Proper holomorphic mappings between circular domains.

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STEVEN R. BELL

1. Introduction

A now classical theorem of H. Cartan [3] states that if $f:\Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded circular domains in $\mathbf{C}^n$ which contain the origin, and if $f(0)=0$, then f is a linear mapping. Cartan's theorem was later generalized by W. Kaup [4] to biholomorphic mappings between domains in a much wider class. In this note, we prove a generalization of Cartan's theorem which allows the mapping f to be proper and non-biholomorphic. To be precise, we prove

THEOREM 1. *Suppose that $f:\Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded circular domains in $\mathbf{C}^n$ which contain the origin, and suppose that $f^{-1}(0)=\{0\}$. Then the mapping f is a polynomial mapping.*

The proof of this theorem uses only elementary properties of the Bergman projection associated to a bounded circular domain. Therefore, before we attempt to prove Theorem 1, it seems worthwhile to recall some basic definitions and to list the rudimental properties of the Bergman projection.

2. Basic definitions and facts

A circular domain contained in $\mathbf{C}^n$ is a connected open set such that if $z=(z_1,\ldots,z_n)$ is in the set, then for any real number θ, the point $e^{i\theta}z=(e^{i\theta}z_1,\ldots,e^{i\theta}z_n)$ is also in the set.

The Bergman projection P associated to a bounded domain D contained in $\mathbf{C}^n$ is the orthogonal projection of $L^2(D)$ onto its closed subspace $H(D)$ consisting of L^2 holomorphic functions. Connected to the projection P is the Bergman kernel function $K(w, z)$. This kernel is determined by the property that

$$P\phi(w)=\int_D K(w,z)\phi(z)\,dV_z$$

for all ϕ in $L^2(D)$. The kernel $K(w, z)$ is defined on $D \times D$, and is holomorphic in w and antiholomorphic in z, and $K(w, z) = \overline{K(z, w)}$. Proofs of these elementary facts can be found in [2] and [5].

We shall require a lemma which is proved in [1].

LEMMA A. *Suppose that $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping between bounded domains Ω_1 and Ω_2 contained in $\mathbf{C}^n$. Let P_i denote the Bergman projection associated to Ω_i, $i = 1, 2$, and let $u = \mathrm{Det}\,[f']$. The Bergman projections transform according to*

$$P_1(u \cdot (\phi \circ f)) = u \cdot ((P_2\phi) \circ f)$$

for all ϕ in $L^2(\Omega_2)$.

The classical Remmert proper mapping theorem states that the mapping f in Lemma A is a branched cover of some finite order m. Since $|u|^2$ is equal to the real Jacobian determinant of f viewed as a mapping of $\mathbf{R}^{2n}$, it follows from a simple change of variables that

$$\|u \cdot (\phi \circ f)\|_{L^2(\Omega_1)} = m^{1/2}\, \|\phi\|_{L^2(\Omega_2)}$$

for all ϕ in $L^2(\Omega_2)$. This fact will be used at a crucial step in the proof of Theorem 1.

It is well known ([2, 5]) that if the mapping f in Lemma A is biholomorphic, then the Bergman kernel functions transform according to

$$u(w) K_2(f(w), f(z)) \overline{u(z)} = K_1(w, z)$$

where K_i denotes the kernel function associated to Ω_i, $i = 1, 2$.

Finally, before we proceed to prove Theorem 1, it is instructive to take a glance at the original proof of Cartan's theorem. Suppose $f : \Omega_1 \to \Omega_2$ is a biholomorphic mapping between bounded circular domains which contain the origin, and suppose $f(0) = 0$. For each real number θ, the mapping F_θ defined by

$$F_\theta(z) = f^{-1}(e^{-i\theta} f(e^{i\theta} z)) \tag{2.1}$$

is an automorphism of Ω_1 such that $F_\theta(0) = 0$ and such that the Jacobian matrix $F'_\theta(0)$ is equal to the identity matrix. Therefore, according to Cartan's lemma, the mapping F_θ is the identity. Now, writing equation (2.1) out in terms of power series reveals that f must be a linear mapping. It is interesting that a very similar argument must be used at a key point in the proof of Theorem 1.

