Extremal eigenvalue problems defined on conformal classes of compact Riemannian manifolds.

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1. Introduction

The aim of this paper is to extend our recent results on eigenvalue problems for certain classes of membranes [3] to conformal classes of compact Riemannian manifolds. We refer to [1] for the definitions and properties of Riemannian manifolds needed here. Let $\mathcal{M}$ be a compact smooth (C^∞) n-dimensional manifold. We shall assume that $n \geq 2$. Denote by $x=(x^1, \ldots, x^n)$ the points of $\mathcal{M}$, by dV the volume element and by $G(x)=(g_{ij}(x))_1^n$ the metric matrix. Consider a new metric on $\mathcal{M}$ given by the matrix $\hat{G}=(\hat{g}_{ij}(x))_1^n$. Assume that this metric is conformal to the given metric. That is

$$\hat{g}_{ij}(x)=\varphi^2(x)g_{ij}(x), \qquad i, j=1, \ldots, n. \tag{1.1}$$

Assume first that φ is a positive smooth function. Denote by $\hat{\Delta}$ the corresponding Laplacian to the matric $\hat{G}$. Consider the eigenvalue problem.

$$\hat{\Delta}u+\mu u=0. \tag{1.2}$$

Denote by

$$0=\mu_0(\varphi)<\mu_1(\varphi)\leq\mu_2(\varphi)\leq \cdots \tag{1.3}$$

the corresponding eigenvalues of $\hat{\Delta}$. The eigenvalues $\mu_k(\varphi)$, $k=0, 1, \ldots,$ are characterized by the min-max principle applied to the Rayleigh ratio

$$\int \varphi^{n-2} \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\, dV \Big/ \int \varphi^n u^2\, dV. \tag{1.4}$$

Here $G^{-1}=(g^{ij})_1^n$. Using this characterization one can define $\{\mu_k(\varphi)\}_0^\infty$ for any non-negative bounded measurable function φ. The precise definition of $\mu_k(\varphi)$ is given in the next section. Denote by C the following set of functions

$$0 \le m(\xi) \le \varphi(\xi) \le M(\xi) \tag{1.5}$$

$$\int \varphi^n \, dV = W, \tag{1.6}$$

where m and M are bounded measurable function. The corresponding set of Riemannian manifolds has an obvious geometric meaning. To see this meaning let us consider the case where m and M are positive and constant and φ is a smooth function. Then the condition (1.5) states that the metrics $\hat{G}$ and G are equivalent. That is

$$md(x, y) \le \hat{d}(x, y) \le Md(x, y), \tag{1.7}$$

where $d(x, y)$ and $\hat{d}(x, y)$ are the distances between the points x and y according to the metrics G and $\hat{G}$ respectively. The condition (1.6) means that the manifold $\hat{\mathcal{M}}$ has a fixed volume W.

By C^* we denote the set of functions φ which belong to C and satisfy the condition

$$(M(\xi) - \varphi(\xi))(\varphi(\xi) - m(\xi)) = 0 \tag{1.8}$$

almost everywhere. A set of corresponding Riemannian manifolds to C^* is a set of non-smooth conformal manifolds to $\mathcal{M}$ which have almost everywhere either the minimal or the maximal distortion and a fixed volume W. The main result of this paper is

THEOREM 1. *Let $\mathcal{M}$ be a compact smooth manifold of dimension $n \ge 2$. Let C and C^* be nonempty sets of functions defined by the conditions* (1.5), (1.6) *and* (1.8), (1.6) *respectively. Let $F(\xi_1, \ldots, \xi_p)$ be a continuous function on R_+^p increasing with respect to each of its arguments. Then*

$$\inf_{C} F(\mu_1(\varphi), \ldots, \mu_p(\varphi)) = \inf_{C^*} F(\mu_1(\psi), \ldots, \mu_p(\psi)). \tag{1.9}$$

The proof of this theorem is given in the next section. In the last section we study in detail the problem $\min \mu_1(\varphi)$, $\varphi \in C$ in the case where $\mathcal{M}$ is a two dimensional sphere S^2 and the functions $m(\xi)$ and $M(\xi)$ are constant. We show that the minimum in question is achieved for a certain function $\varphi^* \in C^*$ which is characterized almost completely. Finally if $m = 0$ then this minimum is completely determined.

