Boundary regularity and the uniform convergence of quasiconformal mappings.

Autor(en): **Näkki, Raimo / Palka, Bruce**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **54 (1979)**

PDF erstellt am: **19.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-41590

Comment. Math. Helvetici 54 (1979) 458–476 Birkhäuser Verlag, Basel

Boundary regularity and the uniform convergence of quasiconformal mappings

RAIMO NÄKKI[(1)] and BRUCE PALKA[(2)]

1. Introduction

Let D be a proper subdomain of $\bar{R}^n$ and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D into $\bar{R}^n$ with uniformly bounded maximal dilatations. Assume that $\langle f_i \rangle$ converges pointwise in D to a homeomorphism f. Then f is itself a quasiconformal mapping and $\langle f_i \rangle$ converges to f uniformly on compact subsets of D. Under what circumstances can it be inferred that this sequence is, in fact, uniformly convergent on all of D? Such information can be useful, among other places, in the study of extremal problems. In this paper we study the above question in two distinct situations. In Section 3 we treat the case where D is an arbitrary domain but where each of the mappings f_i is assumed to admit a continuous extension to the closure of D. We show that $\langle f_i \rangle$ converges uniformly on D if and only if the sequence of boundary mappings converges uniformly on the boundary of D. In Section 4 the assumption that such boundary mappings exist is removed, but it is replaced by the requirement that the domain D be quasiconformally equivalent to a "smoothly bounded" domain. In this case the uniform convergence of $\langle f_i \rangle$ is characterized in terms of a metric regularity condition on the sequence of image domains $f_i(D)$. This approach involves the use of a distortion function first introduced by Warschawski [13] to study conformal mappings in the plane and leads to generalizations of results of Gaier [2]. The paper concludes with some comments regarding the boundary regularity of the limit domain $f(D)$, under the assumption that $\langle f_i \rangle$ converges to f uniformly on D.

2. Preliminaries

Unless otherwise stipulated, the notation and terminology in this paper will conform to the relatively standard notation and terminology employed in the

(1) Part of this research was done at the University of Michigan in 1974–75. The research was supported by the Academy of Finland.

(2) This research was partially supported by NSF Grant MCS 76-06563.

book of Väisälä [12]. In the absence of a statement to the contrary, any set considered is assumed to be a subset of extended euclidean n-space $\bar{R}^n = R^n \cup \{\infty\}$, $n \geq 2$. All topological considerations refer to the usual topology on $\bar{R}^n$, that is, to the metric topology associated with the chordal metric q on $\bar{R}^n$. For x in R^n and for $r>0$ we use $B^n(x, r)$ to denote the open (euclidean) ball of radius r centered at x, we let $S^{n-1}(x, r)$ denote the boundary sphere of $B^n(x, r)$ and we abbreviate

$$B^n(r)=B^n(0, r), \qquad S^{n-1}(r)=S^{n-1}(0, r), \quad B^n=B^n(1), \quad S^{n-1}=S^{n-1}(1).$$

By a *path* in a set G we understand a continuous mapping of an interval on the real line into G. A path is termed *closed* (respectively, *open*) if its domain is a closed (respectively, an open) interval. For sets E, F and G the notation $\Delta(E, F\colon G)$ indicates the family of all closed paths which join E to F through G. The corresponding family of open paths will be denoted by $\Delta_0(E, F\colon G)$. (See [12, pp. 21–23] for the precise description of these path families.) The notation $M(\Delta)$ designates the *conformal modulus* of a family Δ of paths. A homeomorphism f mapping a domain D into $\bar{R}^n$ is said to be K-*quasiconformal*, $1 \leq K < \infty$, provided that

$$K^{-1}M(\Delta) \leq M[f(\Delta)] \leq KM(\Delta) \tag{1}$$

is satisfied for each family Δ of paths in D. A mapping f is *quasiconformal* if it is K-quasiconformal for some K. The smallest such K is referred to as the *maximal dilatation* of f and is denoted by $K(f)$. The terminology *chordal isometry* is used to describe a 1-quasiconformal mapping f of $\bar{R}^n$ onto itself such that

$$q[f(x), f(y)] = q(x, y)$$

for all points x, y of $\bar{R}^n$.

