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The first k-invariant, Quillen's space BG^+ and the construction of Kan and Thurston

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1. Outline

In this paper we shall use the interpretation of group cohomology in terms of crossed n-fold extensions [7].

Let Y be a connected CW-complex with a single 0-cell; we assume $\pi_k(Y)=0$ for $1<k<n$, $n\geqq 2$ (nothing is assumed for $n=2$). Then we have an exact sequence

$$e_Y : 0 \longrightarrow \pi_n(Y) \xrightarrow{\partial_n} \pi_n(Y, Y^{n-1}) \xrightarrow{\partial_{n-1}} \pi_{n-1}(Y^{n-1}, Y^{n-2}) \xrightarrow{\partial_{n-2}}$$

$$\cdots \xrightarrow{\partial_2} \pi_2(Y^2, Y^1) \xrightarrow{\partial_1} \pi_1(Y^1) \longrightarrow \pi_1(Y) \longrightarrow 1$$

where each ∂_j is obtained from the exact homotopy sequences of the corresponding pairs of spaces in the obvious way. Now the action of $\pi_1(Y^1)$ on $\pi_2(Y^2, Y^1)$ (on $\pi_2(Y, Y^1)$ in case $n=2$) and that of $\pi_1(Y)$ on the remaining groups turn e_Y into a crossed n-fold extension [7§3] (that the relative π_2 is a crossed $\pi_1(Y^1)$-module is due to J. H. C. Whitehead, see [6 p. 39]). In view of the main Theorem of [7§7], e_Y represents a class $[e_Y]\in H^{n+1}(\pi_1(Y), \pi_n(Y))$.

THEOREM 1. *The class $[e_Y]$ is the first (non-trivial) k-invariant of Y.*

This offers an answer to a question in [2 p. 301]. Furthermore, we obtain, as a consequence, a description of the n'th cohomology group which recovers the description in [5 p. 75] (Theorem 14.1'):

COROLLARY. *Let A be a $\pi_1(Y)$-module, and let the crossed n-fold extension*

$$0\rightarrow \pi_n(Y)\rightarrow C_{n-1}\rightarrow \cdots \rightarrow C_1\rightarrow F\rightarrow \pi_1(Y)\rightarrow 1$$

represent $[e_Y]$, with $C_{n-2},\ldots, C_1$, F free (e_Y itself is such a representative). Then *the group* $H^n(Y, A)$ (*local coefficients*) *is the cokernel of the induced map* $\mathrm{Hom}_G(C_{n-2}, A) \to \mathrm{Hom}_G(C_{n-1}, A)$ ($C_0 = F$); *here* $G = \pi_1(Y)$ *resp.* $G = F$ (*in low dimensions*), *and* $\mathrm{Hom}_G(F, A) = \mathrm{Der}(F, A)$.

We shall use Theorem 1 to determine the first k-invariant of Quillen's $(\ )^+$-construction: Let G be a group, and let E be a perfect normal subgroup. Now E has a *universal central extension* [11 p. 43]

$$e : 0 \to H_2(E) \to X \to E \to 1.$$

Using the universal property of e, we extend the action of G on E to a unique action of G on X, turning X into a crossed G-module, whence we obtain a crossed 2-fold extension

$$e_{(G,E)} : 0 \longrightarrow H_2(E) \longrightarrow X \xrightarrow{\partial} G \longrightarrow Q \longrightarrow 1$$

where $Q = G/E$. In view of the main Theorem in [7§7], $e_{(G,E)}$ represents a class $[e_{(G,E)}] \in H^3(Q, H_2(E)) = H^3(\pi_1(BG^+), \pi_2(BG^+))$. Here BG^+ is Quillen's space [10].

THEOREM 2. *The class* $[e_{(G,E)}]$ *is the first k-invariant of* BG^+.

By a recent result of Kan and Thurston [8], to any connected space Y there may be associated a group G together with a map $\chi : BG \to Y$ such that (i) the kernel E of the induced map $G \to \pi_1(Y)$ is perfect, and (ii) the map χ extends to a (possibly weak) h-equivalence $BG^+ \to Y$. Theorem 2 shows how the first k-invariant of Y is determined by G and E. Note that $[e_{(G,E)}]$ may be non-zero. As to the vanishing of $[e_{(G,E)}]$ we have

THEOREM 3. *The class* $[e_{(G,E)}]$ *is zero, and* Q (*resp.* G) *acts trivially on* $H_2(E)$ *if and only if the induced map* $H_2(E) \to H_2(G)$ *is a split injection* (*i.e. admits a left inverse*).

A particular example arises in algebraic K-theory: If Λ is a ring with unit, in view of the above, the class of

$$e_\Lambda : 0 \to K_2(\Lambda) \to St(\Lambda) \to GL(\Lambda) \to K_1(\Lambda) \to 1$$

is the first k-invariant of $\mathrm{BGL}(\Lambda)^+$. For commutative rings, this was also announced in [3].

