (see [8], [9]). To each locally faithful finite-dimensional representation τ of G, he constructed an embedding of X in some real projective space, and took the closure, X^*_τ (see our (2.1)). In fact, as he observed, the homeomorphism type of X^*_τ can be described explicitly, and depends on τ only through the orthogonality relations between its restricted highest weight and the simple $\mathbf{R}$-roots of G. The boundary of X^*_τ can be written as a union of so-called *boundary components*. In [9], Satake observed that by taking only the *rational* boundary components (defined suitably), one could use the space X^*_τ to construct a Hausdorff compactification of $\Gamma\backslash X$ in certain cases. This included the case of the Siegel upper half-spaces that he had worked out slightly earlier. The construction was extended somewhat by Borel in [3], where notions from the theory of algebraic groups were added to the discussion.

Assume now that X is Hermitian symmetric. In [2], Baily and Borel compactified all arithmetic quotients of such X. The procedure used the realization of X as a bounded symmetric domain to generate the boundary. In these cases, it is known that the closure of the bounded domain is homeomorphic to X^*_τ for suitably chosen τ (see our (3.11)). Thus, the Baily–Borel compactification of $\Gamma\backslash X$

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is a *generalized Satake compactification.*[1] Moreover, it is proved in [2] that the compactification becomes, via the projective embedding defined by automorphic forms of sufficiently high weight, a normal complex algebraic variety. The rather nasty nature of the singularities of these spaces has led algebraic geometers to seek smooth models. Mumford et al. have an elaborate procedure for constructing desingularizations of the varieties by means of toroidal embeddings [1].

The invention of intersection homology, a theory with Poincaré duality even for singular varieties, by Goresky and MacPherson has opened the possibility of avoiding the complicated construction, and non-canonical nature, of desingularizations. This is especially promising in light of the link we discovered between the L_2 cohomology of $\Gamma\backslash X$ with respect to natural metrics, and the intersection homology of the Baily–Borel compactification [10, (6.11)ff.]. In fact, we have conjectured that they are always isomorphic. Borel has verified this for all groups of $\mathbf{Q}$-rank one. We would, of course, like to prove it in higher rank, where the local topology is more complicated, as well.

The most natural place on which to study the L_2 cohomology is the manifold with corners $\Gamma\backslash\bar{X}$ defined by Borel and Serre in [4], or what comes to almost the same thing, on the maximal Satake compactification (where τ is generic; see [10, §4(a)]). The reason for this is that there are distinguished neighborhoods of compact subsets of the faces of the boundary of $\Gamma\backslash\bar{X}$; with respect to associated coordinates, the metric has explicit asymptotic formulas. Now, the verification of our conjecture is equivalent to certain vanishing assertions for the L_2 cohomology of neighborhoods of points on the Baily–Borel compactification. It seems to be a good idea, then, to express these neighborhoods in terms of the distinguished neighborhoods on $\Gamma\backslash\bar{X}$, for then we could try to patch together the local L_2 cohomology by the methods of [10]. This approach works for arithmetic quotients of the Siegel upper half-space of genus two.

Thus grew the idea of realizing the Baily–Borel compactification as the natural quotient of $\Gamma\backslash\bar{X}$. The possibility of doing this has been conceded for some time, but it had not been carried out, perhaps for lack of incentive. In this paper, we realize all generalized Satake compactifications as quotients of $\Gamma\backslash\bar{X}$. We require two assumptions on the representation τ (see (3.3) and (3.4)) in order to carry out the construction; these hypotheses are met in the cases covered in [2], [3] and [9], and as such, one can regard our discussion as a mild generalization of these works.

We will reconstruct the Satake compactifications in such a way that they really do look like quotients of the manifold with corners. Let ${}_{\mathbf{Q}}A$ denote the identity component of the real points of a maximal $\mathbf{Q}$-split torus of G. By making a choice

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[1] By this, what we mean is a compactification whose topology is induced from that of the closure in some X_τ^* of a Siegel set, by the procedure introduced in [9] (see our (3.9)).

of positive simple roots, one obtains an identification

$$_{\mathbf{Q}}A \simeq (0,\infty)^r,$$

where r is the $\mathbf{Q}$-rank of G. One puts

$$_{\mathbf{Q}}\bar{A} \simeq (0,\infty]^r.$$

Roughly speaking, the basic point in the construction of the manifold with corners $\bar{X}$ is the adjoining of $_{\mathbf{Q}}\bar{A}$ along images of $_{\mathbf{Q}}A$ in X. In effect, one allows the simple roots to go to infinity independently and in all possible ways. For the Satake compactifications, if a simple root goes to infinity, it is irrelevant whether certain other ones do or not. As such, we are led to introduce an $_{\mathbf{Q}}A$-equivariant quotient $_{\mathbf{Q}}A^*_\tau$ of $_{\mathbf{Q}}\bar{A}$. By adjoining $_{\mathbf{Q}}A^*_\tau$ along $_{\mathbf{Q}}A$, and then doing a little more, we define a *crumpled corner* and the manifold with crumpled corners in a way that mimics the construction in [4, §§5–7]. After taking an arithmetic quotient, one sees that one has reproduced the topology of $\Gamma\backslash X^*_\tau$. Having done this, one can write down rather easily a description of the fibers of the quotient mapping from $\Gamma\backslash\bar{X}$.[2] We remark that the smooth compactifications of [1], however, are not in general natural quotients of the manifold with corners.

In §1, we give an exposition of basic facts about restricted root systems, following the treatments in [5] and [8]. In §2, we present a discussion of the Satake compactifications X^*_τ. The one ingredient which could be called new is the introduction of A^* in (2.6). In §3, we carry out the construction of Satake compactifications via manifolds with crumpled corners, as described above.

I wish to express my gratitude to Armand Borel for encouraging this project, and for patiently and critically listening to the details of the construction.

1. Restricted root systems associated to real algebraic groups

(1.1) Let $\mathbf{F}$ be a subfield of the real numbers $\mathbf{R}$. We establish the following abuse of language as convention: "H is an algebraic group over $\mathbf{F}$" shall mean "H is the set of real points of an algebraic group defined over $\mathbf{F}$, and we regard it as a Lie group"; for the set of $\mathbf{F}$-rational points of H, we write $H_{\mathbf{F}}$.

Let G be a semi-simple algebraic group over $\mathbf{F}$, with Lie algebra $\mathfrak{g}$. Let $_{\mathbf{F}}A$ be

[2] Our description of the quotient mapping has been used by Charney and Lee in their calculation of the cohomology of the (Baily–Borel–)Satake compactification of the Siegel modular varieties ($G = \mathrm{Sp}(2n,\mathbf{R})$, $\Gamma = \mathrm{Sp}(2n,\mathbf{Z})$) [6].

the identity component of a maximal $\mathbf{F}$-split torus ${}_{\mathbf{F}}T$ of G, with Lie algebra ${}_{\mathbf{F}}\mathfrak{a}$. Let K be a maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$, which can be chosen so that under the corresponding Cartan decomposition

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p} \tag{1}$$

(orthogonal with respect to the Killing form B), we have ${}_{\mathbf{F}}\mathfrak{a}\subset\mathfrak{p}$. One can enlarge ${}_{\mathbf{F}}\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}_{\mathbf{C}}$ of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$ such that $\mathfrak{h}_{\mathbf{C}}=(\mathfrak{h}_{\mathbf{C}}\cap\mathfrak{k}_{\mathbf{C}})\oplus(\mathfrak{h}_{\mathbf{C}}\cap\mathfrak{p}_{\mathbf{C}})$. Then

$$\mathfrak{h}=i(\mathfrak{h}_{\mathbf{C}}\cap\mathfrak{k})\oplus(\mathfrak{h}_{\mathbf{C}}\cap\mathfrak{p}) \tag{2}$$

is a real form of $\mathfrak{h}_{\mathbf{C}}$ on which the roots of $\mathfrak{g}_{\mathbf{C}}$ are real-valued, and to which the restriction of B is positive-definite.

(1.2) Let $\mathbf{E}\subseteq\mathbf{R}$ or $\mathbf{E}=\mathbf{C}$ denote a field containing $\mathbf{F}$. In the latter case, we write ${}_{\mathbf{C}}\mathfrak{a}$ for $\mathfrak{h}$. We assume that ${}_{\mathbf{E}}\mathfrak{a}$ has been chosen so that ${}_{\mathbf{F}}\mathfrak{a}\subseteq{}_{\mathbf{E}}\mathfrak{a}$. Let

$${}_{\mathbf{F}}\rho_{\mathbf{E}}:{}_{\mathbf{E}}\mathfrak{a}^*\to{}_{\mathbf{F}}\mathfrak{a}^*$$

denote the restriction mapping.

Let ${}_{\mathbf{F}}\Phi\subset{}_{\mathbf{F}}\mathfrak{a}^*$ denote the system of roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to ${}_{\mathbf{F}}\mathfrak{a}$, etc. By selecting positive chambers in ${}_{\mathbf{F}}\mathfrak{a}$ and ${}_{\mathbf{E}}\mathfrak{a}$ consistently, one obtains systems of positive simple roots ${}_{\mathbf{F}}\Delta$ in ${}_{\mathbf{F}}\Phi$, ${}_{\mathbf{E}}\Delta$ in ${}_{\mathbf{E}}\Phi$, with ${}_{\mathbf{F}}\rho_{\mathbf{E}}$ inducing a mapping, denoted by the same symbol,

$${}_{\mathbf{F}}\rho_{\mathbf{E}}:{}_{\mathbf{E}}\Delta\to{}_{\mathbf{F}}\Delta\cup\{0\} \tag{1}$$

(see [5, (6.8)]). For $\beta\in{}_{\mathbf{F}}\Delta\cup\{0\}$, we put

$${}_{\mathbf{E}}\Delta^{\beta}={}_{\mathbf{F}}\rho_{\mathbf{E}}^{-1}(\beta). \tag{2}$$

A subset of ${}_{\mathbf{E}}\Delta$ will be said to be $\mathbf{F}$-*rational* whenever it is of the form

$${}_{\mathbf{F}}\rho_{\mathbf{E}}^{-1}(Y\cup\{0\})=\bigcup_{\beta\in Y\cup\{0\}}{}_{\mathbf{E}}\Delta^{\beta}. \tag{3}$$

for some $Y\subseteq{}_{\mathbf{F}}\Delta$.

(1.3) The Killing form B defines an inner product ${}_{\mathbf{E}}B$ on ${}_{\mathbf{E}}\mathfrak{a}$ for any $\mathbf{E}$. We may then identify ${}_{\mathbf{E}}\mathfrak{a}^*$ with ${}_{\mathbf{E}}\mathfrak{a}$, which in turn identifies ${}_{\mathbf{F}}\mathfrak{a}^*$ as the orthogonal complement of $(\ker {}_{\mathbf{F}}\rho_{\mathbf{E}})$ in ${}_{\mathbf{E}}\mathfrak{a}$, and ρ as orthogonal projection onto ${}_{\mathbf{F}}\mathfrak{a}^*$. With this inner product, ${}_{\mathbf{E}}\Phi$ satisfies the axioms of a root system (see [5, (2.1), (5.8)]). In particular, there is a Weyl group ${}_{\mathbf{E}}W$ acting orthogonally on $({}_{\mathbf{E}}\mathfrak{a}^*, {}_{\mathbf{E}}\Phi)$; for $\mathbf{F} \subset \mathbf{E}$, ${}_{\mathbf{F}}W$ can be seen as the set of restrictions of those elements in ${}_{\mathbf{E}}W$ that leave ${}_{\mathbf{F}}\mathfrak{a}^*$ invariant (see [5, (6.10)]).

