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Autor(en): **Plotnick, Steven P.**

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Vanishing of Whitehead groups for Seifert manifolds with infinite fundamental group

Steven P. Plotnick

1. Introduction

Seifert manifolds are a very well understood class of 3-manifolds, arising naturally in a variety of situations – circle actions, algebraic varieties, branched covers, surgery on knots, plumbing, etc. We have:

THEOREM. *Let M be an orientable irreducible Seifert manifold with infinite fundamental group. Then* $\mathrm{Wh}(\Pi_1 M)=0$ *and* $\tilde{K}_0(\mathbf{Z}[\Pi_1 M])=0$.

This theorem, in the case that M is sufficiently large, follows from deep results of Waldhausen, who proved that $\mathrm{Wh}(\Pi_1 M)=0$ and $\tilde{K}_0(\mathbf{Z}[\Pi_1 M])=0$ when M is an orientable, irreducible, sufficiently large 3-manifold [W]. However, Seifert manifolds provide a very well-known class of closed, orientable, aspherical, non-sufficiently large 3-manifolds – those 3-manifolds M admitting effective circle actions with 3 exceptional orbits, $M/S^1=S^2$, infinite fundamental group and finite first homology group [EJ]. In terms of Seifert invariants, these are the manifolds $M^3=\{b;(o,0,0,0);(p,\beta_1),(q,\beta_2),(r,\beta_3)\}$, satisfying $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1$ and $bpqr+\beta_1 qr+\beta_2 pr+\beta_3 pq\ne 0$ [O]. In particular, this includes all Brieskorn homology spheres.

$$\Sigma(p,q,r)=\{(z_1,z_2,z_3)\in\mathbf{C}^3\cap S^5: z_1^p+z_2^q+z_3^r=0;\qquad p,q,r \text{ pairwise coprime}\},$$

with the exception of $\Sigma(2,3,5)$. Until recently ([T], Section 4.10), these Seifert manifolds were the only known examples of closed, orientable, aspherical, non-sufficiently large 3-manifolds.

The theorem in the case that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ follows from recent work of Farrell and Hsiang, [FH2], who have proved these vanishing results for torsion-free extensions of poly-$\mathbf{Z}$ groups by finite groups. (The euclidean triangle groups contain subgroups of finite index which are free abelian of rank two, so that the 3-manifolds have finite covers which are principal circle bundles over T^2.) Thus, the proof of the theorem reduces to proving the following theorem:

THEOREM. *Let M be the non-sufficiently large Seifert 3-manifold $\{b; (o, 0, 0, 0); (p, \beta_1), (q, \beta_2), (r, \beta_3)\}$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Then $\mathrm{Wh}(\Pi_1 M)=0$ and $\tilde{K}_0(\mathbf{Z}[\Pi_1 M])=0$.*

The proof will be via induction on the hyperelementary subgroups of some finite homomorphic image of $\Pi_1 M$. This requires

(1) that we find an epimorphism $\Pi_1 M \xrightarrow{\varphi} F$, where F is a non-hyperelementary finite group, and
(2) that we can understand the covering spaces of M determined by the subgroups $\varphi^{-1}(G)$, where G is a hyperelementary subgroup of F.

Most of the groups needed in (1) are provided by Fox [F]. To accomplish (2), we view M as an injective Seifert fiber space in the sense of Conner–Raymond [CR]. This allows us to reduce our problem to one concerning the subgroups of finite index in hyperbolic triangle groups. In particular, when does a triangle group contain another triangle group as a subgroup of finite index? This has been answered by Greenberg (see [G] or [K]). The crucial point is that the angles of a hyperbolic triangle determine its area.

It is a pleasure to thank Tom Farrell and Frank Raymond for helpful conversations and encouragement.

2. The strategy and an interesting special case

The technique of hyperelementary induction goes as follows: Suppose we have an epimorphism $\Pi \xrightarrow{\varphi} F$, where F is a finite group. If G is a subgroup of F, $\varphi^{-1}(G)$ will be a subgroup of finite index in Π, and there is the transfer homomorphism $i^*: \mathrm{Wh}(\Pi) \to \mathrm{Wh}(\varphi^{-1}G)$. Let $x \in \mathrm{Wh}(\Pi)$. Induction tells us that $x=0$ if and only if $i^*(x)=0 \in \mathrm{Wh}(\varphi^{-1}G)$ for all hyperelementary subgroups G of F. A group G is hyperelementary if it can be written as $1 \to \mathbf{Z}_k \to G \to P \to 1$, where P is a p-group, and we may assume $(k, p)=1$. All results stated for $\mathrm{Wh}(\Pi)$ carry over to $\tilde{K}_0(\mathbf{Z}\Pi)$.

Very briefly, induction is proved by considering $\mathrm{Wh}(\Pi)$ as a Frobenius module over Swan's Frobenius functor $G_0(F)$, a Grothendieck construction applied to the category of finitely generated, free abelian representations of F, [S1, S2]. $G_0(F)$ is a ring with unit, where tensor product gives the ring structure, and the unit is $\mathbf{Z}$, regarded as a trivial $\mathbf{Z}F$-module. We have Swan's result that the unit in $G_0(F)$ is a sum of elements $i_*(x)$, where the inclusion $i: G \to F$ induces $i_*: G_0(G) \to G_0(F)$ by $i_*(N) = N \otimes_{\mathbf{Z}G} \mathbf{Z}F$, and G ranges over the hyperelementary subgroups of F ([S1], Corollary 4.2). One now uses Frobenius reciprocity to conclude that $x \in \mathrm{Wh}(\Pi)$ is trivial if it transfers to zero in $\mathrm{Wh}(\varphi^{-1}G)$ for all hyperelementary $G \subset F$, as in [FH1], Theorem 3.1.