THEOREM 4. *The class* $[e_\Lambda] \in H^3(K_1(\Lambda), K_2(\Lambda))$ *is zero.*

I am indebted to K. Dennis who provided me with the crucial argument for the proof of Theorem 4. In fact, he has shown that the induced map $H_2(E(\Lambda)) \to H_2(GL(\Lambda))$ is a split injection [4 Cor. 8]. Hence Theorem 4 is a consequence of Theorem 3. I am also grateful to K. Brown; he read an earlier version of the paper and conjectured Theorem 3.

2. Proofs

Proof of Theorem 1. It is known [1] that the first (non-trivial) k-invariant of Y is the Eilenberg- Mac Lane invariant $l_Y \in H^{n+1}(\pi_1(Y), \pi_n(Y))$ [5]. If $\mathbf{C}$ is the crossed standard resolution of $\pi_1(Y)$ [7§9], we may lift the identity map of $\pi_1(Y)$ to

$$\begin{array}{cccccccccccc}
\mathbf{C}: \cdots \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & \cdots & \longrightarrow & F & \longrightarrow & \pi_1(Y) & \longrightarrow 1 \\
& \downarrow{\scriptstyle \xi} & & \downarrow & & & & \downarrow & & \| & \\
e_Y : 0 \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(Y, Y^{n-1}) & \longrightarrow & \cdots & \longrightarrow & \pi_1(Y^1) & \longrightarrow & \pi_1(Y) & \longrightarrow 1.
\end{array}$$

Now ξ is Eilenberg–Mac Lane's cocycle. This, together with the main Theorem in [7 §7] proves Theorem 1.

Proof of Theorem 2. In view of Theorem 1, the Eilenberg–Mac Lane class resp. the first k-invariant l_{BG^+} is represented by

$$\hat{e} : O \to \pi_2(BG^+) \to \pi_2(BG^+, (BG^+)^1) \to \pi_1((BG^+)^1) \to \pi_1(BG^+) \to 1.$$

But $\hat{e}$ is clearly equivalent to

$$e^+ : 0 \to \pi_2(BG^+) \to \pi_2(BG^+, BG) \to \pi_1(BG) \to \pi_1(BG^+) \to 1,$$

since there is an obvious morphism $(1, \ldots, 1) : \hat{e} \to e^+$ of crossed 2-fold extensions. We complete the proof by showing that $\hat{e}$ and $e_{(G, E)}$ are essentially the same crossed 2-fold extension:

The group $\pi_2(BG^+, BG)$ is $\pi_1(Z)$, Z the homotopy fibre of $BG \to BG^+$. Since Z is acyclic (this is just the universal property of the $(\ )^+$-construction), $H_1(\pi_1(Z)) = 0 = H_2(\pi_1(Z))$. It follows from Lemma 2 in [9] (p. 215) that

$$0 \to \pi_2(BG^+) \to \pi_2(BG^+, BG) \to E \to 1$$

is the universal central extension of E.

Proof of Theorem 3. By the Theorem in [7 §10], $[e_{(G,E)}]$ is zero if and only if there is a group extension $1 \to X \xrightarrow{j} D \to Q \to 1$ together with a morphism $(1, \alpha) : (X, D, j) \to (X, G, \partial)$ of crossed modules inducing the identity map of Q. It follows that $[e_{(G,E)}]$ is zero if and only if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_2(E) \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "i"] & 1 \\ 0 \arrow[r] & H_2(E) \arrow[r] & D \arrow[r] & G \arrow[r] & 1, \end{tikzcd} \tag{*}$$

such that conjugation in D induces the crossed G-structure on X; here i denotes the inclusion $E \subset G$.

It is clear that Q (resp. G) acts trivially on $H_2(E)$ if $H_2(E) \to H_2(G)$ is a split injection. Hence we may assume that Q acts trivially on $H_2(E)$. Consider now the commutative diagram

$$\begin{tikzcd} H^2(G, H_2(E)) \arrow[r] \arrow[d] & \mathrm{Hom}\,(H_2(G), H_2(E)) \arrow[d] \\ H^2(E, H_2(E)) \arrow[r] & \mathrm{Hom}\,(H_2(E), H_2(E)) \end{tikzcd} \tag{**}$$

with the obvious maps. Inspection of (**) shows that we have a diagram (*) if and only if $H_2(E) \to H_2(G)$ is a split injection. Hence the condition of the Theorem is necessary. Now, if we have a diagram (*), conjugation in D induces an action of G on X; since this one extends the action of G on E given by conjugation, it agrees with that one we used to define the crossed G-structure on X, as there is only one such G-action on X. This last fact is a consequence of the universal property of the universal central extension. It follows that the condition of the Theorem is also sufficient.

Remark. The same arguments, applied to the Kan–Thurston construction, may be used to prove the following classical result (which seems to be folk-lore):

THEOREM. *For a given connected space Y, the Hurewicz map $\pi_2(Y) \to H_2(Y)$ is a split injection if and only if $\pi_1(Y)$ acts trivially on $\pi_2(Y)$ and if the first k-invariant of Y is zero.*

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