2. Proof of the main result

Let φ be a positive smooth function. Then according to the classical Courant principle $\mu_p(\varphi)$ is characterized as follows:

$$\mu_p(\varphi) = \max_{f_0,\ldots,f_{p-1}} \min_u \int_{\mathcal{M}} \varphi^{n-2} \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} dV \Big/ \int \varphi^n u^2 \, dV, \tag{2.1}$$

where u satisfies the orthogonality conditions

$$\int_{\mathcal{M}} \varphi^n f_j f \, dV = 0, \qquad j = 0, \ldots, p-1. \tag{2.2}$$

However, to prove Theorem 1 one needs another characterization of $\mu_p(\varphi)$. It was named by Pólya and Schiffer as the convoy Principle [7] (see also [2] for the version stated here).

The Convoy Principle

Let φ be a positive smooth function. Let $f_0, \ldots, f_p$ be continuous and differentiable functions, satisfying the conditions

$$\int_{\mathcal{M}} f_i f_j \varphi^n \, dV = \delta_{ij}, \qquad i, j = 0, 1, \ldots, p. \tag{2.3}$$

Let $A(\varphi, f_0, \ldots, f_p) = (a_{ij})_0^p$ be the matrix

$$a_{ij} = \int_{\mathcal{M}} \varphi^{n-2} \left(\sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} \frac{\partial f_i}{\partial x^\alpha} \frac{\partial f_j}{\partial x^\beta} \right) dV. \tag{2.4}$$

Denote by $\mu_0(\varphi, f_0, \ldots, f_p), \ldots, \mu_p(\varphi, f_0, \ldots, f_p)$ the eigenvalues of $A(\varphi, f_0, \ldots, f_p)$ arranged in the increasing order. Then

$$\mu_k(\varphi) = \inf_{f_0,\ldots,f_p} \mu_k(\varphi, f_0, \ldots, f_p), \quad k = 0, \ldots, p. \tag{2.5}$$

The infimum is achieved for the eigenfunctions $u_0 = 1, u_1, \ldots, u_p$ of (1.2).

For an arbitrary non-negative measurable function $\varphi(\not\equiv 0)$ we let (2.5) be the definition of $\mu_p(\varphi)$. It is easy to show that (2.5) holds for any $k<p$ for this choice of φ.

Proof of Theorem 1. First we show that

$$F(\mu_1(\varphi, f_0, \ldots, f_p), \ldots, \mu_p(\varphi, f_0, \ldots, f_p)) \geq \inf_{C^*} F(\mu_1(\psi), \ldots, \mu_p(\psi)) \tag{2.6}$$

for a function φ of the form

$$\varphi^n = \sum_{i=1}^{q} \beta_i \psi_i^n, \ \beta_i \geq 0, \ \psi_i \in C^*, \ i=1, \ldots, q, \ \sum_{i=1}^{q} \beta_i = 1. \tag{2.7}$$

Let χ_S be a characteristic function of the set $S \subset \mathcal{M}$. Thus $\psi \in C^*$ can be represented

$$\psi = m + \chi_S(M-m) \tag{2.8}$$

Clearly

$$\psi^n = m^n + \chi_S(M^n - m^n) \tag{2.9}$$

So S satisfies the condition

$$\int_S (M^n - m^n)\, dV = W - \int_{\mathcal{M}} m^n\, dV \tag{2.10}$$

Let $S_1, \ldots, S_q$ be the sets corresponding to the functions $\psi_1, \ldots, \psi_q$. Thus we can find a partition $T_1, \ldots, T_N$ of $\mathcal{M}$ such that the following condition holds