A family $\mathcal{F}$ of continuous mappings of a set A into $\bar{R}^n$ is said to be *equicontinuous at a point x in A* if, corresponding to each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$q[f(x), f(y)] < \varepsilon$$

for every f in $\mathcal{F}$ and for every y in A satisfying $q(x, y) < \delta$. The family $\mathcal{F}$ is *equicontinuous* if it is equicontinuous at each point of A. Ascoli's Theorem asserts that $\mathcal{F}$ is equicontinuous if and only if $\mathcal{F}$ is a *normal family*, that is, if and only if each sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of A.

If $\langle f_i \rangle$ is a sequence of K-quasiconformal mappings of a domain D into $\bar{R}^n$ and if $\langle f_i \rangle$ converges pointwise in D to a mapping f, then one of the following must hold: (1) the mapping f is a constant mapping; (2) the mapping f assumes precisely two values; (3) the mapping f is a K-quasiconformal mapping of D onto some component of the open set $\ker_{i\to\infty} D_i$,

$$\ker_{i\to\infty} D_i = \bigcup_{i=1}^{\infty} \operatorname{int}\left[\bigcap_{j=i}^{\infty} D_j\right],$$

where we have written $D_i = f_i(D)$. Furthermore, in case (3) it can be shown that $f_i \to f$ uniformly on compact subsets of D. For a detailed discussion of these results the reader is advised to consult the work of Gehring [3], Srebro [11] and Väisälä [12, §21].

3. Uniform convergence and boundary mappings

To what extent is the uniform convergence of a sequence of quasiconformal mappings with uniformly bounded maximal dilatations controlled by the uniform convergence of associated boundary correspondences? For a pointwise convergent sequence whose limit is a homeomorphism the answer to this question is provided by the following theorem.

THEOREM 3.1. *Let D be a proper subdomain of $\bar{R}^n$ and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges pointwise in D to a homeomorphism. Assume that each f_i admits an extension to a continuous mapping $\bar{f}_i$ of $\bar{D}$. Then $\langle f_i \rangle$ converges uniformly on D if and only if $\langle \bar{f}_i \rangle$ converges uniformly on ∂D.*

Proof. Denote by f the limit of the sequence $\langle f_i \rangle$. The necessary part in the theorem is evident. For the sufficiency it is enough to verify that the sequence $\mathcal{F} = \langle \bar{f}_i \rangle$ of extended mappings forms an equicontinuous family. (See, for example, [12, Theorem 20.3].) Since $f_i \to f$ uniformly on compact subsets of D, it is clear that $\mathcal{F}$ is equicontinuous at points of D. We need only demonstrate the equicontinuity of $\mathcal{F}$ at points of ∂D. Fix a point b in ∂D. Because $\langle \bar{f}_i \rangle$ converges uniformly on ∂D, we note that the family $\mathcal{F} \mid \partial D$ of boundary mappings is equicontinuous at b. Now assume that $\mathcal{F}$ fails to be equicontinuous at b. After possible passage to subsequences and relabeling we may assume that there is a sequence $\langle b_i \rangle$ of points in D converging to b such that

$$\lim_{i\to\infty} \bar{f}_i(b) = b' \neq b'' = \lim_{i\to\infty} f_i(b_i). \tag{2}$$

Applying preliminary chordal isometries, we may further assume that b, b' and b'' are finite points and that $f(D)$ contains the point ∞. In order to derive a contradiction we divide the argument into two parts, depending on the "thickness" of ∂D near b. To be precise, let U be the set of r in the open interval $(0, 1)$ such that $S^{n-1}(b, r)$ does not intersect ∂D. Clearly U is an open subset of $(0, 1)$. As a rough indicator of the thickness of ∂D near b we make use of the integral

$$I=\int_U \frac{dr}{r}$$

Two cases arise.

