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Cohomologically complete and pseudoconvex domains

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§0. Introduction

An open subset D of a Stein manifold may be studied from a number of different viewpoints. On the one hand, there are the essentially geometric conditions of q-pseudoconvexity in various forms and, on the other hand, the more analytic aspects of Dolbeault cohomology and extendibility of holomorphic functions or cohomology classes (cf. [1], [2], [3], [4], and [13]).

The object of this paper is to link some of these concepts by an indirect but, we believe, rapid method using certain natural cohomology classes derived in §1. It should be mentioned that these classes can be represented directly by means of certain forms (cf. [5] and [11]) although we shall not use such representatives explicitly in this paper. After briefly reviewing the various notions of q-pseudoconvexity in §2 the "test classes" are used in §3 to compare these notions (Theorem 3.8). In particular, we obtained the interesting conclusion that if D has C^2 boundary with $H^p(D, \mathcal{O})=0$ for $p>q$ then D is q-complete; this gives an answer to a special case of a conjecture proposed in 1962 by Andreotti and Grauert [2].

The test classes also turn out to be useful in discussing extendibility questions for cohomology classes (considered in [3], for example) and such questions are studied in §3, §4 and §5. An interesting application of these methods gives a lower bound on the number of analytic functions needed to define an analytic subvariety just "touching" $\bar{D}$ in terms of extendibility of cohomology classes (Theorem 5.7).

§1. The test classes

Suppose R is a commutative ring with identity and $f_1, f_2, \ldots, f_n \in R$. If we regard these elements as a column vector $f \in R^n = \bigwedge^1 R^n$, then we may define the

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Koszul complex $K^{\cdot}(f)$ by

$$K^{\cdot}(f) = \bigwedge^{\cdot}(R^n), \text{ with differential } dw = f \wedge w.$$

There is an analogous construction for a sheaf of rings and n sections thereof to give a Koszul complex of sheaves $\mathcal{K}^{\cdot}(f)$.

Now suppose M is an n-dimensional Stein manifold with structure sheaf $\mathcal{O}$. If $x \in M$ it is always possible [6, Satz 1, p. 91] to find n holomorphic functions $f_1, f_2, \ldots, f_n$ such that

$$\{x\} = \{y \in M \text{ s.t. } \ f_1(y) = f_2(y) = \cdots = f_n(y) = 0\}.$$

If these functions give local coordinates at x (which can always be arranged), then we shall denote them by $z_1, z_2, \ldots, z_n$. It is easy to see that $\mathcal{K}^{\cdot}(f)$ is exact on M-$\{x\}$ so we may break it up into short exact sequences:—

$$0 \to \mathcal{L}_s(f) \to \mathcal{K}^{n-s-1}(f) \to \mathcal{L}_{s-1}(f) \to 0 \quad \text{for} \quad 0 \leq s \leq n-2,$$

where

$$\mathcal{L}_{-1}(f) \equiv \mathcal{O} \cong \mathcal{K}^n(f) \quad \text{and} \quad \mathcal{L}_{n-2}(f) \equiv \mathcal{K}^0(f) = \mathcal{O}.$$

The connecting homomorphisms of the corresponding long exact sequences on cohomology may be composed to give

$$\alpha_k(f) : H^p(M - \{x\}, \mathcal{L}_s(f)) \to H^{p+k+1}(M - \{x\}, \mathcal{L}_{s+k+1}(f)) \quad \text{for} \quad k \geq 0$$

and in particular (if $n \geq 2$)

$$\alpha_s(f) : \Gamma(M, \mathcal{O}) \cong \Gamma(M - \{x\}, \mathcal{O}) = H^0(M - \{x\}, \mathcal{L}_{-1}(f)) \to H^{s+1}(M - \{x\}, \mathcal{L}_s(f)).$$

Hence we obtain *test classes* $\alpha_s(g, f) \equiv \alpha_s(f)(g)$ in $H^{s+1}(M - \{x\}, \mathcal{L}_s(f))$ for any holomorphic function g on M and for $0 \leq s \leq n-2$. If $g = 1$ then we shall denote these test classes by $\alpha_s(x, f)$.