If $M = \{b; (o, 0, 0, 0); (p, \beta_1), (q, \beta_2), (r, \beta_3)\}$, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then $\Pi_1 M$ has the following structure:

$$1 \to \mathbf{Z} \to \Pi_1 M \xrightarrow{\pi} Q(p, q, r) \equiv \langle q_1, q_2, q_3 \mid 1 = q_1 q_2 q_3 = q_1^p = q_2^q = q_3^r \rangle \to 1,$$

where $\mathbf{Z}$ is the center of $\Pi_1 M$ and $Q(p, q, r)$ is a hyperbolic triangle group [CR].

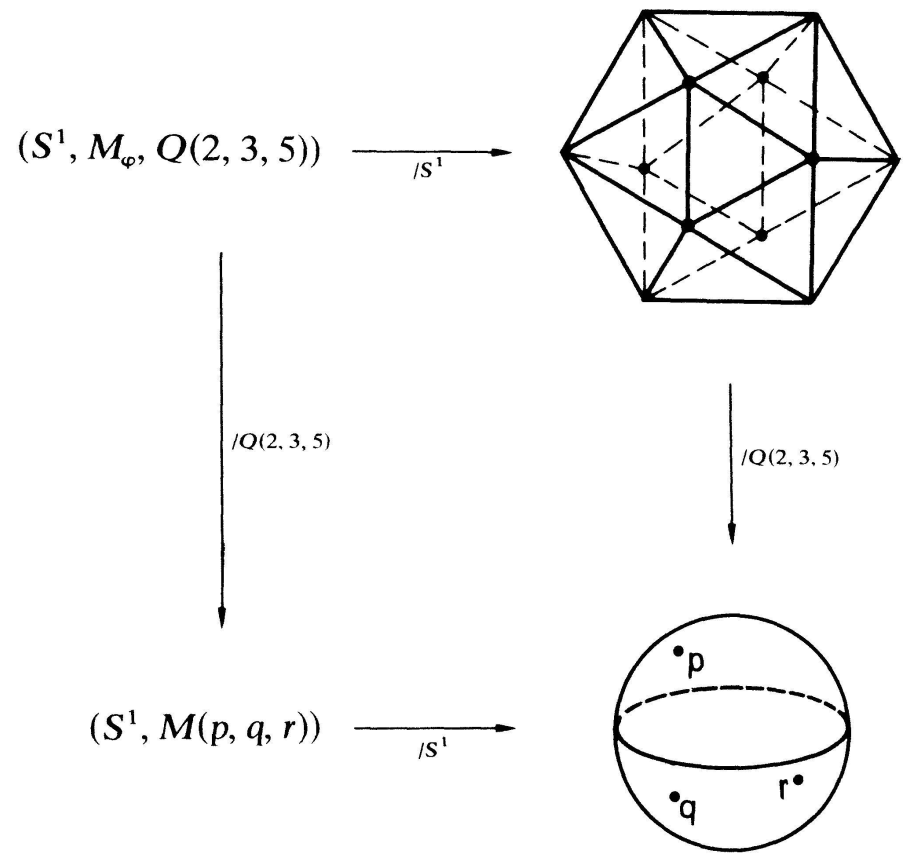
Let us now discuss a special case where $Q(p, q, r)$ has an obvious finite quotient. Assume that $p \equiv 0 \pmod 2$, $q \equiv 0 \pmod 3$, $r \equiv 0 \pmod 5$, and $(p, q, r) \neq (2, 3, 5)$. We have an obvious map φ of $Q(p, q, r)$ onto $Q(2, 3, 5) =$ Icosohedral group, given by

$$\langle q_1, q_2, q_3 \mid 1 = q_1 q_2 q_3 = q_1^p = q_2^q = q_3^r \rangle \xrightarrow{\varphi} \langle p_1, p_2, p_3 \mid 1 = p_1 p_2 p_3 = p_1^2 = p_2^3 = p_3^5 \rangle.$$
$$q_i \mapsto p_i$$

For convenience, we denote M above by $M(p, q, r)$. (The invariants $b, \beta_1, \beta_2, \beta_3$ will not be used.) The map φ defines a subgroup of $\Pi_1 M(p, q, r)$ of index 60, namely $\ker(\varphi \circ \pi)$: .

$$\begin{tikzcd}
 & & & Q(2,3,5) & \\
1 \arrow[r] & \mathbf{Z} \arrow[r] & \Pi_1 M(p,q,r) \arrow[r, "\pi"] & Q(p,q,r) \arrow[r] \arrow[u, "\varphi"] & 1 \\
1 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[u, equal] & \ker(\varphi \circ \pi) \arrow[r] \arrow[u, hook] & \ker \varphi \arrow[r] \arrow[u, hook] & 1.
\end{tikzcd}$$