Case 1. $I=\infty$. In this case choose a continuum A' which is contained in $f(D)$ and which contains ∞ as an interior point. Set $A=f^{-1}(A')$ and let

$$q_0=\tfrac{1}{2}\min\{q(b', b''), q(A')\}, \tag{3}$$

where we write $q(A')$ for the chordal diameter of A'. There exists a $\delta>0$ such that

$$q(T)\geq\delta, \tag{4}$$

whenever T is a topological $(n-1)$-sphere in $\bar{R}^n$ such that both components of $\bar{R}^n\setminus T$ have chordal diameter no smaller than q_0. (See [14].) Now let $S=S^{n-1}(b, r)$ be a sphere which is contained in D and which separates A from b. For large values of i, S separates A from b_i as well. For such i, $f_i(S)$ is a topological $(n-1)$-sphere and one component of $\bar{R}^n\setminus f_i(S)$ contains $\bar{f}_i(b)$ and $f_i(b_i)$, while the other component contains the set $f_i(A)$. We infer using (2) and (3) – together with the fact that $f_i\to f$ uniformly on A – that, for sufficiently large i, each component of $\bar{R}^n\setminus f_i(S)$ has chordal diameter at least q_0. Utilizing (4) we conclude that $q[f_i(S)]\geq\delta$ for sufficiently large i. Since $f_i\to f$ uniformly on S, this implies that

$$q[f(S)]\geq\delta \tag{5}$$

for each such sphere S.

Fix $r_0>0$ so that $B^n(b, r_0)$ is disjoint from A and set $U_0=U\cap(0, r_0)$. Obviously,

$$\int_{U_0}\frac{dr}{r}=\infty. \tag{6}$$

For r in U_0 the sphere $S_r=S^{n-1}(b, r)$ is contained in D and separates A from b. From (5) we obtain

$$\delta\leq q[f(S_r)]\leq\max_{x,y\in S_r}|f(x)-f(y)|=\operatorname{osc}(f, r),$$

whenever r belongs to U_0. It is then a consequence of (6) that

$$\int_{U_0} [\operatorname{osc}(f, r)]^n \frac{dr}{r} = \infty. \tag{7}$$

On the other hand, U_0 is a non-empty open set of real numbers. As such, U_0 is a finite or countably infinite union of disjoint open intervals (r_j, s_j). Each of the spherical rings A_j,

$$A_j = \{x \in R^n : r_j < |x - b| < s_j\},$$

is contained in D. Since the mapping f is K-quasiconformal – here we set $K = \sup_i K(f_i)$ – the n-dimensional version of a lemma due to Gehring [3, p. 18] implies that

$$\int_{U_0} [\operatorname{osc}(f, r)]^n \frac{dr}{r} \leq C \sum_j m_n[f(A_j)] \leq C m_n(R^n \setminus A') < \infty,$$

where m_n denotes Lebesgue measure on R^n and where C is a positive constant depending only on K and n. Recalling (7), we see that in Case 1 we have arrived at the desired contradiction.

Case 2. $I < \infty$. In this case $\int_V (dr/r) = \infty$, where $V = (0, 1) \setminus U$. Let $t > 0$ and $N > 0$ be given and write $B = B^n(b, t)$. We observe that there exists an s in $(0, t)$ such that

$$M[\Delta_0(F, B \cap \partial D: D)] \geq N, \tag{8}$$

whenever F is a connected set in D meeting both $S^{n-1}(b, s)$ and $S^{n-1}(b, t)$. To see this choose s in $(0, t)$ such that

$$\int_{V_0} \frac{dr}{r} \geq \frac{N}{c_n}, \tag{9}$$

where $V_0 = V \cap (s, t)$ and where c_n is the constant in the "cap inequality." (We are referring here to [12, Theorem 10.9].) Let F be a connected set in D which meets both $S^{n-1}(b, s)$ and $S^{n-1}(b, t)$ and let ρ be an admissible density for the path family $\Delta_0(F, B \cap \partial D: D)$. For each r in V_0, the path family $\Delta(F, B \cap \partial D: S_r)$, where $S_r = S^{n-1}(b, r)$, is minorized by the path family $\Delta_0(F, B \cap \partial D: D \cap S_r)$. The "cap inequality" allows us to conclude that

$$\int_{S_r} \rho^n \, dm_{n-1} \geq \frac{c_n}{r}$$

for each r in V_0. Here m_{n-1} denotes $(n-1)$-dimensional Hausdorff measure on R^n. In light of (9) we obtain

$$\int_{R^n} \rho^n \, dm_n \geq \int_{V_0} \left(\int_{S_r} \rho^n \, dm_{n-1} \right) dr \geq N$$

for each such ρ. Consequently, $M[\Delta_0(F, B \cap \partial D \colon D)] \geq N$, establishing (8).