If D is an open subset of M then there is (see [12]) a Stein manifold $E(D)$ called the *envelope of holomorphy* of D which contains D and is characterised by the property than every holomorphic function on D extends uniquely to $E(D)$. $E(D)$ is not necessarily a subset of M because sheeting can occur and in general $E(D)$ will be a Riemann domain over M with projection $\pi : E(D) \to M$, say. This notation will be retained for the rest of the paper. A reason for calling the cohomology classes $\alpha_s(g, f)$ "test classes" is given by the following theorem:

1.1. THEOREM. *Suppose $x \in M-D$ and g is chosen with germ at x not contained in the ideal generated by the germs of the functions $f_1, f_2, \ldots, f_n$ (for example if $g(x) \neq 0$). Then*

$$x \in \pi(E(D)) \Leftrightarrow \alpha_0(g, f)\,|_D \neq 0$$

where $\alpha_0(g, f)\,|_D$ means the image of $\alpha_0(g, f)$ under restriction:

$$H^1(M-\{x\}, \mathcal{L}_0(f)) \to H^1(D, \mathcal{L}_0(f)).$$

Proof. Follows exactly the argument in [5, Theorem 2.1] where the case $g=1$ is proved. □

If $H^p(D, \mathcal{O})=0$ for $1 \leq p \leq n-1$ then it is easy to show that $H^1(D, \mathcal{L}_0(f))=0$ for any f defining a point $x \in M-D$ and hence (as in [5]) we may deduce that D is Stein. This suggests that we could use the test classes $\alpha_s(x, f)\,|_D$ for $0 \leq s \leq n-2$ to measure how far D is from being Stein. It follows from the above theorem that whether $\alpha_0(x, f)\,|_D$ vanishes or not depends only on x and D, and is independent of choice of f. We will show that the same is true of $\alpha_s(x, f)\,|_D$ if we restrict to the case where $f_1, f_2, \ldots, f_n$ form a local coordinate system near x. So suppose f is a general collection of functions defining $x \in M-D$ and z is a collection which give local coordinates near x. Then:

1.2 PROPOSITION. *For any s, $\alpha_s(x, f)\,|_D = 0 \Rightarrow \alpha_s(x, z)\,|_D = 0$.*

Proof. If $\mathcal{J}$ denotes the ideal sheaf of $\{x\}$ and $\mathcal{O}^{n\times n}$ the sheaf of germs of $n\times n$ matrices of holomorphic functions then

$$\begin{array}{ccc} \mathcal{O}^{n\times n} & \longrightarrow & \mathcal{J}^n \\ \Cup & & \Cup \\ A & \longmapsto & Az \end{array}$$

is surjective, and hence the corresponding map on sections over M is surjective too by Cartan's theorem B. This shows that there is a matrix A of holomorphic functions on M such that $Az=f$. A may be regarded as a sheaf homomorphism $A:\mathcal{O}^n \to \mathcal{O}^n$ and hence gives rise to $A:\bigwedge^{\cdot}\mathcal{O}^n \to \bigwedge^{\cdot}\mathcal{O}^n$. $A(z\wedge w)=Az\wedge Aw=f\wedge Aw$ so $A:\mathcal{K}^{\cdot}(z)\to\mathcal{K}^{\cdot}(f)$ is a map of complexes. Less obvious is that A also induces a map $A^*:\mathcal{K}^{\cdot}(f)\to\mathcal{K}(z)$ which in some sense is adjoint to A. A^* is defined by $*A^*=A^t*$ where A^t means A transposed and $*$ is the Hodge $*$-operator. We observe that $A^*:\mathcal{K}^n(f)\to\mathcal{K}^n(z)$ is just the identity and hence, from our definitions, the induced map $A^*: H^{s+1}(M-\{x\}, \mathcal{L}_s(f)) \to H^{s+1}(M-\{x\}, \mathcal{L}_s(z))$ takes $\alpha_s(x, f)$ to $\alpha_s(x, z)$. The same is true over D i.e. $\alpha_s(x, z)|_D = A^*(\alpha_s(x, f)|_D)$. Thus $\alpha_s(x, f)|_D = 0$ gives $\alpha_s(x, z)|_D = 0$ as required. □

1.3. COROLLARY. *Whether $\alpha_s(x, z)|_D$ vanishes or not is independent of choice of functions $z_1, z_2, \ldots, z_n$.* □

This corollary allows us to simplify notation and write $\alpha_s(x)$ instead of $\alpha_s(x, z)$ in the following definition and for the rest of this paper.

1.4. DEFINITION. The *projected q-envelope of holomorphy* of D is the set

$$E_q(D) = D \cup \{x \in M - D \text{ s.t. } \alpha_q(x)|_D \neq 0\} \quad \text{for} \quad 0 \leq q \leq n-2,$$

and we say D is *α-q-complete* if and only if $D = E_q(D)$.