Since $\ker(\varphi \circ \pi)$ contains the center of $\Pi_1 M$, we may lift the S^1 action on $M(p, q, r)$ to the covering space corresponding to $\ker(\varphi \circ \pi)$, which we will call M_φ ([CR] Theorem 4.3). The icosohedral group acts as the group of covering transformations, commuting with the S^1 action. We can divide out by the circle, and view the action of $Q(2, 3, 5)$ on the 2-manifold level. We have the following:



We have the standard action of $Q(2, 3, 5)$ on the 2-sphere. Every principal orbit of $M(p, q, r)$ is covered by 60 principal orbits in M_φ. The situation over exceptional orbits is slightly more complicated, and is reflected in the action of $Q(2, 3, 5)$ on S^2. Points of S^2 that are fixed by an element of $Q(2, 3, 5)$ correspond to orbits in M_φ where the covering transformation is acting in the orbit, as opposed to permuting orbits. Since p corresponds to elements of order 2, over $p \in S^2$ will be 30 points, the midpoints of edges in the triangulation. Over q will be 20 points, the centers of faces, and over r will be 12 points, the vertices of the triangles. Notice that we are letting p, q, r represent both points of S^2 and the order of the isotropy subgroup of the corresponding orbit in $M(p, q, r)$.

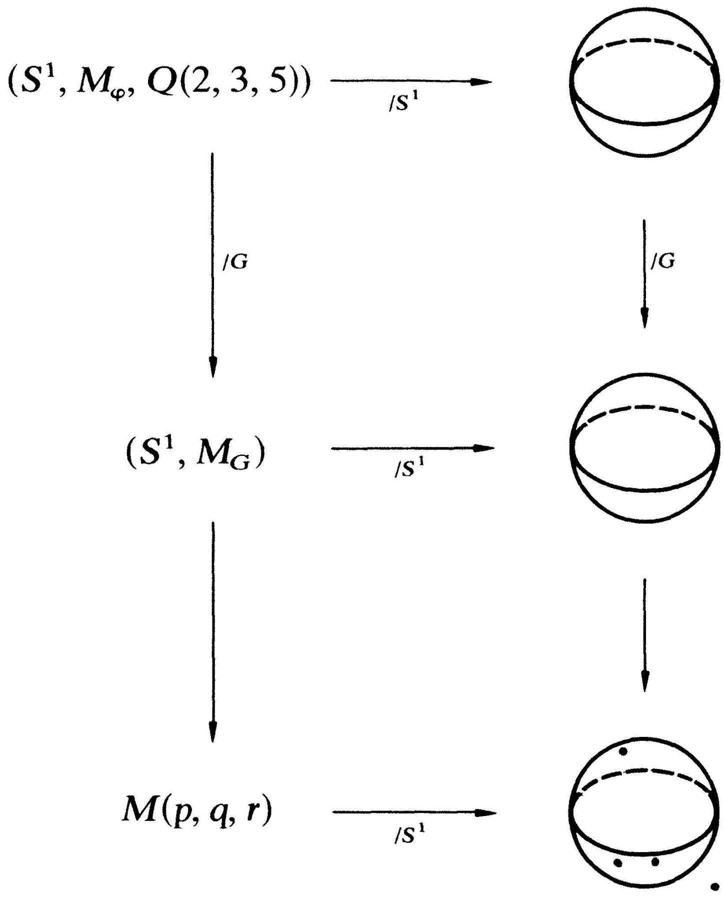
Now, over the $\mathbf{Z}_p$ orbit in $M(p, q, r)$ will be 30 orbits, each 2-fold covering the orbit downstairs. Since we have lifted the S^1 action, each of these orbits will have stabilizer $\cong \mathbf{Z}_{p/2}$. Similarly, there will be 20 orbits with $\mathbf{Z}_{q/3}$ stabilizer, and 12 orbits with $\mathbf{Z}_{r/5}$ stabilizer. Since $(p, q, r) \neq (2, 3, 5)$, there will be *at least* 12 exceptional orbits in M_φ.

To apply induction, we must consider hyperelementary subgroups of $Q(2, 3, 5)$. These are well known to be either cyclic or dihedral. (The tetrahedral subgroup is not hyperelementary.) Actually, we need only consider dihedral subgroups, since, for example, every $\mathbf{Z}_5$ in $Q(2, 3, 5)$ is contained in a dihedral subgroup D_5. Thus, the map $\mathrm{Wh}(\Pi_1 M(p, q, r)) \to \mathrm{Wh}((\varphi \circ \pi)^{-1}\mathbf{Z}_5)$ factors through $\mathrm{Wh}((\varphi \circ \pi)^{-1} D_5)$. Notice that we are referring to $\mathbf{Z}_2 \times \mathbf{Z}_2$ as D_2.