Now fix a point x_0 in D such that $f(x_0)$ is finite and fix open euclidean balls B', B'' and B_0 centered at b', b'' and $f(x_0)$, with the property that these balls have pairwise disjoint closures. We may assume that

$$\bar{f}_i(b) \in B', \qquad f_i(b_i) \in B'', \qquad f_i(x_0) \in B_0$$

hold for each i. Let α be a closed Jordan arc joining ∂B_0 to $\partial B''$ in $R^n \backslash \bar{B}'$ and choose, for each i, a closed Jordan arc α_i in the set $L = \alpha \cup \bar{B}_0 \cup \bar{B}''$ which joins $f_i(x_0)$ to $f_i(b_i)$. Designate by β_i the component of $\alpha_i \cap f_i(D)$ which contains the point $f_i(b_i)$. Also, set

$$M = M[\Delta_0(L, \bar{B}' \colon \bar{R}^n)] < \infty. \tag{10}$$

Next, utilizing the equicontinuity of the family $\mathcal{F} \mid \partial D$ at b, choose a ball $B = B^n(b, t)$ such that $\bar{B}$ does not contain the point x_0 and such that

$$\bar{f}_i(\bar{B} \cap \partial D) \subset B' \tag{11}$$

is valid for each i. If $\beta_i = \alpha_i$, then $f_i^{-1}(\beta_i)$ is a closed Jordan arc joining x_0 and b_i. If $\beta_i \neq \alpha_i$, then β_i is a Jordan arc with an endpoint in the boundary of $f_i(D)$ and, therefore, $f_i^{-1}(\beta_i)$ has an accumulation point in ∂D. In view of (11), any such accumulation point must lie in $\partial D \backslash \bar{B}$. In either case $f_i^{-1}(\beta_i)$ is a connected set in D which contains b_i and which intersects $D \backslash \bar{B}$. Because $b_i \to b$ we can apply (8) with $N = (M+1)K$ and fix an index i such that $M(\Delta) > MK$, where, as earlier, $K = \sup_i K(f_i)$ and where $\Delta = \Delta_0(f_i^{-1}(\beta_i), B \cap \partial D \colon D)$. Since f_i is K-quasiconformal we infer from (1) that $M[f_i(\Delta)] > M$. However, $f_i(\Delta) \subset \Delta_0(L, \bar{B}' \colon \bar{R}^n)$. Consequently, by (10), $M[f_i(\Delta)] \leq M$. Again, in Case 2, a contradiction has been obtained.

We are now able to conclude that $\mathcal{F}$ is equicontinuous at b, as desired. The proof of Theorem 3.1 is complete.

As a consequence of Theorem 3.1 we obtain the following normal family theorem.

THEOREM 3.2. *Let D be a proper subdomain of $\bar{R}^n$ and let $\mathcal{F}$ be a family of continuous mappings of $\bar{D}$ into $\bar{R}^n$ such that each member of $\mathcal{F}$ maps D quasiconformally into $\bar{R}^n$ with uniformly bounded maximal dilatation. Assume that $\mathcal{F}$ contains no sequence which converges pointwise in D to a constant mapping. Then $\mathcal{F}$ is a normal family if and only if each of the restricted families $\mathcal{F} \mid \partial D$ and $\mathcal{F} \mid D$ is a normal family.*

Proof. Only the sufficiency requires proof. Let $\langle f_i \rangle$ be a sequence in $\mathcal{F}$. By assumption, $\langle f_i \rangle$ contains a subsequence which converges uniformly on ∂D and uniformly on compact subsets of D to a limit mapping f. Making use of [12, Theorem 21.1], we infer that f is not a mapping which assumes only two values in D and, by hypothesis, f is not a constant mapping in D. Therefore f is a quasiconformal homeomorphism in D. Using Theorem 3.1, we conclude that $\langle f_i \rangle$ has a subsequence which converges uniformly on D, hence uniformly on $\bar{D}$. Thus $\mathcal{F}$ is a normal family.

It is not difficult to exhibit examples which show that, in general, neither the normality of $\mathcal{F} \mid \partial D$ nor that of $\mathcal{F} \mid D$ is by itself sufficient to insure the normality of $\mathcal{F}$ in Theorem 3.2. However, we note the following companion to Theorem 3.2.