$$\bigcup_{i=1}^{N} T_i = \mathcal{M}, \ T_i \cap T_j = \phi \quad \text{for} \quad i \neq j, \ i, j = 1, \ldots, N, \tag{2.11}$$

each T_i is a measurable set and for a given positive ε

$$\int_{T_i} dV < \varepsilon, \qquad i = 1, \ldots, N \tag{2.12}$$

Furthermore

$$\chi_{S_i} = \sum_{j=1}^{N} \alpha_{ij} \chi_{T_j}, \qquad \alpha_{ij}(1-\alpha_{ij}) = 0, \quad i = 1, \ldots, q, \ j = 1, \ldots, N \tag{2.13}$$

Let

$$\theta_j = m^n + c_j \chi_{T_j}(M^n - m^n) \tag{2.14}$$

where c_j is defined by the equality

$$c_j \int_{T_j} (M^n - m^n)\, dV = W - \int_{\mathcal{M}} m^n\, dV) \tag{2.15}$$

Thus $\theta_j^{1/n}$ satisfies (1.6) and

$$\varphi^n = \sum_{j=1}^{N} \alpha_j \theta_j = m^n + \sum_{j=1}^{N} \alpha_j c_j \chi_{T_j}(M^n - m^n),$$

$$\alpha_j \geq 0, \quad j = 1, \ldots, N, \sum_{j=1}^{N} \alpha_j = 1 \tag{2.16}$$

The assumption that $m \leq \varphi \leq M$ is equivalent to the inequalities

$$\alpha_j \leq c_j^{-1}, \qquad j = 1, \ldots, N \tag{2.17}$$

Let $f_0, \ldots, f_p$ be smooth functions satisfying the condition (2.3) Consider the quadratic form

$$\sum a_{ij}(f_0, \ldots, f_p)\xi_i\xi_j = \int_{\mathcal{M}} \varphi^{n-2} \left[\sum_{k,l=1}^{n} g^{kl} \frac{\partial}{\partial x^k}\left(\sum_{i=0}^{p} \xi_i f_i\right) \frac{\partial}{\partial x^l}\left(\sum_{j=0}^{p} \xi_j f_j\right)\right] dV \tag{2.18}$$

Let

$$\tilde{\varphi} = \sum_{j=1}^{N} \alpha_j [m^{n-2} + c_j \chi_{T_j}(M^{n-2} - m^{n-2})] \tag{2.19}$$

As $0 \leq (n-2)/n < 1$ from the concavity of $\xi^{(n-2)/n}$ we deduce

$$\varphi^{n-2} = \left[\left(1 - \sum_{j=1}^{N} \alpha_j c_j \chi_{T_j}\right) m^n + \left(\sum_{j=1}^{N} \alpha_j c_j \chi_{T_j}\right) M^n\right]^{(n-2)/n}$$

$$\geq \left(1 - \sum_{j=1}^{N} \alpha_j c_j \chi_{T_j}\right) m^{n-2} + \left(\sum_{j=1}^{N} \alpha_j c_j \chi_{T_j}\right) M^{n-2} = \tilde{\varphi} \tag{2.20}$$

Let

$$\tilde{a}_{ij}(\tilde{\varphi}, f_0, \dots, f_p) = \int_{\mathcal{M}} \tilde{\varphi}\left(\sum_{k,l=1}^{n} g^{kl} \frac{\partial f_i}{\partial x^k}\frac{\partial f_j}{\partial x^l}\right) dV, \quad i, j = 0, \dots, p. \tag{2.21}$$

Then the inequality (2.20) implies

$$\sum_{i,j=0}^{p} a_{ij}(\varphi, f_0, \dots, f_p)\xi_i\xi_j \geq \sum_{i,j=0}^{p} \tilde{a}_{ij}(\tilde{\varphi}, f_0, \dots, f_p)\xi_i\xi_j \tag{2.22}$$