(1.4) Let $\mathscr{G} = \mathrm{Aut}\,(\mathbf{C}/\mathbf{E})$. Then $\mathscr{G}$ acts on ${}_{\mathbf{C}}\Phi$ as follows. As $\sigma \in \mathscr{G}$ acts on both $\mathfrak{h}_{\mathbf{C}}$ and $\mathbf{C}$, one can set for $\alpha \in {}_{\mathbf{C}}\Phi$

$$\sigma(\alpha) = \sigma\alpha\sigma^{-1}. \tag{1}$$

Under the identification $\mathfrak{h}^* \simeq \mathfrak{h}$, extended linearly to the complexifications, this becomes the standard action of $\mathscr{G}$ on $\mathfrak{h}_{\mathbf{C}}$.

While $\sigma(\alpha)$ need not be in ${}_{\mathbf{C}}\Delta$ when $\alpha \in {}_{\mathbf{C}}\Delta$, it is clear that $\sigma({}_{\mathbf{C}}\Delta)$ is a base of ${}_{\mathbf{C}}\Phi$. Thus, there exists a unique $w_\sigma \in {}_{\mathbf{C}}W$ such that $w_\sigma[\sigma({}_{\mathbf{C}}\Delta)] = {}_{\mathbf{C}}\Delta$; then

$$\sigma^+(\alpha) = w_\sigma\sigma(\alpha) \tag{2}$$

defines an action of $\mathscr{G}$ on Δ.

The above actions extend linearly to $\mathfrak{h}^*$. With respect to either, ${}_{\mathbf{C}}B$ is invariant.

For $\sigma \in \mathscr{G}$, $\alpha \in {}_{\mathbf{C}}\Delta$, it is clear that

$${}_{\mathbf{E}}\rho_{\mathbf{C}}(\sigma(\alpha)) = {}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha). \tag{3}$$

It follows that if ${}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha) \neq 0$, there exist a unique $\tilde{\alpha}_\sigma \in {}_{\mathbf{C}}\Delta$ such that

$$\sigma(\alpha) = \tilde{\alpha}_\sigma + \sum_{\delta \in {}_{\mathbf{C}}\Delta^0} n_\delta \delta, \tag{4}$$

where the n_δ's are non-negative integers. In fact, $\tilde{\alpha}_\sigma = \sigma^+(\alpha)$ (cf. [5, (6.7)]), and therefore

$${}_{\mathbf{E}}\rho_{\mathbf{C}}(\sigma^+(\alpha)) = {}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha). \tag{5}$$

(1.5) We recall some basic facts about the inner products. From here on, we will write

$$\langle \alpha, \beta \rangle = {}_{\mathbf{E}}B(\alpha, \beta) \tag{1}$$

for $\alpha, \beta \in {}_{\mathbf{E}}\mathfrak{a}^*$. It follows from the basic properties of roots that for $\alpha, \alpha' \in {}_{\mathbf{E}}\Delta$,

$$\langle \alpha, \alpha' \rangle \leq 0 \quad \text{if} \quad \alpha \neq \alpha'. \tag{2}$$

Let $\hat{\mathscr{G}} \subset \mathscr{G}$ be a set of representatives for the image of $\mathscr{G}$ in the permutation group of ${}_{\mathbf{C}}\Phi$. If $\beta \in {}_{\mathbf{E}}\Delta$, and $\beta = {}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha)$ for $\alpha \in \Delta$, the identification in (1.3) gives

$$\beta = (\#\hat{\mathscr{G}})^{-1} \sum_{\sigma \in \hat{\mathscr{G}}} \sigma(\alpha), \tag{3}$$

and thus one obtains for any $h \in \mathfrak{h}^*$

$$(\#\hat{\mathscr{G}})\langle \rho(\alpha), \rho(h) \rangle = \sum_{\sigma \in \hat{\mathscr{G}}} \langle \sigma(\alpha), h \rangle. \tag{4}$$

(1.6) Let S be a subset of an inner product space. The *graph* of S consists of a vertex for each element of S; two vertices are connected by an edge if and only if the inner product of the corresponding elements is non-zero. One says that S is *connected* if its graph is connected. One can speak of the connected components of S, etc.

(1.7) For any $h \in \mathfrak{h}^*$, we have a unique expression

$$h = \sum_{\alpha \in {}_{\mathbf{C}}\Delta} c_\alpha \alpha \qquad (c_\alpha \in \mathbf{R}). \tag{1}$$

We define

$$\operatorname{supp}(h) = \{\alpha \in {}_{\mathbf{C}}\Delta : c_\alpha \neq 0\}. \tag{2}$$

Viewing $\mathbf{E} \subset \mathbf{C}$ as fixed, we put ${}_{\mathbf{C}}\Delta = \Delta$, ${}_{\mathbf{E}}\rho_{\mathbf{C}} = \rho$.

PROPOSITION. *Suppose that* $\rho(\alpha) \neq 0$, $\alpha' \in \Delta - \bigcup_{\sigma \in \hat{\mathscr{G}}} \operatorname{supp}(\sigma(\alpha) - \sigma^+(\alpha))$, *and* $\rho(\alpha') \neq \rho(\alpha)$. *Then* $\langle \rho(\alpha), \rho(\alpha') \rangle = 0$ *if and only if* α' *is orthogonal to* $\sigma^+(\alpha)$ *and* $\operatorname{supp}(\sigma(\alpha) - \sigma^+(\alpha))$ *for every* $\sigma \in \mathscr{G}$.

Proof. We may, of course, replace $\mathscr{G}$ by $\hat{\mathscr{G}}$ in the above statement. By (1.5(3)), we have

$$(\#\hat{\mathscr{G}})\langle \rho(\alpha), \rho(\alpha') \rangle = \sum_{\sigma \in \hat{\mathscr{G}}} \langle \sigma(\alpha), \alpha' \rangle.$$

Using (1.4(4)) with the definition (2), we rewrite this as

$$(\#\hat{\mathscr{G}})\langle\rho(\alpha),\rho(\alpha')\rangle=\sum_{\sigma\in\hat{\mathscr{G}}}\langle\sigma^+(\alpha),\alpha'\rangle$$
$$+\sum_{\sigma\in\hat{\mathscr{G}}}\Bigg(\sum_{\delta\in\operatorname{supp}(\sigma(\alpha)-\sigma^+(\alpha))} n_\delta^\sigma\langle\delta,\alpha'\rangle\Bigg). \tag{3}$$

With the hypotheses on α', we have by (1.5(2)) that all terms in the above sum are non-positive. Since the n_δ^σ's in (3) are all positive, it follows that $\langle\rho(\alpha),\rho(\alpha')\rangle=0$ if and only if all inner products in the right-hand side are zero.

COROLLARY. *$\bigcup_{\sigma\in\hat{\mathscr{G}}}\operatorname{supp}(\sigma(\alpha)-\sigma^+(\alpha))$ is a union of connected components of Δ^0.*

(1.8) PROPOSITION. *A component C of Δ^0 is orthogonal to all $\sigma^+(\alpha)$ if and only if C is not contained in $\bigcup_{\sigma\in\hat{\mathscr{G}}}\operatorname{supp}(\sigma(\alpha)-\sigma^+(\alpha))$.*

Proof. One direction is contained in the statement of the proposition in (1.7). For the other, suppose that for all $\beta\in C$, $\sigma\in\hat{\mathscr{G}}$, we have $\langle\sigma^+(\alpha),\beta\rangle=0$. Then as in (1.7(3)), we have for any $\beta\in C$

$$\sum_{\sigma\in\hat{\mathscr{G}}}\sum_{\delta\in\Delta^0} n_\delta^\sigma\langle\delta,\beta\rangle=(\#\hat{\mathscr{G}})\langle\rho(\alpha),\rho(\beta)\rangle=0. \tag{1}$$

If $C=\{\beta_1,\ldots,\beta_m\}$, we obtain from (1) the system of equations:

$$\sum_{j=1}^m\langle\beta_j,\beta_k\rangle\Bigg(\sum_{\sigma\in\hat{\mathscr{G}}} n_{\beta_j}^\sigma\Bigg)=0 \qquad k=1,\ldots,m. \tag{2}$$

Now, the matrix $[\langle\beta_j,\beta_k\rangle]$ is invertible. Therefore, we must have that $\sum_{\sigma\in\hat{\mathscr{G}}} n_{\beta_j}^\sigma=0$ for all j. Since the $n_{\beta_j}^\sigma$'s are non-negative, they must all equal zero.

One combines (1.7) and (1.8) to obtain

COROLLARY. *For $\alpha,\alpha'\in\Delta-\Delta^0$, $\langle\rho(\alpha),\rho(\alpha')\rangle\neq0$ if and only if for some $\sigma\in\mathscr{G}$, $\Psi\subseteq\Delta^0$, the set $\Psi\cup\{\sigma^+(\alpha),\alpha'\}$ is connected.*

(1.9) PROPOSITION. (i) *If $\Psi\subseteq\Delta$ is connected, then $\rho(\Psi)-\{0\}$ is connected.*

(ii) *If $\Theta\subseteq_{\mathbf{E}}\Delta$ is connected, then $\tilde{\Theta}=\rho^{-1}(\Theta)\cup\{\delta\in\Delta^0:\delta$ is not orthogonal to $\rho^{-1}(\Theta)\}$ has at most $(\#\hat{\mathscr{G}})$ connected components, each of which projects onto Θ.*

Proof. From the corollary in (1.8), (i) follows immediately. Suppose, on the other hand, that $\Theta \subseteq {}_{\mathbf{E}}\Delta$ is connected, and let $\tilde{\Theta}$ be as in (ii). Since $\mathscr{G}$ acts orthogonally on Δ (1.4), we see that $\mathscr{G}$ permutes the connected components of $\tilde{\Theta}$. It follows from the corollary in (1.8) that each component contains at least one element from each $\mathscr{G}$-orbit in $\rho^{-1}(\Theta)$. Since $\tilde{\Theta}$ has at most as many components as does $\rho^{-1}(\Theta)$, we obtain the first assertion of (ii). By (1.4(5)), all components are seen to have the same image in ${}_{\mathbf{E}}\Delta$, namely Θ.

COROLLARY 1. *$\Theta \subseteq {}_{\mathbf{E}}\Delta$ is connected if and only if there is a connected subset $\Psi \subseteq \Delta$ with $\rho(\Psi) - \{0\} = \Theta$.*

COROLLARY 2. *Let $\mathbf{F} \subset \mathbf{E} \subseteq \mathbf{R}$. Then $Y \subseteq {}_{\mathbf{F}}\Delta$ is connected if and only if there is a connected subset $\Theta \subseteq {}_{\mathbf{E}}\Delta$ with ${}_{\mathbf{F}}\rho_{\mathbf{E}}(\Theta) - \{0\} = Y$.*

(*Proof.* Apply Corollary 1 twice, once with $\mathbf{E}$ replaced by $\mathbf{F}$, and recall that ${}_{\mathbf{F}}\rho_{\mathbf{C}} = {}_{\mathbf{F}}\rho_{\mathbf{E}} \circ {}_{\mathbf{E}}\rho_{\mathbf{C}}$.)