THEOREM 3.3. *Let D be a proper subdomain of $\bar{R}^n$ and let $\mathcal{F}$ be a family of continuous mappings of $\bar{D}$ into $\bar{R}^n$ such that each member of $\mathcal{F}$ maps D quasiconformally into $\bar{R}^n$ with uniformly bounded maximal dilatation. Assume that $\mathcal{F}$ contains no sequence which converges pointwise in D to either a constant mapping or a mapping assuming only two values. Then $\mathcal{F}$ is a normal family if and only if the restricted family $\mathcal{F} \mid \partial D$ is a normal family.*

Proof. This result will follow from Theorem 3.2 if it can be demonstrated that $\mathcal{F} \mid D$ is a normal family. Suppose that this is not the case. By Ascoli's Theorem, $\mathcal{F}$ must fail to be equicontinuous at some point x_0 in D. This implies the existence of an $\varepsilon > 0$, a sequence $\langle x_i \rangle$ in D converging to x_0 and a sequence $\langle f_i \rangle$ in $\mathcal{F}$ such that, for each i,

$$q[f_i(x_i), f_i(x_0)] \geq \varepsilon. \qquad (12)$$

If x is a point of $D \backslash \{x_0\}$ and if U is a neighborhood of x contained in D whose closure does not contain x_0, then, for large i, f_i does not assume the values $f_i(x_i)$ and $f_i(x_0)$ in U. By (12) and by [12, Theorem 19.2], the sequence $\langle f_i \rangle$ is equicontinuous in U. Therefore $\langle f_i \rangle$ is equicontinuous in $D \backslash \{x_0\}$. Using Ascoli's Theorem, we may pass to a subsequence and assume that $\langle f_i \rangle$ converges uniformly on compact subsets of $D \backslash \{x_0\}$. We may further assume that $\langle f_i(x_0) \rangle$ converges, that is, we may assume that $\langle f_i \rangle$ converges pointwise in D to a mapping f. In view of the hypotheses, f must be a quasiconformal homeomorphism in D. It then follows that $f_i \to f$ uniformly on compact subsets of D. This, in turn, implies that $q[f_i(x_i), f_i(x_0)] \to 0$, contradicting (12). We conclude that $\mathcal{F} \mid D$ is a normal family, as desired.

Theorems 3.1, 3.2 and 3.3 generalize results in [8] and [9], where only suitably "regular" domains D were considered. (For related ideas see [1] and the paper of Martio [4].) It must be emphasized that the uniform boundedness of the maximal dilatations is crucial in these theorems. This fact is demonstrated by the example in [9, p. 291].

4. Uniform convergence and boundary regularity

The remainder of this paper is concerned primarily with quasiconformal mappings of "smoothly bounded" domains, namely, with domains which are quasiconformally collared. We will see that the uniform convergence of a sequence $\langle f_i \rangle$ of K-quasiconformal mappings of a collared domain D is intimately related to the boundary regularity of the image domains $f_i(D)$ and to the manner in which these domains "fit together."

Quasiconformally collared domains. A proper subdomain D of $\bar{R}^n$ is said to be *quasiconformally collared*, or briefly, *collared*, provided that each boundary point of D has arbitrarily small neighborhoods U such that $U \cap D$ can be mapped quasiconformally onto B^n. A collared domain has only finitely many boundary components, each of which is a compact $(n-1)$-dimensional manifold. Conversely, if D is a domain in $\bar{R}^n$ with only finitely many boundary components, each of which is an $(n-1)$-dimensional C^1-submanifold of $\bar{R}^n$, then D is collared. A plane domain is collared if and only if its boundary consists of a finite number of disjoint Jordan curves. For amplification of the above comments see [5] and [12, §17].

Finite connectedness. A domain D is said to be *finitely connected on the boundary* if each boundary point of D has arbitrarily small neighborhoods U such that $U \cap D$ has only finitely many components.

Uniform domains. A domain D is called a *uniform domain* if, to each $t>0$, there corresponds a $\delta>0$ such that $M[\Delta(F, F^*: D)]\geq\delta$ for each pair of connected sets F and F^* in D with $q(F)\geq t$ and $q(F^*)\geq t$. (See [6], [10].) Every collared domain is a uniform domain and every uniform domain is finitely connected on the boundary. The converse statements are, in general, false. However, if a domain D is quasiconformally equivalent to a collared domain, then D is a uniform domain if and only if D is finitely connected on the boundary. A plane domain with only finitely many boundary components is a uniform domain if and only if it is finitely connected on the boundary.