By construction $\alpha_q(x)|_D = 0 \Rightarrow \alpha_{q+1}(x)|_D = 0$ so we have

$$D \subseteq E_{n-2}(D) \subseteq E_{n-3}(D) \subseteq \cdots \subseteq E_1(D) \subseteq E_0(D) = \pi(E(D)),$$

where the last equality follows from Theorem 1.1. Unfortunately it is not clear that $E_q(D)$ are open except in case $q = 0$ although this does appear to be true in all cases we have checked.

In what follows we shall compare α-q-completeness with other related properties of open subsets of Stein manifolds including cohomological completeness and q-pseudoconvexity.

§2. Hartogs figures and q-pseudoconvexity

Let Δ denote the open unit polydisc in $\mathbb{C}^n$ and define for each integer q in the range $1 \leq q \leq n-1$ the *qth Hartogs figure H_q* by

$$H_q = \{z \in \Delta \text{ s.t. } |z_j| < \tfrac{1}{2} \quad \text{for} \quad j > q\} \cup \bigcup_{j=1}^{q} \{z \in \Delta \text{ s.t. } \tfrac{1}{2} < |z_j| < 1\}.$$

Since each of the sets in this union is Stein it follows that $H^p(H_q, \mathcal{S}) = 0$ for $p > q$ and any coherent analytic sheaf $\mathcal{S}$. By using this cover and comparing coefficients in Laurent series expansions it is easy to show that $H^p(H_q, \mathcal{O}) = 0$ for $1 \leq p < q$ also (see [2, p. 218]).

2.1. PROPOSITION. $E_{q-1}(H_q)=\Delta$ (*yet* $E_q(H_q)=H_q$).

Proof. Writing just $\mathscr{L}_s$ instead of $\mathscr{L}_s(z)$ we have $H^{q+1}(H_q,\mathscr{L}_q)=0$ since $\mathscr{L}_q$ is coherent and so $\alpha_q(x)|_{H_q}=0$ for all $x\in\mathbb{C}^n-H_q$, Thus $E_q(H_q)=H_q$.

On the other hand, consider the exact sequence

$$H^s(H_q,\mathscr{K}^{n-s-1})\to H^s(H_q,\mathscr{L}_{s-1})\to H^{s+1}(H_q,\mathscr{L}_s).$$

$\mathscr{K}^{n-s-1}$ is merely the direct sum of some number of copies of $\mathcal{O}$ so $H^s(H_q,\mathscr{K}^{n-s-1})=0$ for $1\le s<q$ and we can conclude that $\alpha_s(x)|_{H_q}=0$ implies $\alpha_{s-1}(x)|_{H_q}=0$. Thus $E_s(H_q)=E_{s-1}(H_q)$ for $1\le s<q$ and hence $E_{q-1}(H_q)=E_0(H_q)$. However, an application of the Cauchy integral formula shows that the envelope holomorphy of H_q is Δ so $E_0(H_q)=\Delta$ by Theorem 1.1. □

In the above proof, $E_{q-1}(H_q)=E_0(H_q)$ was deduced from $H^p(H_q,\mathcal{O})=0$ for $1\le p<q$ so the same can be said for H_q as in the following definition.

2.2. DEFINITION. An open subset H_q of M is said to be a *general q-Hartogs figure* if and only if $H^p(H_q,\mathcal{O})=0$ for $1\le p<q$. D is said to be *Hartogs q-complete* if and only if for any general $q+1$-Hartogs figure $H_{q+1}\subseteq D$, also $\pi(E(H_{q+1}))\subseteq D$ (or, equivalently, $E_0(H_{q+1})\subseteq D$). Cf. [7, Definition 2.1, p. 35].

We now recall the basic definitions of q-pseudoconvexity and q-completeness.