Suppose we take one of the dihedral subgroups, say G. We have:

$$
\begin{tikzcd}
 & & & 1 & \\
 & & & Q(2,3,5) \supset G \arrow[u] & \\
1 \arrow[r] & \mathbf{Z} \arrow[r] & \Pi_1 M(p, q, r) \arrow[r, "\pi"] & Q(p, q, r) \arrow[r] \arrow[u, "\varphi"'] & 1 \\
1 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[u, equal] & \Pi_1 M_G \arrow[r] \arrow[u, hook] & \varphi^{-1}(G) \arrow[r] \arrow[u, hook] & 1 \\
1 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[u, equal] & \Pi_1 M_\varphi \arrow[r] \arrow[u, hook] & \ker \varphi \arrow[r] \arrow[u, hook] & 1,
\end{tikzcd}
$$

where the middle row corresponds to a covering space between $M(p, q, r)$ and M_φ:



We now show that M_G is sufficiently large. By Waldhausen [W], then, $\mathrm{Wh}(\Pi_1 M_G)=0$. Thus, we can apply induction in a rather trivial fashion: elements of $\mathrm{Wh}(\Pi_1 M(p,q,r))$ have no choice but to transfer to zero in $\mathrm{Wh}(\Pi_1 M_G)=0$. To show M_G is sufficiently large, we need only count at least 4 exceptional orbits.

For example, suppose $G=D_5$. (All dihedral subgroups of $Q(2,3,5)$ of order 10 are conjugate, so it is enough to consider one of them.) Then G is generated by a 5-fold rotation and a 2-fold one, preserving a pentagon inscribed in the icosohedron, perpendicular to the axis of the 5-fold rotation.

The action of G partitions the set of vertices into 2 orbits, one containing 10 points and one containing 2. The circle orbits in M_φ, therefore, are collapsed to 2 orbits in M_G. One of these orbits now 5-fold covers the $\mathbf{Z}_r$ orbit in $M(p,q,r)$, while one of them singly covers the $\mathbf{Z}_r$ orbit. Notice that M_G is a 6-fold irregular cover of $M(p,q,r)$.

Similarly, the 30 $\mathbf{Z}_{p/2}$ orbits are collapsed by G to 4 orbits. Two of these doubly cover the $\mathbf{Z}_p$ orbit, while two of them singly cover. Finally, the 20 $\mathbf{Z}_{q/3}$ orbits are collapsed to 2 orbits, each triply covering the $\mathbf{Z}_q$ orbit in $M(p,q,r)$.

Now count. The one orbit which singly covers the $\mathbf{Z}_r$ orbit gives an exceptional orbit with stabilizer $\mathbf{Z}_r$. The two orbits which singly cover the $\mathbf{Z}_p$ orbit give rise to two more exceptional orbits. If $r/5>1$, we have one $\mathbf{Z}_{r/5}$ orbit. If $p/2>1$, we have two $\mathbf{Z}_{p/2}$ orbits. If $q/3>1$, we have two $\mathbf{Z}_{q/3}$ orbits. In other words, we always have at least 4 exceptional orbits. Hence, M_G is sufficiently large.

The situation with the other dihedral groups is similar. If $G\cong D_2$, G collapses the 30 $\mathbf{Z}_{p/2}$ orbits to 6 orbits which doubly cover and 3 orbits which singly cover, thus giving at least 3 $\mathbf{Z}_p$ orbits in M_G. If $p/2>1$, we have an additional 6 $\mathbf{Z}_{p/2}$ orbits. If $q/3>1$, we get 5 $\mathbf{Z}_{q/3}$ orbits, and if $r/5>1$ we get 3 $\mathbf{Z}_{r/5}$ orbits. Finally, $G\cong D_3$ yields 1 $\mathbf{Z}_q$ orbit and 2 $\mathbf{Z}_p$ orbits, and either 4 $\mathbf{Z}_{p/2}$ orbits, 3 $\mathbf{Z}_{q/3}$ orbits, or 2 $\mathbf{Z}_{r/5}$ orbits. This completes the proof that $\mathrm{Wh}(\Pi_1 M(p,q,r))=0$ in this special case.

On the one hand, it is very nice to be able to determine so much of the structure of these covering spaces. We have produced explicit actions of the icosohedral group on Seifert manifolds. With a little more work we could recover the remaining Seifert invariants of these spaces. In fact, enthusiasts of the binary icosohedral group will be pleased to know that in a similar fashion one may construct free actions of $SL(2,5)$ on Seifert manifolds other than S^3. This is slightly more involved, since we must find a map directly from $\Pi_1 M(p,q,r)$ onto $SL(2,5)\cong \Pi_1\Sigma(2,3,5)$, bypassing $Q(p,q,r)$. A method for doing this (for certain $M(p,q,r)$) may be found in [P, Theorem II.3.5 and Lemma II.3.2]. For instance, there is a free action of $SL(2,5)$ on

$$\{-7;(o,0,0,0);\underbrace{(7,4),(7,4),\ldots,(7,4)}_{12}\}\text{ with quotient }\Sigma(2,3,35)!$$