The function $\eta(r, D)$. A connected set S in a domain D is termed a *cross-set* of D if S is closed in D, if $\bar{S}$ intersects ∂D and if $D\setminus S$ has precisely two components, both of which have boundaries which intersect ∂D. Given such a cross-set S, let $D^*(S)$ be the component of $D\setminus S$ of smaller chordal diameter. Should both components have the same chordal diameter either of the two can be designated $D^*(S)$. For $0<r\leq 1$ we define

$$\eta(r, D)=\sup\nolimits_{q(S)\leq r} q[D^*(S)].$$

The function $\eta(\cdot, D)$ is clearly nondecreasing on $(0, 1]$. Furthermore, $\eta(r, D)=\eta(r, f(D))$ for every chordal isometry f. Roughly speaking $\eta(\cdot, D)$ provides a gauge for measuring "bulges" in D. As such, it is a convenient device for translating certain qualitative information pertaining to the boundary regularity of D into quantitative terms. The function $\eta(\cdot, D)$ was introduced in dimension $n=2$ by Warschawski [13], who used it to study the boundary behavior of conformal mappings.

Given a sequence $\langle D_i\rangle$ of domains we will make use of the functions

$$\eta^*(r)=\eta^*(r, \langle D_i\rangle)=\limsup_{i\to\infty}\eta(r, D_i)$$

and

$$\eta_*(r)=\eta_*(r, \langle D_i\rangle)=\liminf_{i\to\infty}\eta(r, D_i),$$

defined for r in $(0, 1]$.

LEMMA 4.1. *Let D be a domain which is quasiconformally equivalent to a collared domain. Then*

$$\lim_{r\to 0^+}\eta(r, D)=0 \tag{13}$$

if and only if D is finitely connected on the boundary.

Proof. Suppose first that D is finitely connected on the boundary but that (13) is false. Then there is a $d>0$ and a sequence $\langle S_i \rangle$ of cross-sets of D such that $q(S_i)\to 0$, while each component of $D\backslash S_i$ has chordal diameter at least d. Fix a continuum F in D. Since $\bar{S}_i$ meets ∂D, F is contained in one of the components of $D\backslash S_i$, provided that i is sufficiently large. For such i let F_i denote the other component of $D\backslash S_i$. The minorizing principle for the modulus allows us to conclude that $M[\Delta(F, F_i : D)] \leq M[\Delta(F, S_i : D)] \to 0$, as $i\to\infty$. This contradicts the fact, noted earlier, that D is a uniform domain. Therefore (13) must hold.

The reverse implication in Lemma 4.1 is a special case of the following result.

LEMMA 4.2. *Let D be a domain which is quasiconformally equivalent to a collared domain and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges pointwise in D to a homeomorphism f. Assume that*

$$\lim_{r\to 0^+} \eta_*(r) = 0, \tag{14}$$

where η_* corresponds to the sequence $\langle f_i(D) \rangle$ of image domains. Then $f(D)$ is finitely connected on the boundary.

Proof. By composing the mappings f_i and f with some fixed quasiconformal mapping of a collared domain onto D, we can reduce the proof to the situation in which D is itself collared. We assume this to be the case. After the application of a chordal isometry we may also assume that $f(D)$ contains a continuum E which has ∞ as an interior point. Finally, we may assume that E is contained in $D_i = f_i(D)$ for each i. Since D is collared, results in [5] imply that the conclusion of the theorem will follow, if it can be established that f has a continuous extension to $\bar{D}$.

If no such extension exists, there is a point b in ∂D and there are sequences $\langle b_k \rangle$ and $\langle b'_k \rangle$ of points in D converging to b such that

$$q[f(b_k), f(b'_k)] \geq d > 0 \tag{15}$$

for each k. Let

$$\delta = \min\left\{\frac{d}{2}, q(E)\right\}. \tag{16}$$

By hypothesis there is an $r_0>0$ such that

$$\eta_*(r_0) < \delta. \tag{17}$$

Thus there exist arbitrarily large indices i with

$$\eta(r_0, D_i) < \delta. \tag{18}$$

This fact, in conjunction with (15), will lead to a contradiction.