Denote by $\tilde{\mu}_0 \leq \dots \leq \tilde{\mu}_p$ the eigenvalues of the matrix $\tilde{A}(\tilde{\varphi}, f_0, \dots, f_p) = (\tilde{a}_{ij}(\tilde{\varphi}, f_0, \dots, f_p))_0^n$, Now the inequality (2.22) implies [5, Ch. 10]

$$\mu_i(\varphi, f_0, \dots, f_p) \geq \tilde{\mu}_i, \qquad i = 0, \dots, p. \tag{2.23}$$

Consider $(p+1)(p+2)$ equations in unknowns $\beta_1, \dots, \beta_N$

$$\sum_{s=1}^{N} \beta_s \int_{\mathcal{M}} [m^n + c_s\chi_{T_s}(M^n - m^n)]f_if_j = \delta_{ij},$$

$$\sum_{s=1}^{N} \beta_s \int_{\mathcal{M}} [m^{n-2} + c_s\chi_{T_s}(M^{n-2} - m^{n-2})]\left(\sum_{k,l=1}^{n} g^{kl}\frac{\partial f_i}{\partial x^k}\frac{\partial f_j}{\partial x^l}\right) dV$$
$$= \tilde{a}_{ij}(\tilde{\varphi}, f_0, \dots, f_p), \qquad i, j = 0, \dots, p \tag{2.24}$$

Demand also $\sum_s^N \beta_s = 1$ and $\beta_s \leq c_s^{-1}$, $s = 1, \dots, N$. Note that we have an admissible solution $\alpha_1, \dots, \alpha_N$. Suppose that the ε in (2.12) is small enough. Then of course N must be large. Assume that $N > (p+1)(p+2)+1$. In that case there exists a solution $\alpha_1^*, \dots, \alpha_N^*$ such that at most $(p+1)(p+2)+1$ coordinates α_s^*, do not satisfy $\alpha_s^*(c_s^{-1} - \alpha_s^*) = 0$.

Let

$$(\psi^*)^n = \sum_{s=1}^{N} \alpha_s^*\theta_s^* = \sum_{s=1}^{N} \alpha_s^*[m^n + c_s\chi_{T_s}(M^n - m^n)] \tag{2.25}$$

Thus $(M - \psi^*)(\psi^* - m) \neq 0$ on a set S whose measure is less than $[(p+1)(p+2)+1]\varepsilon$.

Furthermore

$$(\psi^*)^{n-2} \neq \sum_{s=1}^{N} \alpha_s^*(m^{n-2} + c_s\chi_{T_s}(M^{n-2} - m^{n-2})]$$

on a set S. Thus, given $\varphi, f_0, \ldots, f_p$ and $\varepsilon_1 > 0$ fixed we can find ε small enough such that

$$\mu_i(\varphi, f_0, \ldots, f_p) \geq \mu_i(\psi^*, f_0, \ldots, f_p) - \varepsilon_i, \qquad i = 0, \ldots, p. \tag{2.26}$$

Furthermore we can find φ^* in the set C^* such that $\varphi^* = \psi^*$ on $\mathcal{M} - S$. This means that

$$\mu_i(\varphi, f_0, \ldots, f_p) \geq \mu_i(\psi^*, f_0, \ldots, f_p) - \epsilon_1, \qquad i = 0, \ldots, p, \tag{2.27}$$

which proves (2.6) for φ of the form (2.7). This in return implies (2.6) for any φ and fixed $f_0, \ldots, f_p$ satisfying the conditions (2.3). From the characterization (2.5) we deduce

$$F(\mu_1(\varphi), \ldots, \mu_p(\varphi)) \geq \inf_{C^*} F(\mu_1(\psi), \ldots, \mu_p(\psi)).$$

This of course is equivalent to (1.9). The proof of the theorem is completed.