2. Satake's compactifications of G/K

(2.1) Let G be a semi-simple real algebraic group, K a maximal compact subgroup of G, and $X = G/K$ the associated symmetric space.

Let $\tau: G \to SL(V)$ be a finite-dimensional representation of G, with finite kernel. There exists an *admissible inner product* on V, which means that

$$\tau(g)\tau(\theta(g))^* = I. \tag{1}$$

where θ denotes the Cartan involution of G with respect to K, and * denotes adjoint with respect to the inner product. It follows that the mapping

$$\tau_0(g) = \tau(g)\tau(g)^* \tag{2}$$

descends to X, and has values in the space $S(V)$ of self-adjoint endomorphisms of V. By taking the quotient by the action of the scalars, one obtains a mapping, which we also denote τ_0

$$\tau_0: X \to \mathbf{P}S(V), \tag{3}$$

which is easily seen to be an embedding. It is G-equivariant with respect to the natural action on X and the projectivization of the action

$$g \cdot M = gMg^* \qquad M \in S(V). \tag{4}$$

The *Satake compactification* determined by τ is the closure of the image of τ_0:

$$X^*_\tau = \overline{\tau_0(X)}. \tag{5}$$

If τ is fixed, we write X^* instead of X^*_τ, the dependence on τ being understood.

(2.2) We prefer an intrinsic description of the topological structure of X^*. From now on, we will assume that τ is irreducible.

To facilitate the discussion to come in §3, we suppose that G is defined over $\mathbf{F} \subseteq \mathbf{R}$. Let $A = {}_{\mathbf{F}}A$ and $\mathfrak{a} = {}_{\mathbf{F}}\mathfrak{a}$ be as in (1.1). Then V has an orthogonal weight-space decomposition with respect to $\mathfrak{a}$:

$$V = \bigoplus V_\mu, \tag{1}$$

where V_μ is the subspace of V on which $\mathfrak{a}$ acts with weight μ.[3] With respect to a basis of weight vectors, τ, and therefore also τ_0, maps A to diagonal matrices. Thus, we see that points in the boundary of X^* which lie in the closure of $\tau_0(A)$ are determined by the behavior of weights, as we shall next describe.

(2.3) The simple roots ${}_{\mathbf{F}}\Delta$ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ define characters on A, whose values are denoted a^α ($a \in A$, $\alpha \in {}_{\mathbf{F}}\Delta$). From these, one obtains a canonical isomorphism

$$\iota : A \to (\mathbf{R}^+)^{{}_{\mathbf{F}}\Delta}, \tag{1}$$

where $\mathbf{R}^+$ denotes the interval $(0, \infty)$. By adjoining $\{\infty\}$ to each factor of $\mathbf{R}^+$, one gets via ι a partial compactification $\bar{A}$ of A.[4]

One puts for $Y \subseteq {}_{\mathbf{F}}\Delta$

$$A_Y = \bigcap_{\beta \in Y} (\ker \beta) \tag{2}$$

(so $A = A_\varnothing$), and for $Y \subseteq \Theta \subseteq {}_{\mathbf{F}}\Delta$

$$A_{Y,\Theta} = A_Y \cap \bigcap_{\beta \notin \Theta} (\ker \beta), \tag{3}$$

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[3] These weights are, in fact, real.

[4] In [4], $\{0\}$ is adjoined to $\mathbf{R}^+$, but G is acting there on the right. Our adjoining $\{\infty\}$ is consistent with [8], and also with current convention, where G is also assumed to act on the left.

so that

$$A_Y = A_{Y,\Theta} \times A_\Theta \quad \text{if} \quad Y \subseteq \Theta. \tag{4}$$

We then have an A-orbit space decomposition

$$\bar{A} = \coprod_{Y \subseteq {}_{\mathbf{F}}\Delta} A'_Y, \tag{5}$$

where

$$A'_Y = \{a \in \bar{A} : a^\beta \neq \infty \text{ if and only if } \beta \in Y\}. \tag{6}$$

The closure $\bar{A}'_Y$ of A'_Y in $\bar{A}$ can be written as

$$\bar{A}'_Y = \coprod_{\Psi \subseteq Y} A'_\Psi. \tag{7}$$

For $Y \subseteq \Theta$, there is an obvious projection

$$p_{\Theta,Y} : A'_\Theta \to A'_Y, \tag{8}$$

determined by setting the characters in $\Theta - Y$ to infinity. This mapping extends continuously to yield

$$\bar{p}_{\Theta,Y} : \bar{A}'_\Theta \to \bar{A}'_Y, \tag{9}$$

with

$$p_{\Theta,Y}|_{A'_\Psi} = p_{\Psi,Y\cap\Psi} \quad \text{if} \quad \Psi \subseteq \Theta. \tag{10}$$

Finally, we remark that as the A-orbit of

$$1_Y = p_{{}_{\mathbf{F}}\Delta,Y}(1), \tag{11}$$

we identify

$$A'_Y = A/A_Y. \tag{12}$$

(2.4) Let $\mu_0 = {}_{\mathbf{F}}\rho_{\mathbf{C}}(\lambda_0)$, where $\lambda_0 \in \mathscr{h}^*$ is the highest weight of τ (relative to Δ), with notation as in (1.1) and (1.2). All weights of τ with respect to ${}_{\mathbf{F}}a$ (cf. (2.2(1))) are of the form

$$\mu = \mu_0 - \sum_{\beta \in {}_{\mathbf{F}}\Delta} m_\beta \beta, \tag{1}$$

where the m_β's are non-negative integers.

As in (1.7(2)), one can define for $\nu \in \mathfrak{a}^*$ the subset $\operatorname{supp}(\nu)$ of ${}_{\mathbf{F}}\Delta$.

PROPOSITION [5, (12.16)]. *Let $Y \subseteq {}_{\mathbf{F}}\Delta$. Then Y equals* $\operatorname{supp}(\mu_0 - \mu)$ *for some weight μ of τ with respect to ${}_{\mathbf{F}}\mathfrak{a}$ if and only if $Y \cup \{\mu_0\}$ is connected.*

In view of the above, the following definition is warranted. One says that a subset Y of ${}_{\mathbf{F}}\Delta$ is *τ-connected* (or *τ-open* [8]) if $Y \cup \{\mu_0\}$ is connected.

COROLLARY. *Let $\mathbf{F} \subset \mathbf{E}$. Then $Y \subseteq {}_{\mathbf{F}}\Delta$ is τ-connected if and only if there is a τ-connected subset $\Theta \subseteq {}_{\mathbf{E}}\Delta$ with ${}_{\mathbf{F}}\rho_{\mathbf{E}}(\Theta) - \{0\} = Y$.*

Remark. An alternate treatment of the proposition, along the lines of [8, (2.3)], can be carried out, by the use of the following analogue of the corollary in our (1.8). If $\beta = {}_{\mathbf{F}}\rho_{\mathbf{C}}(\alpha) \in {}_{\mathbf{F}}\Delta$, then $\langle \beta, \mu_0 \rangle \neq 0$ if and only if for some $\sigma \in \operatorname{Aut}(\mathbf{C}/\mathbf{F})$ and $\Psi \subseteq \Delta^0$, the set $\Psi \cup \{\sigma^+(\alpha), \lambda_0\}$ is connected.

(2.5) One should always keep in mind that (if we divide out the common factor of $a^{2\mu_0}$) $\tau_0(a)|_{V_\mu}$ is equal to the scalar $a^{-2\sum m_\beta \beta}$. Thus, as $a \in A$ approaches a point in A'_Y, $\tau_0(a)$ degenerates to zero on the weight-space V_μ if and only if $\operatorname{supp}(\mu_0 - \mu) \not\subseteq Y$.

Every subset Y of ${}_{\mathbf{F}}\Delta$ contains a largest τ-connected subset, which we call its *τ-connected component.* The assignment of τ-connected components defines a mapping

$$\kappa : 2^{({}_{\mathbf{F}}\Delta)} \to 2^{({}_{\mathbf{F}}\Delta)}. \tag{1}$$

Given $\Theta \subseteq {}_{\mathbf{F}}\Delta$, one puts

$$\omega(\Theta) = \Theta \cup \Theta', \tag{2}$$

where

$$\Theta' = \{\beta \in {}_{\mathbf{F}}\Delta : \beta \text{ is orthogonal to } \Theta \cup \{\mu_0\}\}. \tag{3}$$

The following is immediate:

LEMMA. (i) *Let Y be a τ-connected subset of ${}_{\mathbf{F}}\Delta$. Then $\kappa(\Theta)=Y$ if and only if $Y\subseteq\Theta\subseteq\omega(Y)$.*

(ii) *If μ is a weight of τ with respect to ${}_{\mathbf{F}}\mathfrak{a}$, then* $\operatorname{supp}(\mu_0-\mu)\subseteq\Theta$ *if and only if* $\operatorname{supp}(\mu_0-\mu)\subseteq\kappa(\Theta)$.

Thus, the behavior of $\tau_0(a)$ as a approaches A'_Θ is determined by the projection $p_{\Theta,\kappa(\Theta)}$ of the limit.

(2.6) The preceding observation suggests that we define the following A-equivariant quotient A^* of $\bar{A}$. As underlying set, we take

$$A^*=\coprod_{Y\ \tau\text{-connected}} A'_Y. \tag{1}$$

There is a surjective mapping $p:\bar{A}\to A^*$, defined by

$$p|_{A'_\Theta}=p_{\Theta,\kappa(\Theta)}. \tag{2}$$

We will show that A^*, equipped with the quotient topology induced by p, is a Hausdorff space.

If $a\in\bar{A}$ and $p(a)$ is in A'_Θ, we set

$$Y(a)=\Theta; \tag{3}$$

that is, $Y(a)$ is the τ-connected component of

$$\{\alpha\in{}_{\mathbf{F}}\Delta : a^\alpha<\infty\}.$$

The equivalence relation on $\bar{A}$ induced by p can then be described as:

a and b are identified if and only if

$$Y(a)=Y(b) \text{ and } a^\alpha=b^\alpha \text{ whenever } \alpha\in Y(a). \tag{4}$$

Consider a point $a_0\in A'_\Theta$. Then $a_0^\alpha<\infty$ if and only if α is in the τ-connected subset Θ of ${}_{\mathbf{F}}\Delta$. For each α, let J_α denote a neighborhood of a_0^α in $\mathbf{R}^+\cup\{\infty\}$. Then

$$\{a\in\bar{A}: a^\alpha\in J_\alpha \text{ if } \Theta\cup\{\alpha\} \text{ is } \tau\text{-connected}\} \tag{5}$$

is an open neighborhood of $p^{-1}(a_0)$ in $\bar{A}$. As it is a union of fibers of p, as follows from (4), it projects onto a neighborhood of a_0 in A^*.

Let $b_0 \in A'_\Psi \subset A^*$, with $b_0 \neq a_0$. If $\Theta = \Psi$, then it is clear from (5) that we can find disjoint neighborhoods of a_0 and b_0, since these elements are distinguished by α for some $\alpha \in \Theta$. Otherwise, if $\Psi - \Theta \neq \varnothing$ (as we may assume without loss of generality), there must exist $\alpha \in \Psi - \Theta$ with $\Theta \cup \{\alpha\}$ τ-connected. As $\Psi \cup \{\alpha\} = \Psi$ is τ-connected and α distinguishes a_0 and b_0, we can likewise find disjoint neighborhoods of the form (5). Thus, A^* is Hausdorff.