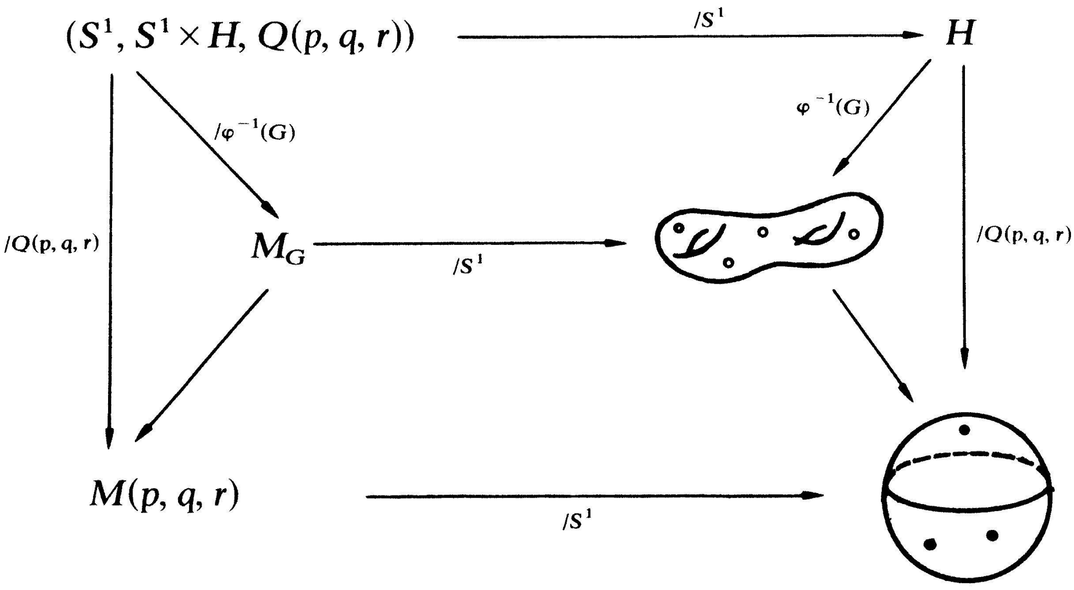
On the other hand, this method is unlikely to go much further in proving vanishing results, except in other special cases. There are always surjections of $Q(p, q, r)$ onto finite groups with torsion free kernels, so that the corresponding cover will be an S^1 bundle over a surface, but these finite groups will not be as well understood as $Q(2, 3, 5)$, and analysis of intermediate covers will be difficult. To prove the theorem, then, we should not study finite groups acting on surfaces, but infinite groups acting on the hyperbolic plane. Our problem will be translated into a question concerning triangle groups. These groups have been extensively studied, and we will use known results to complete the proof.

3. Proof of the theorem

To use induction, we need a surjection $Q(p, q, r) \to F$, where F is a non-hyperelementary finite group. This is provided by the following lemma, the proof of which we defer to section 4.

LEMMA. *The hyperbolic triangle group* $Q(p, q, r)$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, *admits a surjection onto a non-hyperelementary finite group* F.

Assuming the lemma, we now prove the theorem. Let $M(p, q, r)$ be a non-sufficiently large Seifert manifold with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Using the lemma, and letting G be a hyperelementary subgroup of F, we get a diagram of groups as in section 2. We now lift the circle action up to the cover corresponding to the center of $\Pi_1 M(p, q, r)$, where it splits as $(S^1, S^1 \times H)$, [CR, Theorem 7.3]. Here H is the hyperbolic plane, on which $Q(p, q, r)$ acts as a hyperbolic triangle group:



We see that $\varphi^{-1}(G)$ is a discrete subgroup of $PSL(2,\mathbf{R})$ acting on H, and also a subgroup of $Q(p,q,r)$ of index equal to $[F:G]$.

The point is that M_G will almost always be sufficiently large. If M_G were not sufficiently large, $\varphi^{-1}(G)$ would itself be a triangle group. So we are led to ask: When can a triangle group contain another triangle group as a proper subgroup of finite index?

Suppose $Q(p,q,r)$ contains $Q(a,b,c)$ with index N. It is well known (see, for instance, [M]) that the area of a fundamental domain for $Q(p,q,r)$ is equal to $2\pi\left(1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}\right)$, twice the area of a triangle with angles $\frac{\pi}{p},\frac{\pi}{q},\frac{\pi}{r}$. Since all torsion in $Q(p,q,r)$ is conjugate to powers of the generators q_1, q_2, q_3, we see that each a, b, c must divide one of p, q, r. Furthermore, we have the equality

$$\left(1-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right)=N\left(1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}\right).$$

This automatically rules out most possibilities, e.g., when p, q, and r are all at least 6. These facts also imply the crucial point: that $Q(p,q,r)$ can contain only finitely many triangle groups.

In fact, Greenberg [G] has determined all inclusion relations between triangle groups. His result, slightly rephrased, is:

THEOREM 3B [G]: *The following inclusions, and those that follow from them, are all inclusion relations between elliptic triangle groups.*

(1) $Q(m,m,n)\subset Q(2,m,2n)$ *index* 2

(2) $Q(2,n,2n)\subset Q(2,3,2n)$ *index* 3

(3) $Q(3,n,3n)\subset Q(2,3,3n)$ *index* 4

(4) $Q(4,4,5)\subset Q(2,4,5)$ *index* 6

(5) $Q(7,7,7)\subset Q(2,3,7)$ *index* 24.

We might add that Knapp ([K], Theorem 2.3 and Figure 4) has given a nice interpretation of (1), (2), and (3) in terms of assembling triangles associated to the larger triangle group to form a basic triangle in a tesselation associated to the subgroup. Also, it is not hard to find a triangle with angles $\frac{\pi}{7},\frac{\pi}{7},\frac{\pi}{7}$ made up from 24 triangles in the standard tesselation associated to $Q(2,3,7)$. This does not seem possible for case (4).