3. Proof of Theorem 1

Theorem 1 is a relatively simple consequence of two basic lemmas which we now list.

LEMMA B. *Suppose that $K(w, z)$ is the Bergman kernel function associated to a bounded circular domain Ω. Suppose that w and z are points in Ω and that U is any connected neighborhood of the unit circle in* $\mathbf{C}$ *such that tw and $\bar{t}z$ are in Ω for each t in U. Then $K(tw, z) = K(w, \bar{t}z)$ for all t in U.*

Let B_R denote the ball in $\mathbf{C}^n$ of radius R centered at the origin.

LEMMA C. *Suppose that Ω is a bounded circular domain in $\mathbf{C}^n$ which contains the unit ball B_1. Let P denote the Bergman projection associated to Ω. For each multi-index α, there is a function ϕ_α in $C_0^\infty(B_1)$ such that $P\phi_\alpha = z^\alpha$. Furthermore, ϕ_α can be chosen so that if $\phi_{\alpha,\varepsilon}$ is defined via $\phi_{\alpha,\varepsilon}(z) = \varepsilon^{-2n-|\alpha|}\phi_\alpha(z/\varepsilon)$, then $P\phi_{\alpha,\varepsilon} = z^\alpha$ if $0<\varepsilon<1$.*

We shall now prove Theorem 1, assuming the truth of the lemmas. We may suppose, without loss of generality, that Ω_1 and Ω_2 both contain $\bar{B}_1$, the closure of the unit ball.

Let $K(w, z)$ denote the Bergman kernel function associated to Ω_1. Lemma B has as an important consequence the fact that for z close to the origin, the function $K(w, z)$ extends to be holomorphic in w on a large neighborhood of $\bar{\Omega}_1$. Indeed, if R is a large positive number, then $K(w, z)$ extends holomorphically as a function of w to B_R whenever z is in $B_{1/R}$. This follows from the formula

$$K(w, z) = K\left(\frac{w}{R}, Rz\right)$$

which holds for (w, z) in $B_1 \times B_{1/R}$, and which extends to hold for (w, z) in $B_R \times B_{1/R}$ by analytic continuation.

Now notice that if $\phi_{\alpha,\varepsilon}$ is the function of Lemma C associated to Ω_2, and if $u = \text{Det}\,[f']$, then Lemma A yields that

$$u \cdot f^\alpha = u \cdot (z^\alpha \circ f) = P_1(u \cdot (\phi_{\alpha,\varepsilon} \circ f)) \tag{3.1}$$

where P_1 denotes the Bergman projection associated to Ω_1. We may rewrite (3.1) in integral form:

$$u(w)f(w)^{\alpha}=\int_{\Omega_1} K(w,z)u(z)\phi_{\alpha,\varepsilon}(f(z))\,dV_z. \tag{3.2}$$

Equation (3.2) contains the core of the proof of Theorem 1.

We shall now prove that $u\cdot f^{\alpha}$ is a polynomial for each α, including $\alpha=(0,0,\ldots,0)$, by showing that the functions $u\cdot f^{\alpha}$ are entire functions which satisfy an estimate of the form $|u(w)f(w)^{\alpha}|\le C\,|w|^{q}$. Then, since u is a polynomial and $u\cdot f^{\alpha}$ is a polynomial for each α, and since the ring of polynomials is a unique factorization domain, we will conclude that f must be a polynomial mapping.