We remark that p is almost never an open mapping.

The closing remark of (2.5) implies:

PROPOSITION. *The embedding of A by τ_0 in $\mathbf{PS}(V)$ extends to a continuous mapping τ_0^* of A^*.*

In fact, we will later see that τ_0^* is an embedding.

(2.7) For any $t \in A$, we put

$$A(t) = \{a \in A : a^\beta \geq t^\beta \text{ for all } \beta \in {}_{\mathbf{F}}\Delta\}. \tag{1}$$

We let $\bar{A}(t)$ (resp. $A^*(t)$) denote the closure of $A(t)$ in $\bar{A}$ (resp. A^*).

PROPOSITION. *Take $\mathbf{F} = \mathbf{R}$. Then there is a continuous mapping*

$$\phi : G \times A^* \to X^*, \tag{2}$$

defined by $\phi(g, a) = g\tau_0^*(a)g^*$, whose restriction to $K \times A^*(1)$ is already surjective.

Proof. This follows from the proposition of (2.6) and the fact that $KA(1)K = G$.

(2.8) In order to describe the structure of the mapping ϕ (2.7(2)), it is necessary to discuss the role of the parabolic subgroups of G. We continue to assume that $\mathbf{F} = \mathbf{R}$.

To each subset Θ of ${}_{\mathbf{R}}\Delta$ is associated a "standard" parabolic subgroup Q_Θ of G, whose definition we recall. Let $\mathfrak{g}_\alpha$ denote the α root space of $\mathfrak{g}$, if $\alpha \in {}_{\mathbf{R}}\Phi$. Then

$$Q_\Theta = M_\Theta A_\Theta U_\Theta \tag{1}$$

is the connected algebraic subgroup of G with (correspondingly decomposed) Lie algebra

$$\mathfrak{q}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{u}_\Theta, \tag{2}$$

where $\mathfrak{a}_\Theta$ is the Lie algebra of A_Θ (2.3(2)); $\mathfrak{m}_\Theta$ is the sum of the (semi-simple) Lie algebra ℓ_Θ generated by $\{\mathfrak{g}_\beta, \mathfrak{g}_{-\beta} : \beta \in \Theta\}$ and a sub-algebra of $\mathfrak{k}$, which coincides with the orthogonal complement of $\mathfrak{a}_\Theta$ in the centralizer of $\mathfrak{a}_\Theta$; and

$$\mathfrak{u}_\Theta = \bigoplus_{0<\alpha\notin \mathrm{Span}(\Theta)} \mathfrak{g}_\alpha$$

is the nilpotent radical of $\mathfrak{q}_\Theta$. Every parabolic subgroup of G is conjugate, by an element of G, to a unique Q_Θ (see [5, (4.6), (4.13c)]).

(2.9) Let $K_\Theta = K \cap M_\Theta$; put

$$X_\Theta = M_\Theta / K_\Theta, \tag{1}$$

a symmetric space of rank $(\#\Theta)$. If Θ is τ-connected, then X_Θ is embedded in X^* as follows.

Let

$$V_\Theta = \bigoplus_{\mathrm{supp}(\mu_0-\mu)\subseteq\Theta} V_\mu. \tag{2}$$

It is evident that V_Θ is stable under $\tau(M_\Theta)$. In other words, τ determines a representation τ_Θ of M_Θ on V_Θ. It can be seen [8, (2.4)] that τ_Θ has finite kernel, and that all of its irreducible constituents are equivalent. One gets, as in (2.1(3)), an embedding

$$(\tau_\Theta)_0 : X_\Theta \to \mathbf{P}S(V_\Theta). \tag{3}$$

We identify $S(V_\Theta)$ as the linear subspace of $S(V)$ given by transformations which are zero on the weight spaces complementary to V_Θ, and thereby regard $(\tau_\Theta)_0$ as having its image in $\mathbf{P}S(V)$.

We observe that M_Θ contains the subgroup

$$\tilde{A}_{{}_\blacksquare\Delta,\Theta} = M_\Theta \cap A, \tag{4}$$

whose Lie algebra is spanned by the elements of Θ, viewed as elements of $\mathfrak{a}$ via the identification of (1.3). Since $\tilde{A}_{{}_\blacksquare\Delta,\Theta}$, as does $A_{{}_\blacksquare\Delta,\Theta}$, projects isomorphically onto A/A_Θ, we can see from the descriptions of τ_0^* and $(\tau_\Theta)_0$ that

$$\tau_0^*(A'_\Theta) = (\tau_\Theta)_0(\tilde{A}_{_{\mathbf{R}}\Delta,\Theta}) \tag{5}$$

canonically. In particular, τ_0^* is an embedding of A^*; moreover, X_Θ is embedded in X^* as the M_Θ-orbit of $\tau_0^*(1_\Theta)$.

(2.10) The G-translates of the X_Θ's are called the *boundary components* of X^*. It is clear from (2.7) that X^* is the union of its boundary components (note that $X = X_{_{\mathbf{R}}\Delta}$). The best way to index them is by means of their normalizers. The normalizer of X_Θ in G is equal to the normalizer of V_Θ under τ, namely the parabolic subgroup $Q_{\omega(\Theta)}$; thus the normalizer of gX_Θ is the conjugate ${}^gQ_{\omega(\Theta)} := gQ_{\omega(\Theta)}g^{-1}$.

The preceding discussion makes it apparent that as a topological space, X^* depends on τ only to the extent that A^* does, namely on the collection of τ-connected subsets of ${}_{\mathbf{R}}\Delta$. Thus, there are only finitely many topologically distinct Satake compactifications of X.

The preceding can be reformulated as follows. Let $\xi_1, \ldots, \xi_r$ be the fundamental dominant weights of ${}_{\mathbf{R}}\Delta$; i.e., if ${}_{\mathbf{R}}\Delta = \{\alpha_1, \ldots, \alpha_r\}$, then $\{\xi_1, \ldots, \xi_r\}$ is the dual basis of ${}_{\mathbf{R}}a^*$. The restricted highest weight μ_0 of τ can be written

$$\mu_0 = \sum_{j=1}^{r} c_j \xi_j, \tag{1}$$

where the c_j's are non-negative integers. We put

$$\operatorname{supp}^*(\mu_0) = \{\xi_j : c_j \neq 0\}. \tag{2}$$

Equivalently,

$$\xi_j \in \operatorname{supp}^*(\mu_0) \quad \text{if and only if} \quad \langle \alpha_j, \mu_0 \rangle \neq 0. \tag{3}$$

Then we have asserted:

PROPOSITION. *Let μ_0, μ'_0 be the respective restricted highest weights of τ, τ'. If* $\operatorname{supp}^*(\mu_0) = \operatorname{supp}^*(\mu'_0)$, *then the identity mapping of X extends to a homeomorphism of X^*_τ and $X^*_{\tau'}$.*

(2.11) Suppose now that for τ, τ' we have only that $\operatorname{supp}^*(\mu'_0) \subset \operatorname{supp}^*(\mu_0)$. Let Θ be a τ'-connected subset of ${}_{\mathbf{R}}\Delta$. This is equivalent to asserting that every connected component of Θ is not orthogonal to $\mathrm{supp}(\mu_0')$, so Θ is also τ-connected. Thus, for any Θ,

$$\kappa_{\tau'}(\Theta) \subseteq \kappa_\tau(\Theta). \tag{1}$$

It follows that there is an A-equivariant continuous surjection

$$\psi_{\tau,\tau'}: A_\tau^* \to A_{\tau'}^*. \tag{2}$$

PROPOSITION. *The mapping*

$$1 \times \psi_{\tau,\tau'}: K \times A_\tau^* \to K \times A_{\tau'}^*$$

induces a continuous surjection

$$f_{\tau,\tau'}: X_\tau^* \to X_{\tau'}^*.$$

Proof. Since X_τ^* (resp. $X_{\tau'}^*$) is a quotient of the domain (resp. range) of $1 \times \psi_{\tau,\tau'}$, and these are compact spaces, it suffices to show that points identified under ϕ_τ (2.7) are mapped by $1 \times \psi_{\tau,\tau'}$ to points identified under $\phi_{\tau'}$. We should be explicit about these identifications.

(2.12) Throughout this paragraph, we assume that Ξ and Θ are subsets of ${}_{\mathbf{R}}\Delta$ which satisfy $\kappa_\tau(\Xi) = \Theta$. Let I_Θ denote the isometry group of X_Θ. As Q_Ξ normalizes X_Θ, there is a continuous homomorphism

$$h_\Xi : Q_\Xi \to I_\Theta, \tag{1}$$

such that for $\Xi' \subset \Xi$ one has

$$h_\Xi|_{Q_{\Xi'}} = h_{\Xi'}. \tag{2}$$

It is a tautology that $Q_{\omega_\tau(\Theta)}$ acts on X_Θ via $h_{\omega_\tau(\Theta)}$.

Put $K'_\Theta = h_\Theta(K_\Theta)$, the "distinguished" maximal compact subgroup of I_Θ. Since $K_\Xi \supseteq K_\Theta$ and $h_\Xi(K_\Xi)$ is compact, it follows that

$$h_\Xi(K_\Xi) = K'_\Theta. \tag{3}$$

LEMMA (cf. [8, (4.4)]). *Under ϕ_τ, one has the identification* $(k_1, a_1^*) \sim (k_2, a_2^*)$

if and only if the following three statements hold:

(i) a_1^* *and* a_2^* *are in the same* A*-orbit (call it* A'_Θ, *where* Θ *is* τ*-connected) in* A_τ^*,

(ii) $k_2^{-1}k_1 \in K_{\omega_\tau(\Theta)}$,

(iii) *with* a_1^*, a_2^* *regarded as elements of* $\tilde{A}_{\mathbf{R}\Delta,\Theta}$, $(k_2^{-1}k_1)\cdot a_1^* K_\Theta = a_2^* K_\Theta$.

Since $h_\Theta | \tilde{A}_{\mathbf{R}\Delta,\Theta}$ is one-to-one, we may assume without loss of generality that $I_\Theta = M_\Theta$. Then, condition (iii) of the above lemma can be rewritten as

$$\operatorname{Int}[h_{\omega_\tau(\Theta)}(k_2^{-1}k_1)]a_1^* = a_2^*, \tag{4}$$

where Int denotes the action of K_Θ on $\tilde{A}_{\mathbf{R}\Delta,\Theta}$ by inner automorphisms. Also, K_Θ centralizes A_Θ, so the preceding action of K_Θ is canonically isomorphic to that on A/A_Θ.

(2.13) We return to complete the proof of the proposition in (2.11). Suppose that Θ is τ-connected. Then

$$\Psi := \kappa_{\tau'}(\Theta) \subseteq \Theta. \tag{1}$$

As Ψ is the τ'-connected component of Θ, it follows that

$$\Theta - \Psi \subseteq \Psi'_{\tau'} \tag{2}$$

(Ψ' as in (2.5(3))); moreover, it is evident that

$$\Theta'_\tau \subseteq \Theta'_{\tau'} \subseteq \Psi'_{\tau'}. \tag{3}$$

Therefore, we conclude that

$$\omega_\tau(\Theta) \subseteq \omega_{\tau'}(\Psi), \tag{4}$$

so also

$$Q_{\omega_\tau(\Theta)} \subseteq Q_{\omega_{\tau'}(\Psi)}. \tag{5}$$

In particular,

$$K_{\omega_\tau(\Theta)} \subseteq K_{\omega_{\tau'}(\Psi)}. \tag{6}$$

That $f_{\tau,\tau'}$ exists now follows from (6), (2.12(2), (4)), and G-equivariance.