If $Q(p, q, r)$ contains no triangle group, then M_G will be sufficiently large, since either M_G/S^1 is a surface of positive genus, or else M_G/S^1 is S^2 but there are at least 4 exceptional orbits. Thus, elements of $\mathrm{Wh}(\Pi_1 M(p, q, r))$ are forced to transfer to $0 \in \mathrm{Wh}(\Pi_1 M_G) = 0$, for *all* subgroups G, and we have proved the theorem in these cases. The only property of F that we are using is that it is not hyperelementary.

If $Q(p, q, r)$ contains a triangle group $Q(a, b, c)$, we must first show that $\mathrm{Wh}(\Pi_1 M(a, b, c)) = 0$. Perhaps the easiest way to do this is to notice that inclusion between triangle groups give a transitive ordering, and that no triangle group contains a proper subgroup isomorphic to itself, by area considerations. Thus, given $Q(p, q, r)$, we may look at the *finite* set S of all triangle groups properly contained in $Q(p, q, r)$. Minimal elements of S contain no triangle groups, so the previous paragraph applies. By induction on the number of triangle groups contained in a member of S, we see that the necessary Whitehead groups vanish, and the theorem is proved.

Notice that we were fortunate in our choice of $Q(p, q, r) \to Q(2, 3, 5)$ in section 2. For instance, using (1) and (2) above, we see that $Q(15, 15, 15)$ is contained in $Q(2, 3, 30)$ with index 6, and $[Q(2, 3, 5): D_5] = 6$, but in our example the group which arose was $Q(2, 2, 6, 30)$.

Finally, this method fails for the cases $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. The problem is that the angles of a Euclidean triangle do not determine its area. Indeed, $Q(2, 3, 6)$ contains subgroups isomorphic to itself of index $3^i 4^j$, for all i, j, not to mention subgroups isomorphic to $Q(3, 3, 3)$ of arbitrarily large index. Similar remarks apply to $Q(2, 2, 4)$. However, as we have mentioned, these cases are included in the large class of manifolds handled by Farrell and Hsiang.

4. Proof of the lemma

In [F], Fox produced surjections of $Q(a, b, c)$ onto finite permutation groups F for all values of a, b, c, with the added property that the kernel is torsion free. (We are changing p, q, r to a, b, c to conform to the notation in [F].) Given the simple presentation of $Q(a, b, c)$, and the fact that every torsion element is conjugate to a power of some q_i, Fox's proof reduces to finding one permutation, A, of order a, another permutation, B, of order b, such that their product AB has order c. In most cases it will turn out that the finite group generated by A and B in Fox's construction is non-hyperelementary. Notice that we have obvious epimorphisms $Q(a, b, c) \to Q(a', b', c')$ when $a' \mid a$, $b' \mid b$, $c' \mid c$. This will allow us to assume that a, b, and c are prime powers. Also, we may assume $a \leq b \leq c$.

There are two main cases:

(A) There are at least two different primes occurring in prime decompositions of a, b, c. By the above remarks, we may assume that a, b, c are prime, not all equal (except when 2 occurs, since we rule out dihedral groups. In this case, we can assume the numbers are either 2, odd, odd, or 2, 4, odd.)

(1) a, b, c are odd:

(a) $a<b=c$. This is the first part of Fox's case III. Let $A=(u_1\cdots u_a)$, $B=(u_1\cdots u_a\cdots u_b)$. Then $AB=(u_1u_3\cdots u_{a-2}u_au_{a+1}\cdots u_bu_2u_4\cdots u_{a-1})$, another b-cycle. Thus, there is a well-defined homomorphism $Q(a,b,c)\xrightarrow{\varphi}\mathscr{S}_b$ given by $q_1\mapsto A$, $q_2\mapsto B$, $q_3\mapsto(AB)^{-1}$. Suppose that the image of φ, say F, is hyperelementary. We have

$$\begin{array}{ccccccccc} & & & & Q(a,b,c) & & & & \\ & & & & \downarrow{\scriptstyle\varphi} & & & & \\ 1 & \longrightarrow & \mathbf{Z}_k & \longrightarrow & F & \xrightarrow{\pi} & P & \longrightarrow & 1, \end{array}$$

where P is a p-group and $(k,p)=1$. Since $\varphi(q_1)$ has order a in F, $\pi\varphi(q_1)\neq 1$ implies that $a=p$. Similarly, $\pi\varphi(q_2)\neq 1$ implies $b=p$. Since $Q(a,b,c)$ is generated by any two of q_1, q_2, q_3, either F is cyclic (which we will see is not the case) or $\pi\varphi(q_1)=1$, $\pi\varphi(q_2)\neq 1$, $\pi\varphi(q_3)\neq 1$, so that $A\in\mathbf{Z}_k$, a normal abelian subgroup of F. Thus $BAB^{-1}\in\mathbf{Z}_k$ and $[A,BAB^{-1}]=1$. But we easily see that $BAB^{-1}A=(u_1u_3\cdots)$, while $ABAB^{-1}=(u_1u_2\cdots)$. Thus, F could not have been hyperelementary. The proofs in the other cases will be quite similar to this.