First, notice that if $\varepsilon>0$ is taken to be very small, the formula (3.2) implies that $u\cdot f^{\alpha}$ extends to be holomorphic in a large neighborhood of $\bar{\Omega}_1$. Indeed, since $f^{-1}(0)=\{0\}$, the nullstellensatz implies that there are holomorphic functions $a_{ij}(z)$ and positive integers k_i such that $z_i^{k_i}=\sum_{j=1}^{n} a_{ij}(z)f_j(z)$ near $z=0$. Hence, there are positive constants m and c such that f satisfies an estimate of the form $|z|^{m}\le c\,|f(z)|$ for all z in $\bar{\Omega}_1$. Therefore,

$$\operatorname{Supp}(\phi_{\alpha,\varepsilon}\circ f)\subset\{z:|f(z)|\le\varepsilon\}\subset\{z:|z|\le(c\varepsilon)^{1/m}\}.$$

Hence, if R is a large positive number and if ε is chosen so that $(c\varepsilon)^{1/m}<1/R$, formula (3.2) in conjunction with the fact that $K(w,z)$ extends to $B_R\times B_{1/R}$ reveals that $u\cdot f^{\alpha}$ extends to be holomorphic on B_R. Therefore, we conclude that $u\cdot f^{\alpha}$ is an entire function.

We must now show that $|u(w)f(w)^{\alpha}|<C\,|w|^{q}$. Fix a point w in $\mathbf{C}^n$. Pick ε so that $(c\varepsilon)^{1/m}=|w|^{-1}$, i.e., let $\varepsilon=c^{-1}\,|w|^{-m}$. Note that $\operatorname{supp}(\phi_{\alpha,\varepsilon}\circ f)\subset B_{|w|^{-1}}$ and that

$$\begin{aligned}\|u\cdot(\phi_{\alpha,\varepsilon}\circ f)\|_{L^2(\Omega_1)}&=(\text{constant})\,\|\phi_{\alpha,\varepsilon}\|_{L^2(\Omega_2)}\\&=(\text{const.})\,\varepsilon^{-n-|\alpha|}\,\|\phi_{\alpha}\|_{L^2(B_1)}\\&=(\text{const.})\,|w|^{m(n+|\alpha|)}.\end{aligned}$$

We now use formula (3.2) and Lemma B to obtain that

$$\begin{aligned}|u(w)f(w)^{\alpha}|&=\left|\int_{\Omega_1} K\!\left(\frac{w}{|w|},|w|\,z\right)u\cdot(\phi_{\alpha,\varepsilon}\circ f)\,dV_z\right|\\&\le(\text{const.})\left(\operatorname*{Sup}_{\bar{B}_1\times\bar{B}_1}|K(\zeta,\xi)|\right)\|u\cdot(\phi_{\alpha,\varepsilon}\circ f)\|_{L^2(\Omega_1)}\\&\le(\text{const.})\,|w|^{m(n+|\alpha|)}\end{aligned}$$

where the constant is independent of w. This completes the proof of Theorem 1.

4. Proofs of the lemmas

Lemma B is well known. We shall present a proof for the sake of completeness.

Proof of Lemma B. If θ is a real number, the mapping Φ defined via $\Phi(z)=e^{i\theta}z$ is an automorphism of the domain Ω. Therefore, the Bergman kernel function $K(w, z)$ satisfies the identity

$$\mathrm{Det}\,[\Phi'(w)]K(\Phi(w), \Phi(z))\,\mathrm{Det}\,[\overline{\Phi'(z)}]=K(w, z).$$

If we replace z by $e^{-i\theta}z$ in this formula we obtain

$$K(e^{i\theta}w, z)=K(w, e^{-i\theta}z).$$

Now $K(tw, z)$ and $K(w, \bar{t}z)$ are holomorphic functions of t on U which agree on the unit circle. Hence, $K(tw, z)=K(w, \bar{t}z)$ for all t in U.

Proof of Lemma C. Let $K(w, z)$ denote the Bergman kernel function associated to Ω. We shall use the shorthand notation,

$$K^{\bar{\alpha}}(w, z)=\frac{\partial^{\alpha}}{\partial\bar{z}^{\alpha}}K(w, z)$$

and

$$K^{\alpha}(w, z)=\frac{\partial^{\alpha}}{\partial w^{\alpha}}K(w, z)$$

for multi-indices α. We shall also abreviate the operators $\partial^{\alpha}/\partial z^{\alpha}$ and $\partial^{\alpha}/\partial\bar{z}^{\alpha}$ by ∂^{α} and $\bar{\partial}^{\alpha}$, respectively.