3. Quotients by arithmetic groups

Throughout this section, we assume that G is defined over $\mathbf{Q}$.

(3.1) We choose the maximal compact subgroup K and the maximal split tori of (1.1), for $\mathbf{F}=\mathbf{Q}, \mathbf{R}$, so that ${}_{\mathbf{Q}}A \subseteq {}_{\mathbf{R}}A$; of course we now retain the subscripts for distinction. Superseding the convention of (1.7), we put $\rho = {}_{\mathbf{Q}}\rho_{\mathbf{R}}$. *To simplify the notation, if $Y \subseteq {}_{\mathbf{Q}}\Delta$, we denote the $\mathbf{Q}$-rational subset $\rho^{-1}(Y \cup \{0\})$ of ${}_{\mathbf{R}}\Delta$ by $\tilde{Y}$; if $\Theta \subseteq {}_{\mathbf{R}}\Delta$, we put $\hat{\Theta} = \rho(\Theta) - \{0\} \subseteq {}_{\mathbf{Q}}\Delta$.* We observe that $(\tilde{Y})\hat{} = Y$, and that $\tilde{\hat{\Theta}}$ is the smallest $\mathbf{Q}$-rational subset of ${}_{\mathbf{R}}\Delta$ which contains Θ. Also,

$$ {}_{\mathbf{Q}}A_Y = {}_{\mathbf{R}}A_\Theta \cap {}_{\mathbf{Q}}A \quad \text{whenever} \quad \hat{\Theta} = Y. \tag{1}$$

Let $\Theta \subseteq {}_{\mathbf{R}}\Delta$. The standard parabolic subgroup Q_Θ is defined over $\mathbf{Q}$ if and only if Θ is $\mathbf{Q}$-rational [5, (6.3)]. In this case, $\Theta = \tilde{Y}$ for some $Y \subseteq {}_{\mathbf{Q}}\Delta$, and we put

$$ {}_{\mathbf{Q}}Q_Y = Q_\Theta. \tag{2}$$

One redecomposes Q_Θ "rationally":

$$ {}_{\mathbf{Q}}Q_Y = ({}_{\mathbf{Q}}M_Y)({}_{\mathbf{Q}}A_Y)({}_{\mathbf{Q}}U_Y), \tag{3}$$

where: ${}_{\mathbf{Q}}U_Y = U_\Theta$; ${}_{\mathbf{Q}}A_Y$ is as in (2.3(2)); the Lie algebra ${}_{\mathbf{Q}}\mathfrak{m}_Y$ of ${}_{\mathbf{Q}}M_Y$ is orthogonal to the Lie algebra ${}_{\mathbf{Q}}\mathfrak{a}_Y$ of ${}_{\mathbf{Q}}A_Y$, $({}_{\mathbf{Q}}M_Y)({}_{\mathbf{Q}}A_Y) = M_\Theta A_\Theta$ is the Levi subgroup of Q_Θ stable under the Cartan involution of G that fixes K, and if ${}_{\mathbf{R}}\mathfrak{a}$ (or equivalently, the corresponding torus ${}_{\mathbf{R}}T$ of G) is defined over $\mathbf{Q}$, then ${}_{\mathbf{Q}}M_Y$ is isogenous to the product of M_Θ and (the real points of) the maximal $\mathbf{Q}$-anisotropic sub-torus of ${}_{\mathbf{R}}T$.

In what follows, the representation τ, or more properly its equivalence class in the sense of (2.10), used in constructing X^* shall be considered fixed. We also write κ and ω (see (2.5(1), (2))) for subsets of ${}_{\mathbf{Q}}\Delta$, ${}_{\mathbf{R}}\Delta$ and ${}_{\mathbf{C}}\Delta$, since it is always clear which root system one is discussing.

(3.2) Let X_Θ ($\Theta \subseteq {}_{\mathbf{R}}\Delta$ τ-connected) be one of the "standard" boundary components (2.9) of X^*. As was observed in (2.10), the normalizer N_Θ in G of X_Θ is $Q_{\omega(\Theta)}$. One can likewise identify the centralizer Z_Θ of X_Θ as

$$ Z_\Theta = \{h \in N_\Theta : \tau_\Theta(h) \text{ is a multiple of } I\}, \tag{1}$$

where I denotes the identity transformation of V_Θ. Then

$$G_\Theta = N_\Theta / Z_\Theta \tag{2}$$

is an algebraic group over $\mathbf{R}$, with Lie algebra isomorphic to ℓ_Θ.

For a general boundary component gX_Θ, the preceding discussion can be repeated if only we conjugate all groups above by g.

DEFINITION (cf. [2, (3.5)]). *One says that a boundary component is rational if*

(i) *its normalizer N is defined over $\mathbf{Q}$, and*

(ii) *its centralizer Z contains a normal subgroup Z' of N that is defined over $\mathbf{Q}$, such that Z/Z' is compact.*

What (ii) above asserts is that N/Z is the quotient of an algebraic group over $\mathbf{Q}$ by a normal compact subgroup.

(3.3) For X_Θ, the condition (i) in the definition in (3.2) requires that $\omega(\Theta)$ be a $\mathbf{Q}$-rational subset of ${}_{\mathbf{R}}\Delta$. We will impose the following *quasi-rationality* hypothesis on τ:

ASSUMPTION 1. If Y is a τ-connected subset of ${}_{\mathbf{Q}}\Delta$, then the τ-connected component of $\tilde{Y}$ contains $\rho^{-1}(Y)$.

In case G is split over $\mathbf{R}$ (or in general, if we replace ${}_{\mathbf{R}}\Delta$ by ${}_{\mathbf{C}}\Delta$), the above condition is equivalent to the assertion that $\kappa(\tilde{Y})$ be invariant under the action (1.4(2)) of $\mathscr{G} = \mathrm{Aut}\,(\mathbf{C}/\mathbf{Q})$, since this action is transitive on the fibers of ρ other than ${}_{\mathbf{C}}\Delta^0$ [5, (6.4(2))]. If τ is *projectively rational* over $\mathbf{Q}$ in the sense of [5, (12.3a)], e.g. if τ is defined over $\mathbf{Q}$, then Assumption 1 holds, for its highest weight λ_0 is $\mathscr{G}$-invariant [5, (12.6)].

PROPOSITION. (i) *If $\Theta \subseteq {}_{\mathbf{R}}\Delta$ is τ-connected, and $\omega(\Theta)$ is $\mathbf{Q}$-rational, then $\Theta = \kappa(\widetilde{\Theta^\wedge})$.*

(ii) *Suppose that τ satisfies Assumption 1. If Θ is the τ-connected component of $\widetilde{\Theta^\wedge}$, then $\omega(\Theta)$ is $\mathbf{Q}$-rational, and $\omega(\widetilde{\Theta^\wedge}) = \omega(\Theta)$.*

Proof. We set $Y = \Theta^\wedge$, and let $\Psi = \kappa(\tilde{Y})$. By construction, $\Theta \subseteq \Psi$. If $\Theta \neq \Psi$, $\tilde{Y} \subseteq \omega(\Theta)$ is impossible, so we have (i). It is clear (cf. (2.13)) that we always have

$$\omega(\Psi) \supseteq \widetilde{\omega(Y)}.$$

If equality failed to hold, there would exist $\beta \in \omega(\Psi)$ with

$$0 \neq \rho(\beta) \nsubseteq \omega(Y),$$

and then $Y \cup \{\rho(\beta)\}$ is τ-connected. Thus, by the corollary of (2.4), there is $\beta' \in {}_{\mathbf{R}}\Delta$ with $\rho(\beta') = \rho(\beta)$ such that $\Psi \cup \{\beta'\}$ is τ-connected. Under Assumption 1,

$$\Psi = \kappa(\tilde{Y}) \supseteq \rho^{-1}(Y),$$

so $\Psi \cup \{\beta\}$ is also τ-connected (see Remark of (2.4)), a contradiction. Thus

$$\omega(\Psi) = \widetilde{\omega(Y)}.$$

We therefore obtain:

COROLLARY. *When Assumption* 1 *holds, the boundary components of* X^* *which satisfy condition* (i) *in the definition in* (3.2) *are precisely those of the form* gX_Θ, *where* $g \in G_{\mathbf{Q}}$ *and* $\Theta = \kappa(\tilde{Y})$ *for some* τ*-connected* $Y \subseteq {}_{\mathbf{Q}}\Delta$.

(3.4) It is sometimes the case that for a given representation τ, condition (ii) in the definition in (3.2) is a consequence of (i). For our construction, *we will assume that* τ *is as such* (*Assumption* 2). In other words, if τ satisfies Assumption 2, then the corollary of (3.3) describes the set of rational boundary components. *Whenever we need to be explicit, we will for the sake of simplicity assume that a boundary component is standard; the general case is covered by acting by* $G_{\mathbf{Q}}$, *or equivalently, by making a different choice of* ${}_{\mathbf{Q}}A$.

We now show:

PROPOSITION (cf. [3, (4.3)]). *If* τ *is defined over* $\mathbf{Q}$, *then* Z_Θ (3.2(1)) *is defined over* $\mathbf{Q}$ *whenever* $\omega(\Theta)$ *is* $\mathbf{Q}$*-rational, and thus Assumption* 2 *is satisfied.*

Proof. It is enough to show that the subspaces V_Θ (2.9(2)), with Θ as in (i) of the proposition of (3.3), of V are defined over $\mathbf{Q}$. For that, it suffices to show that if λ is a weight of V with respect to ${}_{\mathbf{C}}\mathfrak{a}$, and supp $(\lambda - \lambda_0)$ – which is τ-connected by (2.4) – is contained in

$$\Psi = \kappa({}_{\mathbf{R}}\rho_{\mathbf{C}}^{-1}(\Theta \cup \{0\})), \tag{1}$$

then so is supp $(\sigma(\lambda) - \lambda_0)$ for all $\sigma \in \mathscr{G}$. As $\sigma^+(\lambda_0) = \lambda_0$, we write

$$\sigma(\lambda) - \lambda_0 = \sigma^+(\lambda - \lambda_0) + (\sigma(\lambda) - \sigma^+(\lambda)) \tag{2}$$

From this and (1.4(4)), we see that

$$\operatorname{supp}(\sigma(\lambda)-\lambda_0)\subseteq \sigma^+(\Psi)\cup {}_{\mathbf{Q}}\rho_{\mathbf{C}}^{-1}(0).$$

and hence, by the proposition in (2.4),

$$\operatorname{supp}(\sigma(\lambda)-\lambda_0)\subseteq \kappa[\sigma^+(\Psi)\cup {}_{\mathbf{Q}}\rho_{\mathbf{C}}^{-1}(0)]. \tag{3}$$

In view of the corollary in (2.4), if $\Theta=\kappa(\tilde{Y})$ we can rewrite (1) as

$$\Psi=\kappa({}_{\mathbf{Q}}\rho_{\mathbf{C}}^{-1}(Y\cup\{0\})).$$

Since λ_0 is invariant under $\mathscr{G}$, we see that Ψ, as the τ-connected component of a $\mathscr{G}$-stable set, is $\mathscr{G}$-stable. It follows that (3) gives

$$\operatorname{supp}(\sigma(\lambda)-\lambda_0)\subseteq \Psi,$$

as desired.