(b) $a\leq b<c$. This is again case III of [F]. Following Fox, let $m=\left[\dfrac{c-1}{b+1}\right]$. Then $m\geq 1$ and $m(b+1)+1\leq c\leq(m+1)(b+1)-1$.

(i) $m(b+1)+1\leq c\leq m(a+b-2)+1$. We can find $s_1,\ldots,s_m$, odd, satisfying $1\leq s_i\leq a-2$ and $\sum s_i=m(a+b-1)+1-c$. Let $p_1,\ldots,p_{m+1}$, $u_1^1,\ldots,u_{a-s_1-1}^1,\ldots,u_1^m,\ldots,u_{a-s_m-1}^m$, $v_1^1,\ldots,v_{s_1}^1,\ldots,v_1^m,\ldots,v_{s_m}^m$, $w_1^1,\ldots,w_{b-s_1-1}^1,\ldots,w_1^m,\ldots,w_{b-s_m-1}^m$, be c distinct symbols. Then

$$A=\prod_{i=1}^{m}(p_iu_1^iu_2^i\cdots u_{a-s_i-1}^iv_1^iv_2^i\cdots v_{s_i}^i)$$

and

$$B=\prod_{i=1}^{m}(v_1^iv_2^i\cdots v_{s_i}^iw_{b-s_i-1}^i\cdots w_2^iw_1^ip_{i+1})$$

have order a and b respectively, with AB a cycle of order c (clever fellow, that Fox!). By the same reasoning we have already used, if the group generated by A and B is hyperelementary, AB lies in the cyclic normal subgroup $\mathbf{Z}_k$. But so does $BA = A^{-1}ABA$, and we compute

$$ABBA = (p_2 u_2^2 \cdots)$$
$$BAAB = (p_2 v_3^1 \cdots).$$

(ii) $m(a+b-2)+2 \leq c \leq (m+1)(b+1)-1$. Again, following Fox, write $c - m(a+b-2)-2 = t(a-1)+s$, $1 \leq s < a-1$, and we can write $t-1 = q(b-2)+r$, $0 \leq r < b-2$. Let

$$B = (v_1^1 \cdots v_s^1 w_{b-s-1}^1 \cdots w_1^1 p_2) \cdot \prod_{i=2}^{m} (v_1^i w_{b-2}^i \cdots w_1^i p_{i+1})$$

$$A = (p_1 u_1^1 \cdots u_{a-s-1}^1 v_1^1 \cdots v_s^1) \cdot \prod_{i=2}^{m} (p_i u_1^i \cdots u_{a-2}^i v_1^i) \cdot A',$$

where

$$A' = (p_{m+1} p_2^0 \cdots p_a^0) \cdot \prod_{i=m-q+1}^{m} (A_1^i \cdots A_{b-2}^i) \cdot A_1^{m-q} \cdots A_r^m,$$

and $A_j^i = (w_j^i p_{j2}^i \cdots p_{ja}^i)$, where we are using c distinct symbols. Then AB is a single c cycle. Again, we must show $[AB, BA] \neq 1$:

$$\text{Compute } ABBA = (p_2 u_2^2 \cdots), \quad BAAB = \begin{cases} (p_2 v_4^1 \cdots) & s \geq 5 \\ (p_2 w_{b-4}^1 \cdots) & s = 3 \\ (p_2 u_1^1 \cdots) & s = 1. \end{cases}$$

(2) $a = 2$:

(a) $b < c$, both odd. This case II of [F]. Write $c - a + 1 = qb + r$, $0 \leq r < b$, $q \geq 1$. Let

$$A = \prod_{i=1}^{q-1} (u_b^i u_1^{i+1}) \cdot \prod_{j=1}^{r} (u_j^1 v_j) \cdot (u_b^q w_2)$$

$$B = \prod_{i=1}^{q} (u_1^i \cdots u_b^i), \quad \text{where} \quad u_1^1, \ldots, u_b^1, u_1^2, \ldots, u_b^2, \ldots,$$

$u_1^q, \ldots, u_b^q, v_1, \ldots, v_r, w_2$ are c distinct symbols.

Then AB is a single c-cycle. Since all three primes are distinct, the group F generated by A and B can be of the form $1 \to \mathbf{Z}_k \to F \to P \to 1$ only if it is actually cyclic, hence abelian. But $AB = (w_2 u_b^q \cdots)$ and $BA = (w_2 u_1^q \cdots)$.

(b) $b = c$, both odd, case I of [F]. Let $A = (u_1 v_1)(w_1 w_2)$ and $B = (u_1 w_1 \cdots w_{c-1})$, so that $AB = (u_1 w_2 \cdots w_{c-1} v_1)$. If $A \in \mathbf{Z}_k$, so is BAB^{-1}. But $A(BAB^{-1}) = (u_1 v_1 w_2 w_3 w_1)$, whereas $(BAB^{-1})A = (u_1 w_1 w_3 w_2 v_1)$ if $c \geq 5$. If $c = 3$, $Q(2, 3, 3)$ is the (non-hyperelementary) tetrahedral group of order 12.

(c) $bc \equiv 0 \pmod 2$.