3. Compact surfaces conformally equivalent to the two dimensional sphere

Let us consider two dimensional compact Riemannian manifolds, i.e. $n = 2$. As in the Rayleigh ratio $\varphi^{n-2} = 1$ we have that $\mu_p(\varphi)$ are the eigenvalues of the equation

$$\Delta u + \mu \varphi^2 u = 0 \tag{3.1}$$

where Δ is the original Laplacian. Let $\mathcal{M}$ be the unit sphere S^2.

$$S^2 = \left\{ x \mid x = (x^1, x^2, x^3), \sum_{i=1}^{3} (x^i)^2 = 1 \right\}. \tag{3.2}$$

Assume that $0 \leq m < M$ are constants. In that case we demonstrate that $\min_C \mu_1(\varphi)$ is achieved for a certain function φ^* which is characterized in the sequel. This is done by using the symmetrization principle. See [8] and [4] for use of the symmetrization method to establish bounds for the appropriated eigenvalues. Let f be a measurable function on S^2 with respect to the natural measure dV on the unit sphere. The point (Schwarz) symmetrization of f with respect to a given point O is defined as follows. Denote by $d(O, P)$ the spherical distance between the points O and P. Then the functions f_+ and f_- are equimeasurable to f, f_+ and f_- depends only on the distance $d(O, P)$, and $f_+(f_-)$ is increasing (decreasing) functions of $d(O, P)$. Recall, that f and g are called equimeasurable if for any real α the sets $f > \alpha$ and $g > \alpha$ have the same (spherical) measure. We have the classical inequalities (see for details [4]).

$$\int_{S^2} f_+g_- \, dV = \int_{S^2} f_-g_+ \, dV \le \int_{S^2} fg \, dV \le \int_{S^2} f_+g_+ \, dV = \int_{S^2} f_-g_- \, dV, \tag{3.3}$$

$$\left.\begin{aligned} &\int_{S^2} |\nabla f_+|^2 \, dV \\ &\int_{S^2} |\nabla f_-|^2 \, dV \end{aligned}\right\} \le \int_{S^2} |\nabla f|^2 \, dV. \tag{3.4}$$

Here by $|\nabla f|$ we mean the natural gradient on S^2, i.e.

$$|\nabla f|^2 = \sum_{i,j=1}^{2} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}.$$

THEOREM 2. *Let S^2 be the unit sphere in R^3 of the form* (3·2). *let $M > m \ge 0$ be constants. Denote by C a nonempty set of measurable functions on S^2 satisfying the conditions* (1.5) *and* (1.6) *Consider the problem* min $\mu_1(\varphi)$ *on C, where $\mu_1(\varphi)$ is the first nontrival eigenvalue of* (3.1) *on S^2. Then this minimum is achieved for a function $\varphi^* = \varphi^*(x_3)$ of the form*

$$\varphi^*(x_3) = M \quad \text{for} \quad -1 \le x_3 \le h_1, \qquad h_2 \le x_3 \le 1,$$

$$\varphi^*(x_3) = m \quad \text{for} \quad h_1 < x_3 < h_2, \tag{3.5}$$

The eigenvalue $\mu_1(\varphi^*)$ is the first nontrivial eigenvalue of the problem.

$$\frac{d}{dt}\left((1-t^2)\frac{du}{dt}\right) + \mu\varphi^*(t)^2 u = 0, \tag{3.6}$$

$$\sqrt{1-t^2}\, u'(t) = 0 \quad \text{for} \quad t = \pm 1. \tag{3.7}$$

The difference $h_2 - h_1$ is determined by the equation (1.6).

$$2\Pi\{m^2(h_2-h_1) + M^2[2-(h_2-h_1)]\} = W. \tag{3.8}$$

Furthermore, the corresponding solution u of (3.6) ($\mu = \mu_1(\varphi^*)$) *has to satisfy either* *the condition*

$$u(h_2) = -u(h_1) \tag{3.9}$$

if

$$-1 < h_1 \leq h_2 < 1, \tag{3.10}$$

or the condition

$$0 < u(-1) \leq -u(h_2) \tag{3.11}$$

if

$$h_1 = -1 \tag{3.12}$$

(*Note that* $\varphi^*(-x_3)$ *is also extremal thus if* (3.10) *does not hold we may assume* (3.12)).