Since D is collared, there is a neighborhood U of b and a homeomorphism h of $U \cap \bar{D}$ onto $\{x = (x_1, \dots, x_n) \in B^n : x_n \geq 0\}$ such that $h(b) = 0$ and such that h is quasiconformal in $U \cap D$ [5]. We may choose U so small that $\bar{U} \cap f^{-1}(E) = \phi$. Because $f_i^{-1} \to f^{-1}$ uniformly on E [12, §21] we may, in fact, assume that

$$\bar{U} \cap f_i^{-1}(E) = \phi \tag{19}$$

for each i. Let $K_0 = KK'$, where $K = \sup_i K(f_i)$ and $K' = K(h \mid U \cap D)$. A lemma due to Gehring [3, p. 18] implies that, given ρ in $(0, 1)$ and given a K_0-quasiconformal mapping g of $B^n_+ = \{x \in B^n : x_n > 0\}$ into $R^n \setminus E$, there exists t in (ρ^2, ρ) such that

$$q[g(S^{n-1}_+(t))] \leq \left[\frac{C_0 m_n(R^n \setminus E)}{\log 1/\rho}\right]^{1/n}, \tag{20}$$

where $S^{n-1}_+(t) = S^{n-1}(t) \cap B^n_+$ and where C_0 is a positive constant depending only on n and K_0. Fix ρ in $(0, 1)$ so that the right-hand side of (20) is smaller than r_0 and choose a neighborhood $U_0 \subset U$ of b so that

$$h(U_0 \cap D) \subset B^n_+ \cap B^n(\rho^2). \tag{21}$$

Finally, fix an index k for which b_k and b'_k belong to $U_0 \cap D$.

Choose an index i for which (18) is satisfied and for which

$$q[f(b_k), f_i(b_k)] < \frac{d}{4}, \qquad q[f(b'_k), f_i(b'_k)] < \frac{d}{4}. \tag{22}$$

We apply (20) and choose t in (ρ^2, ρ) such that $q(S) < r_0$, where $S = (f_i \circ h^{-1})[S^{n-1}_+(t)]$, a cross-set of D_i. By (19) and (21), one component of $D_i \setminus S$ contains E, while the other component contains the points $f_i(b_k)$ and $f_i(b'_k)$. Using (16), (15) and (22), we infer that each component of $D_i \setminus S$ has chordal diameter no smaller than δ and conclude that $\eta(r_0, D_i) \geq \delta$, contradicting (18). Therefore the mapping f extends to a continuous mapping of $\bar{D}$, as desired, and the proof of Lemma 4.2 is complete.

The next lemma begins to reveal the relevance of functions such as η_* to questions concerning uniform convergence.

LEMMA 4.3. *Under the other hypotheses of Lemma 4.2, assume that* (14) *remains valid when the sequence* $\langle f_i \rangle$ *is replaced by an arbitrary subsequence. Then* $\langle f_i \rangle$ *converges uniformly on* D.

Proof. As in the proof of Lemma 4.2 we need only consider the case where D is itself collared. Under this assumption and the assumption that (14) holds for the sequence $\langle f_i \rangle$ we have just seen that f admits an extension to a continuous mapping $\bar{f}$ of $\bar{D}$. Now suppose that $\langle f_i \rangle$ fails to converge uniformly on D. After possibly passing to a subsequence of $\langle f_i \rangle$ and relabeling, we may assume the existence of a sequence $\langle b_i \rangle$ in D, which converges to some point b, such that

$$q[f_i(b_i), f(b_i)] \geq d > 0 \tag{23}$$

for all i. By hypothesis, the passage to a subsequence of $\langle f_i \rangle$ does not alter the validity of (14). Since $f_i \to f$ uniformly on compact sets in D, b must be a point in ∂D. The remainder of this argument is quite similar to the proof of Lemma 4.2 and much of the notation used there can be carried over verbatim for use in the present situation. In particular, we assume that E, D_i, δ, r_0, U, h, ρ and U_0 are defined as in the proof of Lemma 4.2 so that (16), (17), (19), (20) and (21) hold. Furthermore, because $\bar{f}$ is continuous at b, we may assume that U was chosen sufficiently small that

$$q[f(U \cap D)] < \frac{d}{4}. \tag{24}$$

Now fix a point b_0 in $U_0 \cap D$ and then fix an index i for which (18) is satisfied, for which b_i belongs to $U_0 \cap D$ and for which