Remark. If ${}_{\mathbf{Q}}A={}_{\mathbf{R}}A$, then Assumption 2 is satisfied for any τ, for one knows that the Lie algebra of the isometry group of X_Θ is defined over $\mathbf{Q}$ (see (3.8)). Of course, Assumption 1 is also satisfied, for trivial reasons.

We will not address the issue of determining for which τ Assumption 2 holds (see [2, §3] for some discussion). The importance of this assumption is that it permits a nice description of the set of rational boundary components. The case studied in [2] is the only general instance we know in which Assumption 2 is satisfied, beyond those described in the above proposition; it would be nice to have a reasonable, more general representation-theoretic condition which guarantees it.

(3.5) Let ${}_{\mathbf{Q}}X^*$ denote the union of all rational boundary components of X^*. It is best to regard ${}_{\mathbf{Q}}X^*$ at this point as only a set, i.e. without a topology. We will eventually define a natural surjective mapping of the manifold with corners $\bar{X}$ constructed by Borel and Serre in [4] onto ${}_{\mathbf{Q}}X^*$.

We recall the definition of $\bar{X}$. The torus ${}_{\mathbf{Q}}A$, or more precisely a certain isomorphic image, operates on X via the so-called *geodesic action* [4, §3]. For a $\mathbf{Q}$-parabolic subgroup $P={}_{\mathbf{Q}}Q_Y$ of G, one defines the *corner* $X(P)$ [4, §5]:

$$X({}_{\mathbf{Q}}Q_Y)={}_{\mathbf{Q}}\bar{A}_Y\times^{{}_{\mathbf{Q}}A_Y}X, \tag{1}$$

where ${}_{\mathbf{Q}}\bar{A}_Y$ is the closure of ${}_{\mathbf{Q}}A_Y$ in ${}_{\mathbf{Q}}\bar{A}$. We should mention that $X(P)$ is intrinsic to P – if $P={}_{\mathbf{Q}}Q_Y$, ${}_{\mathbf{Q}}A_Y$ is a lifting of the maximal $\mathbf{Q}$-split torus of the center of ${}_{\mathbf{Q}}Q_Y/{}_{\mathbf{Q}}U_Y$. Letting

$$e({}_{\mathbf{Q}}Q_Y)={}_{\mathbf{Q}}A_Y\backslash X, \tag{2}$$

one has for any $\mathbf{Q}$-parabolic subgroup P

$$X(P)=\coprod_{\substack{Q\ \mathbf{Q}\text{-parabolic}\\ Q\supseteq P}} e(Q) \tag{3}$$

in a natural way [4, (5.1)]. Moreover, if $P'\subset P$, there is a natural embedding of $X(P)$ in $X(P')$ as an open subset. One then defines

$$\bar{X}=\bigcup X(P)=\coprod e(Q), \tag{4}$$

where the unions are taken over the set of all $\mathbf{Q}$-parabolic subgroups of G. The space $\bar{X}$ is Hausdorff, and hence is a manifold with corners [4, (7.8)]. The action of $G_{\mathbf{Q}}$ on X extends to an action on $\bar{X}$ as a group of diffeomorphisms [4, (7.6)].

(3.6) Inspired by (2.6), we will define for a $\mathbf{Q}$-parabolic subgroup P the *crumpled corner* $X^*(P)$ in three steps. First, let

$$X_1^*({}_{\mathbf{Q}}Q_Y)={}_{\mathbf{Q}}A_Y^*\times^{{}_{\mathbf{Q}}A_Y}X, \tag{1}$$

where ${}_{\mathbf{Q}}A_Y^*$ denotes the closure of ${}_{\mathbf{Q}}A_Y$ in ${}_{\mathbf{Q}}A^*$. The surjection p of (2.6(2)) induces

$$p_1^*(P):X(P)\to X_1^*(P). \tag{2}$$

From (2.6(1)), we see that

$$X^*({}_{\mathbf{Q}}Q_Y)=\coprod_{\substack{\Xi\supseteq\kappa(Y)\\ \Xi\ \tau\text{-connected}}} e({}_{\mathbf{Q}}Q_\Xi), \tag{3}$$

and

$$p_1^*(P)\,|_{e({}_{\mathbf{Q}}Q_\Psi)}:e({}_{\mathbf{Q}}Q_\Psi)\to e({}_{\mathbf{Q}}Q_\Xi) \qquad \Xi=\kappa(\Psi) \tag{4}$$

is the quotient mapping $\nu_{\Psi,\Xi}$ by the geodesic action of ${}_{\mathbf{Q}}A_\Xi/{}_{\mathbf{Q}}A_\Psi\simeq{}_{\mathbf{Q}}A_{\Psi,\Xi}$ (see [4, (5.1(8))]).

It is clear that P operates as a group of homeomorphisms of $X_1^*(P)$. For $P={}_{\mathbf{Q}}Q_Y$, let Y be a cross-section to the geodesic action of ${}_{\mathbf{Q}}A_Y$ (see [4, (5.4)]). Then, we have a homeomorphism

$$X_1^*({}_{\mathbf{Q}}Q_Y)={}_{\mathbf{Q}}A_Y^*\times Y. \tag{5}$$

It follows immediately that $X_1^*(P)$ is a Hausdorff space.

If $P\subset P'$, then $X_1^*(P')$ is naturally embedded as an open subset of $X_1^*(P)$, in analogy with the corresponding assertion for corners in (3.5). We should be aware, however, that $X_1^*({}_{\mathbf{Q}}Q_Y)$ depends only on $\kappa(Y)$; if $\kappa(Y)=\kappa(Y')$, then $X_1^*({}_{\mathbf{Q}}Q_Y)$ and $X_1^*({}_{\mathbf{Q}}Q_{Y'})$ are, by (3) and (4), canonically homeomorphic.

The next step is to "derationalize" $X_1^*(P)$. There is a geodesic action of ${}_{\mathbf{R}}A$ on X (which restricts to that of ${}_{\mathbf{Q}}A$), and therefore we can define ${}_{\mathbf{R}}e(Q)$ for any $\mathbf{R}$-parabolic subgroup Q of G, such that whenever $\Xi\subseteq{}_{\mathbf{Q}}\Delta$ there is a projection

$$d_\Xi: e({}_{\mathbf{Q}}Q_\Xi)\to{}_{\mathbf{R}}e(Q_{\hat{\Xi}}), \tag{6}$$

given as the quotient by the geodesic action of ${}_{\mathbf{R}}A_{\hat{\Xi}}/{}_{\mathbf{Q}}A_\Xi$.

Let $\hat{\Xi}$ be a τ-connected subset of ${}_{\mathbf{Q}}\Delta$. Then we have

$$Q_{\kappa(\hat{\Xi})}\subseteq Q_{\hat{\Xi}}={}_{\mathbf{Q}}Q_\Xi, \tag{7}$$

from which there is a projection (cf. (4))

$${}_{\mathbf{R}}\nu_{\hat{\Xi},\kappa(\hat{\Xi})}:{}_{\mathbf{R}}e(Q_{\hat{\Xi}})\to{}_{\mathbf{R}}e(Q_{\kappa(\hat{\Xi})}). \tag{8}$$

We define the space

$$X_2^*({}_{\mathbf{Q}}Q_Y)=\coprod_{\substack{\Xi\supseteq Y\\ \Xi\ \tau\text{-connected}}}{}_{\mathbf{R}}e(Q_{\kappa(\hat{\Xi})}) \tag{9}$$

and mapping

$$p_2^*(P):X_1^*(P)\to X_2^*(P), \tag{10}$$

where

$$p_2^*({}_{\mathbf{Q}}Q_Y)|_{e({}_{\mathbf{Q}}Q_\Xi)}={}_{\mathbf{R}}\nu_{\hat{\Xi},\kappa(\hat{\Xi})}\circ d_\Xi \tag{11}$$

and $X_2^*(P)$ is given the quotient topology induced by $p_2^*(P)$. Thus, $X_2^*(P)$ is obtained from $X_1^*(P)$ by collapsing the orbits of further geodesic actions. We note that ${}_{\mathbf{R}}A_{\kappa(\Xi)}/{}_{\mathbf{Q}}A_{\Xi}$ is a subgroup of ${}_{\mathbf{R}}A_{\kappa(\Xi')}/{}_{\mathbf{Q}}A_{\Xi'}$, whenever $\Xi' \subset \Xi$.

Finally, for a parabolic subgroup Q, let U_Q denote its unipotent radical. The projection of Q onto Q/U_Q induces a principal U_Q-bundle (see [4, (7.2)])

$$r_Q : {}_{\mathbf{R}}e(Q) \to {}_{\mathbf{R}}\hat{e}(Q), \tag{12}$$

and moreover, one can identify

$${}_{\mathbf{R}}\hat{e}(Q_\Theta) = X_\Theta. \tag{13}$$

Let

$$X^*({}_{\mathbf{Q}}Q_Y) = \coprod_{\substack{\Xi \supseteq Y \\ \Xi\ \tau\text{-connected}}} {}_{\mathbf{R}}\hat{e}(Q_{\kappa(\Xi)}); \tag{14}$$

use (12) to define a surjection

$$p_3^*(P) : X_2^*(P) \to X^*(P), \tag{15}$$

and equip $X^*(P)$ with the quotient topology. Since also $U_{Q'} \subset U_Q$ whenever $Q \subset Q'$, we can see that $X^*(P)$ (and for a similar reason, $X_2^*(P)$) is a Hausdorff space, as a consequence of the following lemma (cf. [10, (4.2)]):

LEMMA. *Let S be a stratified Hausdorff space, with strata $\{S_j\}$, and let Y be a homogeneous space for the Lie group H. Let, for each j, H_j be a closed normal subgroup of H such that $H_i \supseteq H_j$ whenever S_i is in the closure of S_j. On the product $S \times Y$ define an equivalence relation by: $(s, y_1) \sim (s, y_2)$ if and only if $y_1 \in H_j y_2$, where $S_j \ni s$. Then the quotient space is Hausdorff.*

We remark that the quotient mapping, under the conditions of the above lemma, is seldom an open mapping.

(3.7) For $P \subset P'$, we have $X^*(P')$ embedded as an open subset of $X^*(P)$, as was the case with the $X_1^*(P)$'s. We form the identification space

$${}_{\mathbf{Q}}\tilde{X}^* = \bigcup X^*(P), \tag{1}$$

in which P runs over all $\mathbf{Q}$-parabolic subgroups of G. Observe that if $h \in {}_{\mathbf{Q}}Q_Y \dot{-} {}_{\mathbf{Q}}Q_{\kappa(Y)}$, then the distinct [5, (5.18)] crumpled corners $X^*({}_{\mathbf{Q}}Q_{\kappa(Y)})$ and

$X^*({}^h_{\mathbf{Q}}Q_{\kappa(Y)}) = hX^*({}_{\mathbf{Q}}Q_{\kappa(Y)})$ are identified homeomorphically. Thus, we can see that an efficient way to describe ${}_{\mathbf{Q}}\bar{X}^*$ as a set is to allow P in (1) to range only over those parabolic subgroups of the form ${}^g_{\mathbf{Q}}Q_{\omega(Y)}$, where $g \in G_{\mathbf{Q}}$ and Y is τ-connected (cf. (2.10)); sometimes, it is also useful to allow repetitions and let P range over the $G_{\mathbf{Q}}$-conjugates of the ${}_{\mathbf{Q}}Q_Y$'s.