If $x_0\in\partial D$, the boundary of D, it is always possible to find a neighbourhood U of x_0 and a *defining function* of class C^2 $\phi\colon U\to\mathbb{R}$ such that

$$D\cap U=\{x\in U \text{ s.t. } \phi(x)<\phi(x_0)\}.$$

If ϕ can be chosen to be non-singular at x_0 we say that D has C^2 boundary at x_0 and if this is true at all points of ∂D we say D has C^2 boundary. If ϕ is a non-singular defining function near x_o and $z_1,z_2,\ldots,z_n$ are local coordinates centred on x_0 then we consider

$$(\mathscr{H}\phi)(x_0)\equiv\left(\frac{\partial^2\phi(x_0)}{\partial z_i\partial\bar{z}_j}\right)^n_{i,j=1},\qquad\text{the } \textit{complex Hessian}.$$

The Levi form $(\mathscr{L}\phi)(x_0)$ is defined to be the restriction of $(\mathscr{H}\phi)(x_0)$ to the holomorphic tangent space to ∂D at x_0. It is easy to check that the number of negative, zero, and positive eigenvalues of $(\mathscr{L}\phi)(x_0)$ are independent of choice of ϕ and coordinates. Thus they are invariants of D near x_0 and we will denote them by $n(x_0)$, $z(x_0)$, and $p(x_0)$ respectively. If D has C^2 boundary we say D is *weakly* (*resp. strongly*) *q-pseudoconvex* if and only if $n(x) \leq q$(resp. $n(x)+z(x) \leq q$) for all $x \in \partial D$ (cf. [4, p. 209]).

If D does not have C^2 boundary we can still make the following definition [13, p. 433]; D (which only need be a complex analytic manifold) is said to be *q-complete* if and only if we can find an *exhaustion function* $\phi : D \to \mathbb{R}$ of class C^2 such that

(1) $\{x \in D \text{ s.t. } \phi(x) \leq c\}$ is compact for all $c \in \mathbb{R}$.

(2) $(\mathcal{H}\phi)(x)$ has at least $n-q$ positive eigenvalues for all $x \in D$.

Finally we say [13, p. 443] D is *cohomologically q-complete* if and only if $H^p(D, \mathcal{S}) = 0$ for all $p > q$ and all coherent analytic sheaves $\mathcal{S}$.

Using Theorem 1.1 and well-known results (e.g. [9, esp. Ch IX] or [10]) the following are equivalent:

(a) D is 0-complete,

(b) D is cohomologically 0-complete,

(b′) $H^p(D, \mathcal{O}) = 0$ for $p > 0$,

(c) D is α-0-complete,

(d) D is Hartogs 0-complete.

Indeed, they are all equivalent to:

(e) D is Stein,

and in this case we say D is a domain of holomorphy as if $M = \mathbb{C}^n$. If D has C^2 boundary then [10, p. 49] (a) is equivalent to:

(a′) D is weakly 0-pseudoconvex.

§3 Inextendibility of cohomology classes

We now want to investigate what happens if we replace 0 by q in the statement at the end of the previous section. We start with some well-known or simple implications.

3.1. THEOREM. *If D is q-complete then it is also cohomologically q-complete.*

Proof. [2, p. 250]. The notation there is slightly different. □

3.2. LEMMA. *If $H^p(D, \mathcal{O}) = 0$ for $p > q$ then $H^s(D, \mathcal{L}_{s-1}) = 0$ for $s > q$.*

Proof. The conclusion is vacuous unless $q \leq n-2$ so we may assume that this is the case and, in particular, that $H^{n-1}(D, \mathcal{O}) = 0$. The long exact cohomology sequence of $0 \to \mathcal{L}_s \to \mathcal{K}^{n-s-1} \to \mathcal{L}_{s-1} \to 0$ together with the hypotheses show that

$$H^s(D, \mathscr{L}_{s-1}) \cong H^{s+1}(D, \mathscr{L}_s) \quad \text{for} \quad s > q$$

Thus $H^s(D, \mathscr{L}_{s-1}) \cong H^{n-1}(D, \mathscr{L}_{n-2}) = H^{n-1}(D, \mathcal{O}) = 0 \quad \text{for} \quad s > q.$ □

3.3. PROPOSITION. *If $H^p(D, \mathcal{O}) = 0$ for $p > q$ then D is α-q-complete.*

Proof. Lemma 3.2 implies $H^{q+1}(D, \mathscr{L}_q) = 0$ so $\alpha_q(x)|_D = 0$ for all $x \in M - D$. Hence (see Definition 1.4) $E_q(D) = D$ so D is α-q-complete. □

3.4. PROPOSITION. *If D is α-q-complete then it is Hartogs q-complete.*

Proof. As remarked just before Definition 2.2 we know that for any general $q+1$-Hartogs figure H_{q+1}, $E_0(H_{q+1}) = E_q(H_{q+1})$. Hence, if $H_{q+1} \subseteq D$,

$$\pi(E(H_{q+1})) = E_0(H_{q+1}) = E_q(H_{q+1}) \subseteq E_q(D) = D$$

where the last equality is by assumption. By Definition 2.2, D is Hartogs q-complete. □