(i) $Q(2, 3, 4)$. This is the octahedral group, isomorphic to $\mathcal{S}_4$, of order 24. The only possibility is $1 \to \mathbf{Z}_3 \to \mathcal{S}_4 \to \text{order } 8 \to 1$, but 3-cycles in $\mathcal{S}_4$ do not generate normal subgroups.

(ii) $Q(2, 4, 5)$, case I of [F]. Let $A = (u_1 v_1)$, $B = (u_1 w_1 w_2 w_3)$, so $AB = (u_1 w_1 w_2 w_3 v_1)$. If A, B generate a hyperelementary group, AB and BA must lie in the cyclic, normal subgroup. But $ABBA = (v_1 w_2)(w_1 w_3)(u_1)$, $BAAB = (u_1 w_2)(v_1)(w_1 w_3)$.

(iii) $Q(2, 4, c)$, $c \geq 7$, odd, case II of [F]. Write $c - 1 = 4q + r$, $0 \leq r < 4$, $q \geq 1$. Let

$$A = \prod_{i=1}^{q-1} (u_4^i u_1^{i+1}) \cdot \prod_{j=1}^{r} (u_j^1 v_j) \cdot (u_4^q w_2)$$

and

$$B = \prod_{i=1}^{q} (u_1^i \cdots u_4^i),$$

so that AB is a c-cycle. Again, we must show that $[AB, BA] \neq 1$. Compute that $BAAB$ fixes w_2, whereas $ABBA$ does not. This completes the proof of the lemma in case (A).

(B) The numbers a, b, c are powers of the same prime, say p. Fox's permutations work in most of these cases, but not when $a = b = c = p$, in which case the group produced is just $\mathbf{Z}_p \times \mathbf{Z}_p$. Instead, we will use different permutations.

(1) p odd, $\neq 3$. It will suffice to consider the case $Q(p, p, p)$. Let $B = (1\,2 \cdots p)(p+1 \cdots 2p)$ and $A = (2\,3\,p+1\,4\,5 \cdots p)(1\,p+2\quad p+3 \cdots (2p-1)\,2p)$. Then B and A have order p, and $AB = (1\,3\,5 \cdots (p-2)\ p\,(p+2) \cdots (2p-3)\,(2p-1))\,(2\,(p+1)\,(p+3) \cdots (2p-2)\,2p\,4\,6 \cdots (p-1))$, also order p.

(a) $p = 3k + 2$. $AB^2 = (1\quad p+1\quad p+4 \quad\cdots\quad 2p-1\quad 4\quad 7\quad \cdots\quad p-1\quad p+2\quad p+5 \cdots 2p\ p+3\ p+6 \cdots 2p-2)(2\,5\,8 \cdots p\ 3\,6\,9 \cdots p-2)$, the product of a $(p+k+1)$-cycle and a $(2k+1)$-cycle. The argument is completed by observing that $(B^2A)(AB^2) = (1\,6 \cdots)$, whereas $(AB^2)(B^2A) = (1\ p+7 \cdots)$.

(b) $p=3k+1$. $AB^2=(1\ \ p+1\ \ p+4 \cdots 2p\ \ p+3\ \ p+6 \cdots 2p-1\ \ 4\,7 \cdots p\ \ 3\,6 \cdots p-1\ \ p+2\ \ p+5 \cdots 2p-2)(2\,5\,8 \cdots p-2)$, the product of a $(p+2k+1)$-cycle and a k-cycle. Again, we compute that $(B^2A)(AB^2)=(1\,6 \cdots)$, whereas $(AB^2)(B^2A)=(1\ p+7 \cdots)$.

(2) $p=3$. The above permutations do not yield an obvious solution. Since we do not need the case $Q(3,3,3)$ in our theorem, we will content ourselves with covering all other cases, i.e., showing that $Q(3,3,9)$ surjects onto a non-hyperelementary finite group. In this case, we use case III of [F]. We let $A=(1\,2\,3)(4\,5\,6)$, $B=(3\,7\,4)(6\,8\,9)$, so that $AB=(1\,2\,3\,7\,5\,6\,8\,9\,4)$. Now observe that $x=BA^2B^2A=(1)(8\,5\,6\,4\,3\,7\,2)(9)$, order 7, $y=B^{-1}(BA^2B^2A)B=A^2B^2AB$ also has order 7, but $xy=(2\,4 \cdots)$, whereas $yx=(2\,8 \cdots)$.

(3) $p=2$. Consider $Q(2,4,8)$. Using 12 symbols, let $A=(4\,5)(8\,9)(11\,1)$, $B=(1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12)$, with $AB=(1\,2\,3\,5\,6\,7\,9\,10)(4\,11\,12\,8)$. The element $y=(AB^2AB^3AB^3)^2=(2\,7\,12)(3\,9\,8)(4\,10\,5)$ has order 3, and $B^{-1}yB=(2\,12\,7)(3\,9\,8)(1\,6\,11)$. If the group generated by A and B is hyperelementary, these elements of order 3 must lie in a cyclic subgroup. The elements y and $B^{-1}yB$ do commute, but the cannot be different powers of the same permutation – this fact is self-evident after a few moments reflection about cycle structures in $\mathcal{S}_{12}$. This completes the proof of the lemma.

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University of Chicago

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