Proof. We decompose the proof into 2 steps. (i) Let $\varphi \in C$. Let v be the eigenfunction of (3.1) corresponding to $\mu_1(\varphi)$. As $\int_{S^2} v\varphi^2\, dV = 0$ the function v changes its sign. Let I_1 and I_2 be the sets where $v \geq 0$ and $v < 0$ respectively. Denote by v_1, φ_1 and ν_2, φ_2 the restrictions of ν, φ to the sets I_1 and I_2 respectively. We extend v_1, φ_1 and v_2, φ_2 to S^2 by assuming $v_1 = \varphi_1 = v_2 = \varphi_2 = 0$ outside the domains I_1 and I_2 respectively. Let v_1^*, φ_1^*, v_2^*, φ_2^* denote the decreasing symmetrization of v_1, φ_1, v_2, $-\varphi_2$ with respect to the point $x_3 = 1$. Let ξ_3 be the unique number such that the measure of the $x_3 \geq \xi_3$ is equal to the measure of I_1. So $v_1^*(x_3) = \varphi_1^*(x_3) = 0$ for $-1 \leq x_3 \leq \xi_3$, $v_2^*(x_3) = \varphi_2^*(x_3) = 0$ for $\xi_3 \leq x_3 \leq 1$. According to (3.3) and (3.4) we have

$$\begin{aligned} &\int_{I_1} v^2\varphi^2\, dV \leq \int_{\xi_3 \leq x_3 \leq 1} (v_1^*)^2(\varphi_1^*)^2\, dV, \\ &\int_{I_2} v^2\varphi^2\, dV \leq \int_{-1 \leq x_3 < \xi_3} (v_2^*)^2(\varphi_2^*)^2\, dV, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\int_{I_1} |\nabla v|^2\, dV \geq \int_{\xi_3 \leq x_3 \leq 1} |\nabla v_1^*|^2\, dV, \\ &\int_{I_2} |\nabla v|^2\, dV \geq \int_{-1 \leq x_3 < \xi_3} |\nabla v_2^*|^2\, dV \end{aligned} \tag{3.14}$$

Let $\varphi(x_3, h_1, h_2) = \varphi(h_1, h_2)$ be defined by (3.5). The numbers $-1 \le h_1 \le \xi_3 \le h_2 \le 1$ are uniquely determined by the conditions

$$\int_{\xi_3 \le x_3 \le 1} (\varphi_1^*)^2 \, dV = \int_{\xi_3 \le x_3 \le 1} \varphi(h_1, h_2)^2 \, dV,$$

$$\int_{-1 \le x_3 < \xi_3} (\varphi_2^*)^2 \, dV = \int_{-1 \le x_3 < \xi_3} \varphi(h_1, h_2)^2 \, dV. \tag{3.15}$$

From the classical lemma of Neyman and Pearson we deduce

$$\int_{-\xi_3 \le x_3 < 1} (\varphi_1^*)^2 (v_1^*)^2 \, dV \le \int_{\xi_3 \le x_3 \le 1} \varphi(h_1, h_2)^2 \, (v_1^*)^2 \, dV,$$

$$\int_{-1 \le x_3 < \xi_3} (\varphi_2^*)^2 (v_2^*)^2 \, dV \le \int_{-1 \le x_3 < \xi_3} \varphi(h_1, h_2)^2 \, (v_2^*)^2 \, dV. \tag{3.16}$$