Let θ be a radially symmetric function in $C_0^{\infty}(B_1)$ such that $\int\theta=1$. For small $\varepsilon>0$, let $\theta_{\varepsilon}(z)=\varepsilon^{-2n}\theta(z/\varepsilon)$. Since holomorphic functions assume their average values, it follows that if $h(w)$ is a function in $H(\Omega)$, then

$$\partial^{\alpha}h(0)=\int_{\Omega}\partial^{\alpha}h\overline{\theta_{\varepsilon}(z)}\,dV_z=\int_{\Omega}h(-1)^{|\alpha|}\overline{\bar{\partial}^{\alpha}\theta_{\varepsilon}}\,dV=\int_{\Omega}K^{\alpha}(0, z)h(z)\,dV_z.$$

Therefore, the Bergman projection of $(-1)^{|\alpha|}\bar{\partial}^{\alpha}\theta_{\varepsilon}$ is equal to $K^{\bar{\alpha}}(w, 0)$ as a function of w.

Now suppose that w and z are in B_1. If we differentiate the formula $K(tw, z)=K(w, \bar{t}z)$ with respect to $\bar{z}$, we obtain that

$$K^{\bar{\alpha}}(tw, z)=t^{|\alpha|}K^{\bar{\alpha}}(w, \bar{t}z). \qquad (4.1)$$

The formula (4.1) holds for all t in the unit disc of $\mathbf{C}$. If we set $z=0$ in (4.1), we see that

$$K^{\bar{\alpha}}(tw,0)=t^{|\alpha|}K^{\bar{\alpha}}(w,0).$$

This implies that $K^{\bar{\alpha}}(w,0)$ is a homogeneous polynomial of degree $|\alpha|$ in w.

We now claim that the set of homogeneous polynomials $H^N=\{K^{\bar{\alpha}}(w,0):|\alpha|=N\}$ forms a basis for the set of all homogeneous polynomials of degree N. Indeed, the functions in H^N are linearly independent because if

$$\sum_{|\alpha|=N} c_\alpha K^{\bar{\alpha}}(w,0)=0$$

then $\sum_{|\alpha|=N}\bar{c}_\alpha\partial^\alpha h(0)=0$ for every h in $H(\Omega)$, which is absurd. Furthermore, the cardinality of H^N is equal to the dimension of the vector space of all homogeneous polynomials of degree N. Hence, each monomial z^α can be written in the form

$$z^\alpha=\sum_{|\beta|=|\alpha|} c_\beta K^{\bar{\beta}}(z,0).$$

Therefore,

$$z^\alpha=P\Big(\sum_{|\beta|=|\alpha|} c_\beta(-1)^{|\beta|}\bar{\partial}^\beta\theta_\varepsilon\Big).$$

If we set $\phi_\alpha=\sum_{|\beta|=|\alpha|}c_\beta(-1)^{|\beta|}\bar{\partial}^\beta\theta$, then the conditions of Lemma C are met.

Remark. Formula (3.2) can be used to prove the following generalization of a result of Kaup [4] on biholomorphic mappings between Reinhardt domains.

THEOREM 2. *Suppose $f:\Omega_1\to\Omega_2$ is a proper holomorphic mapping between bounded circular domains in $\mathbf{C}^n$. Suppose further that Ω_2 contains the origin and that the Bergman kernel function $K(w,z)$ associated to Ω_1 is such that for each compact subset G of Ω_1, there is an open set $U=U(G)$ containing $\bar{\Omega}_1$ such that $K(w,z)$ extends to be holomorphic on U as a function of w for each z in G. Then f extends holomorphically to a neighborhood of $\bar{\Omega}_1$.*

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Mathematics Department
Princeton University
Princeton, New Jersey 08544

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