To proceed further we discuss inextendibility of cohomology classes. Suppose $x \in \partial D$ and $\mathscr{S}$ is a coherent analytic sheaf. Following Andreotti and Norguet [3, p. 199], we introduce

$$H^p(D, x, \mathscr{S}) \equiv \lim_{\rightarrow} H^p(D \cap U, \mathscr{S})$$

$$H^p_+(D \cup \{x\}, \mathscr{S}) \equiv \lim_{\rightarrow} H^p(D \cup U, \mathscr{S})$$

$$H^p_x(\mathscr{S}) \equiv \lim_{\rightarrow} H^p(U, \mathscr{S})$$

where the direct limits are taken over all open neighbourhoods U of x. Notice that

$$H^p_x(\mathscr{L}) = \begin{cases} \mathscr{S}_x & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

There are restriction maps:

$$\mu : H^p(D, \mathscr{S}) \rightarrow H^p(D, x, \mathscr{S})$$

$$\rho : H^p_+(D \cup \{x\}, \mathcal{S}) \to H^p(D, \mathcal{S})$$

$$\lambda : H^p_x(\mathcal{L}) \to H^p(D, x, \mathcal{L}) \quad \text{etc.}$$

and an exact sequence (the *Mayer-Vietoris sequence*):

$$\ldots \to H^p_+(D \cup \{x\}, \mathcal{S}) \to H^p(D, \mathcal{S}) \oplus H^p_x(\mathcal{S}) \to H^p(D, x, \mathcal{S}) \to H^{p+1}_+(D \cup \{x\}, \mathcal{S}) \to \ldots$$

induced by the usual Mayer-Vietoris sequence. We say that $\xi \in H^p(D, \mathcal{S})$ is *extendible* through $x \in \partial D$ if and only if $\xi \in \operatorname{im} \rho$. From the Mayer-Vietoris sequence we see that if $p > 0$ then ξ is extendible if and only if $\mu(\xi) = 0$. We say that D is a *q-domain of holomorphy* if and only if, for all $x \in \partial D$, there exists $p \leq q$ and a cohomology class $\xi \in H^p(D, \mathcal{O})$ which does not extend through x (cf. [1, p. 138]). For $q = 0$ this is just the usual definition of domain of holomorphy but for general q we shall see (Example 4.3) that there are many domains which are not q-domains of holomorphy for any q.

Inextendibility of cohomology classes is closely related to pseudoconvexity. Suppose $x \in \partial D$ and ϕ is a C^2 defining function in a neighbourhood U of x. The following is due to Andreotti and Grauert:

3.5. PROPOSITION. *If the complex Hessian $(\mathcal{H}\phi)(y)$ has at least $k(\geq 2)$ negative eigenvalues for all $y \in U$ then there are arbitrarily small open neighbourhoods Q of x with*

(1) $H^p(D \cap Q, \mathcal{O}) = 0$ *for* $1 \leq p < k-1$
(2) *The restriction map $\Gamma(Q, \mathcal{O}) \to \Gamma(D \cap Q, \mathcal{O})$ surjective.*

Proof. [2, Proposition 12, p. 222]. □

3.6. COROLLARY 1. *If D has C^2 boundary and it is Hartogs q-complete then it is also weakly q-pseudoconvex.*

Proof. If $x \in \partial D$ and ϕ is a C^2 defining function, non-singular at x, whose Levi form has at least $q+1$ negative eigenvalues then by considering instead the defining function $\psi = -e^{-c\phi}$ for c sufficiently large we can arrange that $\mathcal{H}(\psi)$ has at least $q+2$ negative eigenvalues near x. Thus, if Q is chosen as in the proposition, $D \cap Q$ is a general $q+1$-Hartogs figure by (1) with $\pi(E(D \cap Q)) \not\subseteq D$ by (2). This is in contradiction with D being Hartogs q-complete so our original assumption must be false and consequently $n(x) \leq q$ as required. □

3.7. COROLLARY 2. *If D has C^2 boundary and is a q-domain of holomorphy then D is also weakly q-pseudoconvex.*

Proof. If D is not weakly q-pseudoconvex then there is a point $x \in \partial D$ with $n(x) > q$. Then, arguing as in the first corollary, Proposition 3.5 shows that (combining statements (1) and (2))

$$\lambda : H^p_x(\mathcal{O}) \to H^p(D, x, \mathcal{O}) \quad \text{is surjective for } p \leq q.$$