Combining the inequalities (3.13), (3.14) and (3.16) we obtain

$$\mu_1(\varphi) = \int_{I_1} |\nabla v|^2 \, dV \Big/ \int_{I_1} V^2 \varphi^2 \, dV \ge \int_{S^2} |\nabla v_1^*|^2 \, dV \Big/ \int_{S^2} (v_1^*)^2 \varphi(h_1, h_2)^2 \, dV,$$

$$\mu_1(\varphi) = \int_{I_2} |\nabla v|^2 \, dV \Big/ \int_{I_2} v^2 \varphi^2 \, dV \ge \int_{S^2} |\nabla v_2^*|^2 \, dV \Big/ \int_{S^2} (v_2^*)^2 \varphi(h_1, h_2)^2 \, dV \tag{3.17}$$

Now the convoy principle implies that $\mu_1(\varphi) \ge \mu_1(\varphi(h_1, h_2))$.
(ii) Introducing the parameter $t = x_3$ we easily deduce that $\mu_1(\varphi(h_1, h_2))$ is the first nontrivial eigenvalue of (3.6) with the free boundary conditions (3.7). Furthermore in terms of the variable t the condition (1.6) for $\varphi(h_1, h_2)$ is equivalent to (3.8). Thus $\min_C \mu_1(\varphi) = \min \mu_1(\varphi(h_1, h_2))$. In view of (3.8) $\mu_1(\varphi(h_1, h_2))$ depends only on one parameter, for example h_1. Using the classical Sturm-Liouville theory, one can show that $\min \mu_1(\varphi(h_1, h_2))$ is achieved for some

$$\varphi^* = \varphi(h_1^*, h_2^*).$$

Suppose first that $-1 < h_1^* < h_2^* < 1$ (the case $h_1^* = h_2^*$ is trivial). Let

$$\varphi_\varepsilon = \varphi(h_1^* - \varepsilon, h_2^* - \varepsilon) \tag{3.18}$$

for an arbitrary small enough ε. Choose a constant δ such that

$$\int_{-1}^{1} \varphi_\varepsilon^2 (u+\delta)\, dt = 0 \tag{3.19}$$

Thus

$$\delta = \frac{2\Pi}{W} \varepsilon (M^2 - m^2)[u(h_1^*) - u(h_2^*)] + o(\epsilon)\epsilon \tag{3.20}$$

Note as $M>0$ u is strictly monotonic in $(-1, 1)$ and therefore $u(h_1^*) - u(h_2^*) \neq 0$. From the minimal characterization of $\mu_1(\varphi_\varepsilon)$ we have

$$\mu_1(\varphi_\varepsilon) \leq \frac{\int_{-1}^{1} (1-t^2)[(u+\delta)']^2\, dt}{\int_{-1}^{1} \varphi_\varepsilon^2 (u+\delta)^2\, dt}. \tag{3.21}$$

Assume the normalization

$$\int_{-1}^{1} (\varphi^*)^2 u^2\, dt = 1, \qquad u(-1) > 0. \tag{3.22}$$

Then

$$\mu_1(\varphi_\varepsilon) \leq \mu_1(\varphi^*)\{1 + \varepsilon (M^2 - m^2)[u^2(h_1^*) - u^2(h_2^*)]\} + o(\epsilon)\epsilon \tag{3.23}$$

From the inequality $\mu_1(\varphi_\varepsilon) \geq \mu_1(\varphi^*)$ and the inequality above we conclude

$$0 \leq \varepsilon\{(M^2 - m^2)[u^2(h_1^*) - u^2(h_2^*)] + o(\epsilon)\}. \tag{3.24}$$

As ε has arbirtary sign we conclude

$$u^2(h_1^*) = u^2(h_2^*). \tag{3.25}$$

Since in that case u is strictly monotonic, we deduce that $u(h_1^*) = -u(h_2^*)$ which proves (3.9).

Suppose now that $-1 = h_1^* < h_2^* < 1$. According to the part (i) of the proof for the extremal φ^*, the function u must vanish in the interval $[h_1^*, h_2^*]$. so $u(h_2^*) \leq 0$.