By construction, ${}_{\mathbf{Q}}\bar{X}^*$ is a quotient space of $\bar{X}$ under the mapping

$$p^*: \bar{X} \to {}_{\mathbf{Q}}\bar{X}^*, \tag{2}$$

where $p^* = p_3^* p_2^* p_1^*$. It is also clear from the construction that the action of $G_{\mathbf{Q}}$ on $\bar{X}$ respects the fibers of p^*. It follows that $G_{\mathbf{Q}}$ acts on ${}_{\mathbf{Q}}\bar{X}^*$ as a group of homeomorphisms.

We remark that the construction of ${}_{\mathbf{Q}}\bar{X}^*$ requires neither Assumption 1 nor Assumption 2.

(3.8) We will examine more carefully the structure of the quotient mapping p^*. If one pursues the consequences of the definitions of the geodesic action and the manifold with corners, one sees that, *a priori*, the equivalence relation on $\bar{X}$ that determines the identifications in (3.7(1)) could be unexpectedly large.[5] It will become apparent that things are in actuality fairly nice, because we have in (3.6(4)) that

$$\Xi = \kappa(\Psi); \tag{1}$$

in particular, Ξ is a union of connected components of Ψ.

Let $\Lambda = \omega(\Xi)$, with Ξ τ-connected. According to [4, (7.2)], $e({}_{\mathbf{Q}}Q_\Lambda)$ is a principal ${}_{\mathbf{Q}}U_\Lambda$ fibration over the symmetric space Y_Λ of ${}_{\mathbf{Q}}M_\Lambda$, and moreover, this fibration extends to one of $\overline{e({}_{\mathbf{Q}}Q_\Lambda)}$ over the manifold with corners $\bar{Y}_\Lambda$.

Because of (1), with Ψ taken to be Λ, a certain decomposition is possible. First, we can write a product decomposition of identity components

$$({}_{\mathbf{Q}}M_\Lambda)^0 = (H_\Lambda)^0({}_{\mathbf{Q}}L_\Lambda)^0, \tag{2}$$

where ${}_{\mathbf{Q}}L_\Lambda$ is a semi-simple algebraic group over $\mathbf{Q}$ whose Lie algebra ${}_{\mathbf{Q}}\ell_\Lambda$ is generated by the $\mathbf{Q}$-root spaces

$$\{{}_{\mathbf{Q}}\mathfrak{g}_\beta, {}_{\mathbf{Q}}\mathfrak{g}_{-\beta} : \beta \in \Lambda\}; \tag{3}$$

[5] See Appendix.

H_Λ is defined, and is anisotropic, over $\mathbf{Q}$; and $H_\Lambda \cap {}_{\mathbf{Q}}L_\Lambda$ is finite (see [2, (2.2)]). From (2), we get an induced decomposition

$$Y_\Lambda \simeq W_{\varnothing,\Lambda} \times W_\Lambda, \tag{4}$$

where $W_{\varnothing,\Lambda}$ and W_Λ are the symmetric spaces of H_Λ and ${}_{\mathbf{Q}}L_\Lambda$ respectively. Next, (1) implies the almost-direct product decomposition

$$({}_{\mathbf{Q}}L_\Lambda)^0 = ({}_{\mathbf{Q}}L_\Xi)^0({}_{\mathbf{Q}}L_Y)^0, \tag{5}$$

with $Y = \Lambda - \Xi$. Therefore

$$W_\Lambda \simeq W_\Xi \times W_Y. \tag{6}$$

We put $\Theta = \kappa(\tilde{\Xi})$. We also let $c(\tilde{\Xi})$ denote the union of the connected components of $\tilde{\Xi}$ that meet $\rho^{-1}(\Xi)$; it is a subset of Θ since we have made Assumption 1 (3.3). We then put

$$\Delta^0(\Xi) = \Theta - c(\tilde{\Xi}). \tag{7}$$

We can restate (3), for Ξ instead of Λ, as: the Lie algebra of ${}_{\mathbf{Q}}L_\Xi$ is $\ell_{c(\tilde{\Xi})}$ (2.8); we then write

$${}_{\mathbf{Q}}L_\Xi = {}_{\mathbf{R}}L_{c(\tilde{\Xi})}. \tag{8}$$

By [2, (3.6,iii)], condition (ii) in the definition of "rational boundary component" for X_Θ is equivalent to the existence of a normal subgroup B of $N_\Theta = {}_{\mathbf{Q}}Q_\Lambda$, defined over $\mathbf{Q}$ and containing $({}_{\mathbf{R}}L_\Theta)^0({}_{\mathbf{Q}}U_\Lambda)$, such that $B/({}_{\mathbf{R}}L_\Theta)^0({}_{\mathbf{Q}}U_\Lambda)$ is compact. It is enough to find a normal $\mathbf{Q}$-subgroup B' of ${}_{\mathbf{Q}}M_\Lambda$ that contains $({}_{\mathbf{R}}L_\Theta)^0$, with $B'/({}_{\mathbf{R}}L_\Theta)^0$ compact. (It is useful to recall that X_Θ is the symmetric space of ${}_{\mathbf{R}}L_\Theta$, or of its identity component.) Now, there is an almost-direct product

$$({}_{\mathbf{R}}L_\Theta)^0 = ({}_{\mathbf{Q}}L_\Xi)^0({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0, \tag{9}$$

and moreover

$$({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0 \subset (H_\Lambda)^0. \tag{10}$$

Thus, we see that X_Θ is a rational boundary component if and only if there is a normal algebraic subgroup H'_Λ of H_Λ, defined over $\mathbf{Q}$, such that $({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0 \subset H'_\Lambda$

and $H'_\Lambda/({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0$ is compact. By our Assumption 2 (3.4), we have the existence of such H'_Λ.

Write $(H_\Lambda)^0$ as the almost-direct product of $\mathbf{Q}$-groups

$$(H_\Lambda)^0=(H'_\Lambda)^0\cdot(H''_\Lambda)^0, \tag{11}$$

and decompose accordingly

$$W_{\varnothing,\Lambda}\simeq W'_\Lambda\times W''_\Lambda. \tag{12}$$

We have

$$X_\Theta\simeq W_\Xi\times W'_\Lambda. \tag{13}$$

The following is now apparent:

PROPOSITION. *With notation as above:*
(i) $\overline{e({}_{\mathbf{Q}}Q_\Lambda)}\simeq(\bar W_\Xi\times W'_\Lambda)\times\bar W_Y\times W''_\Lambda\times{}_{\mathbf{Q}}U_\Lambda$,
(ii) *The subset of $\bar X$ that crumples onto ${}_{\mathbf{R}}\hat e(\mathbf{Q}_\Theta)$ under p^* is the subset*

$$\coprod_{\substack{\kappa(\Psi)=\Xi\\ h\in({}_{\mathbf{Q}}Q_\Lambda)_{\mathbf{Q}}/({}_{\mathbf{Q}}Q_\Psi)_{\mathbf{Q}}}} e({}^h_{\mathbf{Q}}Q_\Psi)\ \simeq X_\Theta\times\bar W_Y\times W''_\Lambda\times{}_{\mathbf{Q}}U_\Lambda, \tag{14}$$

of $\overline{e({}_{\mathbf{Q}}Q_\Lambda)}$, and the mapping p^* is given by projecting onto the first factor.

COROLLARY. *The fibers of p^* over ${}_{\mathbf{R}}\hat e(Q_\Theta)$ are naturally isomorphic to $\bar W_Y\times W''_\Lambda\times{}_{\mathbf{Q}}U_\Lambda$. They are closed in $\overline{e({}_{\mathbf{Q}}Q_\Lambda)}$.*

(3.9) We recall the set ${}_{\mathbf{Q}}X^*$ introduced in (3.5). One puts a topology on ${}_{\mathbf{Q}}X^*$ as follows (cf. [9, §2], [3, §4]).

Let $\mathscr{S}$ be a *generalized Siegel set* in X, which we should feel free to take as large as is necessary. By definition,

$$\mathscr{S}=(C\,{}_{\mathbf{Q}}A(t))x_0 \tag{1}$$

where C is a compact subset of ${}_{\mathbf{Q}}M_\varnothing\,{}_{\mathbf{Q}}U_\varnothing$, and x_0 is the basepoint (the coset K) of X. One can choose $\mathscr{S}$ so that

$$X=G_{\mathbf{Q}}\mathscr{S} \tag{2}$$

(see [11]; cf. [3, (2.4)]).

Let Γ be an *arithmetic* subgroup of G. This means that Γ is commensurable with the set of elements of G that are represented by matrices with integer entries under a general rational finite-dimensional representation of G. Then there exists a finite subset F_Γ of $G_{\mathbf{Q}}$ such that

$$\Omega = \Omega_\Gamma := F_\Gamma \mathscr{S} \tag{3}$$

is a *fundamental set* in X for Γ, namely

$$X = \Gamma\Omega, \text{ and for all } g \in G, \{\gamma \in \Gamma : \gamma\Omega \cap g\Omega \neq \varnothing\} \text{ is finite} \tag{4}$$

(see [11, §§10, 12]; compare [3, (2.4)], [2, (4.3)]).

Let Ω^* denote the closure of Ω in X^* (with respect to the "usual" topology of (2.1)); since the closure $\mathscr{S}^*$ of $\mathscr{S}$ is contained in ${}_{\mathbf{Q}}X^*$, we have

$$\Omega^* \subset {}_{\mathbf{Q}}X^*. \tag{5}$$

LEMMA (cf. [3, (4.3)]). (i) *Ω^* intersects only finitely many rational boundary components.*

(ii) *${}_{\mathbf{Q}}X^* = \Gamma \cdot \Omega^*$ if Ω is sufficiently large.*

(iii) *There is a finite subset Γ_0 of Γ such that if $\gamma \in \Gamma$ and $\gamma\Omega^* \cap \Omega^* \neq \varnothing$, then there exists $\gamma_0 \in \Gamma_0$ with $\gamma_0 x = \gamma x$ for all $x \in \gamma\Omega^* \cap \Omega^*$.*

Proof. By construction, $\mathscr{S}^*$ meets only the standard boundary components. From the definition of Ω (3), we have (i). Given (2) and (4), Assumption 2 (3.4), and the corollary of (3.3), in order to prove (ii) it suffices to verify that $G_{\mathbf{Q}} \cdot \mathscr{S}^*$ contains all of the standard rational boundary components. For this, and also for (iii) – since $\Gamma_\Theta := \Gamma \cap N_\Theta$ is arithmetic and, with Assumption 2, its image in the automorphism group of X_Θ is arithmetically defined (cf. [2, (3.4)–(3.6)]) – it is enough to know that $\Omega^* \cap X_\Theta$ is a fundamental set in X_Θ, for rational boundary components X_Θ. When $\Xi \neq \varnothing$, this follows from the fact that $\mathscr{S}^* \cap X_\Theta$ is a Siegel set in X_Θ, and all Siegel sets in X_Θ are of this form (cf. [2, (4.5)] and [4, (6.2)]). To see this, note that for a Siegel set (1) in X, $\mathscr{S}^* \cap X_\Theta$ consists of the translates of the base point of X_Θ by the Siegel set $\tilde{\mathscr{S}} = C_{\mathbf{Q}}A(t)(K \cap N_\Theta)$ in N_Θ. We are taking, to replace the centralizer (see (3.8)),

$$Z'_\Theta = H''_\Lambda({}_{\mathbf{Q}}L_Y)({}_{\mathbf{Q}}A_\Lambda)({}_{\mathbf{Q}}U_\Lambda);$$

remembering the remaining factors from N_Θ, we see that the projection of $\tilde{\mathscr{S}}$ in

N_Θ/Z'_Θ is in fact a Siegel set, and the desired conclusion holds. In the remaining case where $\Xi=\varnothing$, where the quotient group is anisotropic, both the projection of $\tilde{\mathscr{S}}$ and $\Gamma_\Theta\backslash X_\Theta$ are compact; if $\mathscr{S}$ is large enough, we get a fundamental set.