From the Mayer-Vietoris sequence it follows immediately that

$$\rho : H^p_+(D \cup \{x\}, \mathcal{O}) \to H^p(D, \mathcal{O}) \text{ is surjective for } p \leq q$$

so, by definition, D is not a q-domain of holomorphy. □

To complete this section we collect the results to prove the main theorem:

3.8. THEOREM. *Consider the following statements for an open subset D of a Stein manifold:*

(a) *D is q-complete*
(a′) *D is weakly q-pseudoconvex (if ∂D is of class C^2)*
(b) *D is cohomologically q-complete*
(b′) *$H^p(D, \mathcal{O}) = 0$ for $p > q$*
(c) *D is α-q-complete*
(d) *D is Hartogs q-complete*
(e) *D is a q-domain of holomorphy*

Then, (a)⇒(b)⇒(b′)⇒(c)⇒(d) and if, D has C^2 boundary (a), (a′), (b), (b′), (c), and (d) are all equivalent and follow from (e).

Proof. (a)⇒(b) is Theorem 3.1, (b)⇒(b′) is trivial, (b′)⇒(c) is Proposition 3.3, and (c)⇒(d) is Proposition 3.4. If D has C^2 boundary then (d)⇒(a′) and (e)⇒(a′) are the corollaries to Proposition 3.5. Hence, the only missing implication is (a′)⇒(a). This follows from the same argument used in case $q = 0$ [10, p. 50] and an exhaustion function for D can be constructed from log(distance to ∂D). □

§4. Counterexamples

We now discuss some counterexamples to missing implications in Theorem 3.8 when D does not have C^2 boundary. The most interesting examples of such domains are complements of analytic subvarieties.

4.1. PROPOSITION. *Suppose V is an analytic subvariety of M with irreducible components V_j such that, for any open Stein subset U of M, $H^p(U-V_j, \mathcal{O})=0$ for $p\neq 0$ or d_j-1, where d_j is the codimension of V_j. (This happens, for example, if the V_j are geometrically complete [8, Theorem 23]). Let $D=M-V$. Then*

$$E_q(D)=M-\bigcup_{d_j\leq q+1} V_j$$

Proof. If $x\in V_j$ and $d_j\leq q+1$ then $\alpha_q(x)|_{M-V_j}$ is zero since $H^{q+1}(M-V_j, \mathcal{L}_q)=0$ by lemma 3.2. Thus, further restricting to D also gives zero and we conclude that $E_q(D)\subseteq M-\bigcup_{d_j\leq q+1} V_j$. To show the reverse inclusion suppose $x\in V_k-\bigcup_{d_j\leq q+1} V_j$ for some k with $d_k>q+1$. Choose a Stein neighbourhood U of x which avoids $\bigcup_{d_j\leq q+1} V_j$. Then $H^p(U-V_k, \mathcal{O})=0$ for $1\leq p<q+1$ so as in Proposition 2.1 we may conclude that $E_q(U-V_k)=E_0(U-V_k)$. But by the Riemann removable singularities theorem $E_0(U-V_k)=U$ so, as $D\supseteq U-V_k$,

$$E_q(D)\supseteq E_q(U-V_k)=E_0(U-V_k)=U \ \ x \text{ as required. } \square$$

4.2. EXAMPLE. Let $M=\mathbb{C}^4$, $V=V_1\cup V_2$, and $D=M-V$, where

$$V_1=\{z\in\mathbb{C}^4 \text{ s,t, } z_1=z_2=0\}, \qquad V_2=\{z\in\mathbb{C}^4 \text{ s.t. } z_3=z_4=0\}.$$

Then, by [13, pp. 445–447], $H^2(D,\mathcal{O})\neq 0$ so D is not cohomologically 1-complete. By Proposition 4.1, however, we see that $E_1(D)=D$ so D is α-1-complete (and thus Hartogs 1-complete too). Also, since V may be defined by 3 functions (z_1z_3, z_2z_4, $z_2z_3+z_1z_4$), it follows [13, Proposition 2.6, p. 435] that D is 2-complete. Thus D is α-1-complete but nothing better than 2-complete and cohomologically 2-complete. In the notation of Theorem 3.8 (c)$\not\Rightarrow$(b′).