We can use the function φ_ε for $\varepsilon<0$. The formula (3.20) is valid as $u(h_1^*)-u(h_2^*)>0$, so for a small negative ε (3.24) holds. Thus

$$u^2(-1)\leq u^2(h_2^*).$$

As $u(-1)>0$ and $u(h_2^*)\leq 0$ we deduce that $u(-1)\leq -u(h_2^*)$. The proof of the theorem is completed.
We conjecture

Conjecture. Let the assumptions of Theorem 2 *hold. Then the extremal function* φ^* *given by* (3.5) *is an even function of* x_3, i.e. $h_2=-h_1$. Note that if φ^* is even then the corresponding eigenfunction u is odd and the condition (3.9) trivially holds. We prove the above conjecture in case that $m=0$.

THEOREM 3. *Let the assumptions of* Theorem 2 *hold. Assume furthermore that* $m=0$. *Then* $\min_C \mu_1(\varphi)=\mu_1(\varphi^*)$ *where* φ^* *is an even function of the form* (3.5).

Proof. We claim that $-1<h_1\leq h_2<1$. Otherwise we may assume that $\varphi^*(t)=0$ for $-1=h_1\leq t\leq h_2$. As u satisfies (3.6) and (3.7) we deduce that $u(t)=u(-1)>0$ for $-1\leq t\leq h_2$. This contradicts the condition (3.11). Thus (3.10) holds. To avoid the trivial case assume that $h_1<h_2$. Suppose that $\varphi^*(t)$ is not symmetric. As $\varphi^*(-t)$ is also extremal we may assume

$$h_2<-h_1. \tag{3.26}$$

Let ξ be the unique zero of u. According to the proof of Theorem 2

$$h_1\leq\xi\leq h_2 \tag{3.27}$$

We claim that

$$\xi>0. \tag{3.28}$$

Let

$$V(t)=-\frac{(1-t^2)u'(t)}{u(t)},\qquad U(t)=\frac{(1-t^2)u'(-t)}{u(-t)} \tag{3.29}$$

Then V and U satisfy the differential equations

$$V' = \mu\varphi^*(t)^2 + \frac{V}{1-t^2}, \qquad U' = \mu\varphi^*(-t)^2 + \frac{U^2}{1-t^2}, \qquad \mu = \mu_1(\varphi^*), \tag{3.30}$$

with the initial conditions

$$V(-1) = U(-1) = 0. \tag{3.31}$$

Furthermore

$$V(t) < \infty,\ -1 \le t < \xi,\ U(t) < \infty,\ -1 \le t < -\xi,\ V(\xi) = U(-\xi) = \infty, \tag{3.32}$$

and

$$\varphi^*(t) = \varphi^*(-t),\ -1 \le t \le h_1,\ \varphi^*(-t) \mathrel{\underset{\neq}{>}} \varphi^*(t),\ h_1 < t < h_2. \tag{3.33}$$

Combining (3.27), (3.32) and the inequality above, we get $-\xi < \xi$ which proves (3.28). Consider the equation (3.6) for $h_1 \le t \le h_2$. Thus $(1-t^2)u' = a$ for $h_1 \le t \le h_2$. So

$$u(t) = \frac{a}{2}\left[\ln\frac{1+t}{1-t} - \ln\frac{1+\xi}{1-\xi}\right]$$

as $h_1 \le \xi \le h_2$. Now (3.9) implies that

$$\ln\frac{(1+h_2)(1+h_1)}{(1-h_2)(1-h_1)} = 2\ln\frac{1+\xi}{1-\xi}.$$

From (3.26) and the equality above we deduce that $\xi < 0$. This contradicts (3.28). The contradiction above establishes the theorem.

In conclusion, let us recall the result due to Hersch [6].

$$\lambda_1(\varphi) \le 8\Pi/W \tag{3.34}$$

for any non-negative bounded φ which satisfies the condition (1.6) with $n = 2$. This means $\max_C \lambda_1(\varphi) = \lambda_1(\varphi^{**})$, where φ^{**} is a constant function equal to $(W/4\Pi)^{1/2}$.

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