With the above lemma proved, we can now assert:

PROPOSITION [9, §2]. *There exists a unique topology on ${}_{\mathbf{Q}}X^*$ for which*
(i) *the relative topology induced on Ω^* is the given ("usual") one,*
(ii) *Γ acts as a group of homeomorphisms of ${}_{\mathbf{Q}}X^*$,*
(iii) *if $x, x' \in {}_{\mathbf{Q}}X^*$ are in different Γ-orbits, then there exist neighborhoods U, U' of x, x' respectively, which satisfy $\Gamma U \cap U' = \varnothing$;*
(iv) *let Γ_x be the isotropy group in Γ of $x \in {}_{\mathbf{Q}}X^*$. Then x has a neighborhood base consisting of Γ_x-invariant open sets U for which $\gamma U \cap U = \varnothing$ if $\gamma \notin \Gamma_x$.*
It follows that ${}_{\mathbf{Q}}X^*$ becomes a Hausdorff space, and that the quotient $\Gamma\backslash_{\mathbf{Q}}X^*$ is compact Hausdorff. Moreover, the topology on ${}_{\mathbf{Q}}X^*$ does not, in fact, depend on Γ.

From the construction of the topology [9, p. 562], one can be more explicit about (i) and (iv). Let $x \in \Omega^*$. A fundamental system of neighborhoods of x in ${}_{\mathbf{Q}}X^*$ can be described as follows. Let $\{U_j\}$ be a neighborhood base for x in Ω^*. Then $\{\Gamma_x U_j\}$ forms a neighborhood base of x in ${}_{\mathbf{Q}}X^*$.
We can compare compactifications determined by different τ's:

COROLLARY. *If τ and τ' are related as in* (2.11), *then the identity mapping of $\Gamma\backslash X$ extends to a continuous surjection*

$$\Gamma\backslash_{\mathbf{Q}}X^*_\tau \to \Gamma\backslash_{\mathbf{Q}}X^*_{\tau'}.$$

(3.10) We have been aiming toward the following result:

THEOREM. *The identity mapping of X extends to a continuous bijection of ${}_{\mathbf{Q}}\tilde{X}^*$ onto ${}_{\mathbf{Q}}X^*$.*

Proof. From the construction of ${}_{\mathbf{Q}}\tilde{X}^*$, it is apparent that as a set, ${}_{\mathbf{Q}}\tilde{X}^*$ is the union of all rational boundary components of X^*. In other words, there is an obvious one-to-one mapping of ${}_{\mathbf{Q}}\tilde{X}^*$ onto ${}_{\mathbf{Q}}X^*$. Henceforth, we identify ${}_{\mathbf{Q}}\tilde{X}^*$ and ${}_{\mathbf{Q}}X^*$ as sets. In order to show that this mapping is continuous, we need only verify that the sets described at the end of (3.9) are open in the topology of ${}_{\mathbf{Q}}\tilde{X}^*$. Since $G_{\mathbf{Q}}$ acts by homeomorphisms, we can see that

$$\Gamma_x[(p^*)^{-1}(U_j)\cap\tilde{\mathscr{S}}]$$

is open in $\bar{X}$ by checking that the topology induced on $\mathscr{S}^*$ is the usual one. (Note that for $x \in {}_{\mathbf{R}}\hat{e}(\mathbf{Q}_\Theta)$, Γ_x contains, and is commensurable with, the centralizer of the boundary component.) Now, $\mathscr{S}^*$ is the one-to-one continuous image of the (compact) closure of $\mathscr{S}$ in the crumpled corner $X^*({}_{\mathbf{Q}}Q_\varnothing)$. The latter inherits the usual topology, and hence $\mathscr{S}^*$ does as well. By taking larger and larger $\mathscr{S}$, we see that $\Gamma_x(p^*)^{-1}(U_j) = (p^*)^{-1}(\Gamma_x U_j)$ is open in $\bar{X}$. This completes the proof.

COROLLARY. *For any arithmetic subgroup Γ of G, the Satake compactification $\Gamma\backslash{}_{\mathbf{Q}}X^*$ is a quotient of the compact manifold*[6] *with corners $\Gamma\backslash\bar{X}$. The fibers of the quotient mapping can be deduced from* (3.8(8)) *by passing to the quotient by the action of ${}_{\mathbf{Q}}Q_\Lambda \cap \Gamma$.*

Remark. One can see that in general the topology of ${}_{\mathbf{Q}}\tilde{X}^*$ is finer than that of ${}_{\mathbf{Q}}X^*$, since it contains neighborhoods of points x that lack uniformity under the action of Γ_x.

(3.11) Our motivation has been to realize the compactification of $\Gamma\backslash X$, when X is Hermitian, constructed by Baily and Borel in [2] as a quotient of the manifold with corners. In order to apply the results of the preceding section, we must know that the Baily–Borel compactification is, as a topological space, of the form $\Gamma\backslash{}_{\mathbf{Q}}X^*_\tau$ for some representation τ of G. In some instances, the answer is already in the earlier [9]. The issue clearly lies prior to the taking of the quotient by Γ.

We first assume that X is irreducible. Then the $\mathbf{R}$-root system of G is either of classification type C_r, or is the non-reduced system BC_r, for some r (see [2, (1.2)]). These systems contain one simple root α_r that is respectively longer or shorter than the other simple roots. The construction in [2, §4] uses maximal $\mathbf{Q}$-parabolic subgroups, the closure X^c of the realization of X as a bounded domain, and a construction like that in the proposition of (3.9). We appeal to:

PROPOSITION [7, §3], [8, (5.2)]. *Let τ be any representation of G whose restricted highest weight is a multiple of the fundamental dominant weight dual to α_r. Then the Satake compactification X^*_τ is homeomorphic to X^c.*

If X is reducible, one makes the two constructions on each irreducible factor and takes the product; i.e., X^c is homeomorphic to X^*_τ, where τ is the tensor product of the chosen representations for the factors. For such τ, Assumptions 1 and 2 of (3.3) and (3.4) hold (see [2, (2.9), (3.7)]).

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[6] There are finite quotient singularities if Γ contains torsion elements.

To discuss ${}_{\mathbf{Q}}X^*_\tau$, we may assume without loss of generality that ${}_{\mathbf{Q}}\Delta$ is irreducible. The root system is then of type C_r or BC_r ([2, (2.9(a))]). Let Ξ be a τ-connected subset of ${}_{\mathbf{Q}}\Delta$. Then $\omega(\Xi)$ omits but one simple root. Therefore, ${}_{\mathbf{Q}}Q_{\omega(\Xi)}$ is maximal $\mathbf{Q}$-parabolic; conversely, every standard maximal $\mathbf{Q}$-parabolic subgroup is of this form. We see that the rational boundary components of X^c [2, (3.5)] and X^*_τ correspond. We obtain:

THEOREM. *The Baily–Borel compactification is a quotient of the manifold with corners* $\Gamma\backslash\bar{X}$.

We remark that although the Baily–Borel compactification is also the quotient of the smooth compactifications defined in [1, Ch. III, §5], one cannot realize the latter as quotients of $\Gamma\backslash\bar{X}$ (we allow no identifications in $\Gamma\backslash X$). An example of one which cannot be so realized is the Hirzebruch resolution (see [1, Ch. I, §5] for the definition) of a Hilbert modular surface.

Appendix: Comparison of geodesic actions

The geodesic action is denoted $a\circ x$ for $a\in{}_{\mathbf{Q}}A$ and $x\in X$ (or $x\in e({}_{\mathbf{Q}}Q_\Xi)$). We recall from [4, (3.2)] two basic properties of its definition. *For simplicity, we will state things for X, though there are parallel assertions for the general case.*

First, the geodesic action of ${}_{\mathbf{Q}}A_\Xi$ commutes with translations by ${}_{\mathbf{Q}}Q_\Xi$. Secondly, if x_0 is the basepoint of X associated to the choice of K, then the geodesic action of ${}_{\mathbf{Q}}A$ on x_0 coincides with the usual translation by ${}_{\mathbf{Q}}A$. One sees that if $a\in{}_{\mathbf{Q}}A_\Xi$ and $x=qx_0$ for $q\in{}_{\mathbf{Q}}Q_\Xi$, there is the formula

$$a\circ x=qax_0=(qaq^{-1})(qx_0). \tag{1}$$

This makes sense for all $x\in X$, since the parabolic subgroups act transitively on X.

We compare the geodesic actions of ${}_{\mathbf{Q}}A_\Xi$ and ${}^g_{\mathbf{Q}}A_\Xi$, with $g\in{}_{\mathbf{Q}}Q_\Psi$ ($\Xi\subset\Psi$), on $e({}_{\mathbf{Q}}Q_\Psi)$. Write $x\in e({}_{\mathbf{Q}}Q_\Psi)$ as gq_gx_0 (more precisely, the projection of this element of X onto $e({}_{\mathbf{Q}}Q_\Psi)$), with $q_g\in{}_{\mathbf{Q}}Q_\Xi$. Then

$${}^g_{\mathbf{Q}}A_\Xi\circ x={}^g_{\mathbf{Q}}A_\Xi\circ(gq_gg^{-1})gx_0=(gq_gg^{-1}){}^g_{\mathbf{Q}}A_\Xi(gx_0)=gq_g\,{}_{\mathbf{Q}}A_\Xi x_0. \tag{2}$$

If we select $p\in{}_{\mathbf{Q}}Q_\Xi$ such that $x=px_0$, we can then rewrite (2) as

$${}^g_{\mathbf{Q}}A_\Xi\circ x=(gq_gp^{-1}){}_{\mathbf{Q}}A_\Xi\circ x. \tag{3}$$

We see that the two geodesic orbits are ${}_{\mathbf{Q}}Q_{\Psi}$-translates of each other. More specifically:

PROPOSITION. *Let* $x \in e({}_{\mathbf{Q}}Q_{\Psi})$, $\Xi \subset \Psi \subseteq {}_{\mathbf{Q}}\Delta$, $g \in {}_{\mathbf{Q}}Q_{\Psi}$. *Then the projections* $\bar{x}$ *and* $\bar{x}_g$ *of* x, *in* $e({}_{\mathbf{Q}}Q_{\Xi})$ *and* $e({}^{g}_{\mathbf{Q}}Q_{\Xi})$ *respectively, are related by the formula*

$$\bar{x}_g = g(q_g p^{-1} \bar{x}), \tag{4}$$

where q_g *and* p *are defined above.*

Of course, in the construction of ${}_{\mathbf{Q}}\tilde{X}^*$ (3.7(1)), the images of certain $\bar{x}$ and $\bar{x}_g$ are identified. We observe that, in general, the choices of q_g and p, and therefore also the point $\bar{x}_g$, depend not only on $\bar{x}$, but also on x. This is the source of the remark in the second sentence of (3.8).

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