This example also provides a negative answer to the following: Is it true that if D is a q-domain of holomorphy then, for all $x\in\partial D$ and $\xi\in H^p(D,\mathcal{O})$ $p>q$, ξ is extendible through x? The answer is yes if $q=0$ or D has C^2 boundary for then $H^p(D,\mathcal{O})=0$ for $p>q$. However, we claim that, in this example, D is a 1-domain of holomorphy but that there is an element of $H^2(D,\mathcal{O})$ which does not extend through the origin. To see that D is a 1-domain of holomorphy consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^1(D,\mathcal{K}^2)\cdot & \xrightarrow{\Phi} & H^1(D,\mathcal{L}_0) & \xrightarrow{\delta} & H^2(D,\mathcal{L}_1) \\
\downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\
H^1(D,x,\mathcal{K}^2) & \rightarrow & H^1(D,x,\mathcal{L}_0) & \rightarrow & H^2(D,x,\mathcal{L}_1)
\end{array} \qquad \text{for any } x\in V.$$

Now since $E_0(D)=\mathbb{C}^4$ yet $E_1(D)=D$ we have $\alpha_0(x)|_D\neq 0$ yet $\alpha_1(x)|_D=0$. But $\alpha_1(x)|_D=\delta(\alpha_0(x)|_D)$ so there is an element $\xi\in H^1(D,\mathcal{K}^2)$ such that $\phi(\xi)=\alpha_0(x)|_D$. For any polydisc neighbourhood Δ of x $\alpha_0(x)|_{D\cap\Delta}$ is again non-zero by Theorem 1.1 and the Riemann removable singularities theorem. Therefore $\mu(\xi)\neq 0$ and so, identifying $\mathcal{K}^2$ with $\mathcal{O}^6$, one of the components of ξ does not extend through x. Thus D is a 1-domain of holomorphy. On the other hand the Mayer-Vietoris theorem shows that

$$H^2(D,\mathcal{O})\cong H^3(\mathbb{C}^4-\{0\},\mathcal{O})$$

and there is a cohomology class ($\alpha_2(0)$ for example) in $H^3(\mathbb{C}^4-\{0\},\mathcal{O})$ which does not extend through the origin. It is clear that the corresponding class in $H^2(D,\mathcal{O})$ also fails to extend.

4.3. EXAMPLE. Finally we give an example of an open set D in $\mathbb{C}^n$ $(n\geq 2)$ which is not a q-domain of holomorphy for any q. Simply take $D=\mathbb{C}^n-\bar{\Delta}$ where $\bar{\Delta}$ is the closed unit polydisc. Let $x=(1,0,\ldots,0)$. Then D has flat boundary near x so $H^p(D,x,\mathcal{O})=0$ for all $p\geq 1$ and hence every class $\xi\in H^p(D,\mathcal{O})$ extends through x. Moreover every analytic function on D extends across $\bar{\Delta}$ by Hartogs' theorem.

This D does not have C^2 boundary but the corners can clearly be smoothed without destroying the example.

§5 The analytic touching number

There are some further results about inextendibility of cohomology classes which can be deduced by these methods.

5.1. DEFINITION. If $x\in\partial D$ define the *inextendibility index* $k(x)$ by

$$k(x)=\min\{q \text{ s.t. } H^q_x(\mathcal{O})\to H^q(D,x,\mathcal{O}) \text{ is not surjective}\}.$$

$k(x)$ is well-defined by the following:

5.2. LEMMA. *If* $\mu(\alpha_0(x)|_D)=0$ *in* $H^1(D,x,\mathcal{L}_0)$ (*for example if* $H^q(D,x,\mathcal{O})=0$ *for all* $q\geq 1$) *then* $H^0_x(\mathcal{O})\to H^0(D,x,\mathcal{O})$ *is not surjective, i.e.* $k(x)=0$.

Proof. Apply Theorem 1.1 to $U\cap D$ where U is any neighbourhood of x so small that $\alpha_0(x)|_{U\cap D}=0$. $\square$

5.3. PROPOSITION. *If D has C^2 boundary at $x \in \partial D$ then $k(x) \geq n(x)$.* (*Cf.* [3, *théorème* 1, *p.* 203]).

Proof. Follows from Proposition 3.5 along the same lines as the proof of corollary 3.7. □

The following question clearly has a positive answer if $q = 0$: Is it true that in the highest non-vanishing cohomology group $H^q(D, \mathcal{O})$ there is a class which does not extend through at least one point of ∂D? If D has C^2 boundary we can give a positive answer for arbitrary q:

5.4. THEOREM. *Suppose that $H^p(D, \mathcal{O}) = 0$ for $p > q$ but that $H^q(D, \mathcal{O}) \neq 0$. Suppose also that there is a point $x \in \partial D$ such that $k(x) \geq q$ (which is always the case, by Theorem 3.8 and Proposition 5.3, if D has C^2 boundary). Then there exists $\xi \in H^q(D, \mathcal{O})$ which does not extend through x.*

Proof. This is clearly true if $q = 0$ so suppose $q \geq 1$. By Lemma 3.2 $H^{q+1}(D, \mathcal{L}_q) = 0$ so in the following commutative diagram with exact rows

$$\begin{array}{ccccc}
H^q(D, \mathcal{K}^{n-q-1}) & \xrightarrow{\phi} & H^q(D, \mathcal{L}_{q-1}) & \rightarrow & H^{q+1}(D, \mathcal{L}_q) = 0 \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\
H^q(D, x, \mathcal{K}^{n-q-1}) & \rightarrow & H^q(D, x, \mathcal{L}_{q-1}) & \rightarrow & H^{q+1}(D, x, \mathcal{L}_q)
\end{array}$$

it is possible to find $\xi \in H^q(D, \mathcal{K}^{n-q-1})$ s.t. $\phi(\xi) = \alpha_{q-1}(x)|_D$. We claim that $\mu(\xi) \neq 0$ and, by the Mayer-Vietoris sequence, this will show ξ does not extend through x and so prove the theorem since $\mathcal{K}^{n-q-1}$ is merely a direct sum of copies of $\mathcal{O}$. To prove the claim we will show $\mu(\alpha_{q-1}(x)|_D) \neq 0$. If $\mu(\alpha_0(x)|_D) = 0$ then, by Lemma 5.2, $k(x) = 0$. But, by hypothesis, $k(x) \geq q$ so for $q \geq 1$ (as supposed) we may conclude that $\mu(\alpha_0(x)|_D) \neq 0$. Thus, for some $s \geq 1$, $\mu(\alpha_{s-1}(x)|_D) \neq 0$ yet $\mu(\alpha_s(x)|_D) = 0$. Then the exact sequence

$$H^s(D, x, \mathcal{K}^{n-s-1}) \rightarrow H^s(D, x, \mathcal{L}_{s-1}) \rightarrow H^{s+1}(D, x, \mathcal{L}_s)$$

shows $H^s(D, x, \mathcal{K}^{n-s-1}) \neq 0$ and hence $H^s(D, x, \mathcal{O}) \neq 0$. Therefore $s \geq k(x) \geq q$. Hence $\mu(\alpha_{q-1}(x)|_D) \neq 0$ as required. □

We now introduce another invariant motivated by the following:

5.5. PROPOSITION. *Suppose D has C^2 boundary at $x \in \partial D$. Then there exists a neighbourhood U of x and s analytic functions $g_1, g_2, \ldots, g_s$ on U for $s =$*

$n(x)+z(x)+1$ $(=n-p(x))$ *such that, writing*

$$V=\{y\in U \text{ s.t. } g_1(y)=g_2(y)=\cdots=g_s(y)=0\},$$

we have $V\cap\bar{D}=\{x\}$ *i.e.* V *touches* $\bar{D}$ *at* x.

Proof. [4, Prop. 6, p. 209]. □

5.6. DEFINITION. The *analytic touching number* $a(x)$ of $x\in\partial D$ is the least s for which we can find s analytic functions $g_1, g_2, \ldots, g_s$ near x which define a subvariety touching $\bar{D}$ at x as in the above proposition.

Proposition 5.5 shows that $a(x)\le n(x)+z(x)+1$. A lower bound is provided by the following:

5.7. THEOREM. $a(x)\ge k(x)+1$.

Proof. Suppose $g_1, g_2, \ldots, g_s$ are as in Definition 5.6. We suppose without loss of generality that U, their common domain of definition, is Stein. Then, by [8, Theorem 23, p. 154],

$$H^p(U-V,\mathcal{O})=0 \quad \text{for} \quad p\ge s.$$

Thus, by lemma 3.2, $H^s(U-V, x, \mathcal{L}_{s-1})=0$ and hence $\mu(\alpha_{s-1}(x)\,|_D)=0$. It follows, as in the proof of Theorem 5.4, that $k(x)\le s-1$. □

The inequalities proved in this section for D with a C^2 boundary may be summarised as

$$n(x)\le k(x)\le a(x)-1\le n(x)+z(x).$$

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