$$q[f_i(b_0), f(b_0)] < \frac{d}{4}. \tag{25}$$

We apply (20) to obtain t in (ρ^2, ρ) such that $q(S) < r_0$, where $S = (f_i \circ h^{-1})[S_+^{n-1}(t)]$, a cross-set of D_i. By (19) and (21), one component of $D_i \backslash S$ contains E, while the other component contains the points $f_i(b_o)$ and $f_i(b_i)$. Using (16), (23), (24) and (25) we see that each component of $D_i \backslash S$ has chordal diameter at least δ and conclude that $\eta(r_0, D_i) \geq \delta$, contradicting (18). This contradiction shows that $\langle f_i \rangle$ converges uniformly on D, as asserted.

We are now in a position to state one of the main results in this section.

THEOREM 4.4. *Let D be a domain which is quasiconformally equivalent to a collared domain and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges pointwise in D to a homeomorphism. Assume that*

$$\lim_{r \to 0^+} \eta^*(r) = 0, \tag{26}$$

where η^* corresponds to the sequence $\langle f_i(D) \rangle$ of image domains. Then $\langle f_i \rangle$ converges uniformly on D.

Proof. It is clear that (26) implies the validity of (14) for the sequence $\langle f_i \rangle$ and for any of its subsequences. Lemma 4.3 implies the stated conclusion.

The essential content of condition (26) is that, as i gets larger, the domains $f_i(D)$ display progressively greater "regularity" near their boundaries. The idea of using a function such as η^* to express this notion quantitatively originated with Warschawski [13] and was developed by Gaier [2]. Condition (26) is not, in general, a necessary condition for uniform convergence to occur. For example, we may take $f_1 = f_2 = \cdots = f$, where f maps a collared domain D quasiconformally onto a domain D' which is not finitely connected on the boundary. Trivially $f_i \to f$ uniformly on D but

$$\lim_{r \to 0^+} \eta^*(r) = \lim_{r \to 0^+} \eta(r, D') \neq 0$$

by Lemma 4.1. On the other hand, we now show that condition (26) is indeed necessary in the one situation where we may reasonably expect it be so, considering Lemma 4.2.

THEOREM 4.5. *Let D be a domain which is quasiconformally equivalent to a collared domain and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges pointwise in D to a homeomorphism f. Assume that $f(D)$ is finitely connected on the boundary. Then $\langle f_i \rangle$ converges uniformly on D if and only if $\lim_{r \to 0^+} \eta^*(r) = 0$, where η^* corresponds to the sequence $\langle f_i(D) \rangle$ of image domains.*

Proof. We need only verify the necessity. As earlier, we are free to assume that D is actually a collared domain and, therefore, a uniform domain. Because $f(D)$ is finitely connected on the boundary, f has a continuous extension to $\bar{D}$ [5]. In particular, f is uniformly continuous on D. This fact, together with the uniform convergence of $\langle f_i \rangle$, implies that to each $\varepsilon > 0$ there correspond a $t > 0$ and an index i_0 such that

$$q[f_i(x), f_i(y)] < \varepsilon/2, \tag{27}$$

whenever $i \geq i_0$ and x, y are points in D satisfying $q(x, y) < t$.

Choose a continuum E in $f(D)$. We may assume that E is contained in $D_i = f_i(D)$ for each i. Then

$$d = \inf_i q(E, \partial D_i) > 0, \tag{28}$$

where $q(E, \partial D_i)$ designates the chordal distance between E and ∂D_i. Fix ε in $(0, q(E))$ and let t and i_0 be such that (27) holds. Next let $K = \sup_i K(f_i)$, let $\delta > 0$ be a constant corresponding to the domain D and the number t in the definition of a uniform domain and let $M = \delta K^{-1}$. Condition (27) and the K-quasiconformality of f_i imply that

$$M[\Delta(E, F : D_i)] \geq M, \tag{29}$$

whenever $i \geq i_0$ and F is a connected set in D_i with $q(F) \geq \varepsilon$. Finally, it is easily verified that there is an $s > 0$ such that

$$M[\Delta(E, F : \bar{R}^n)] < M \tag{30}$$

for each set F in $\bar{R}^n$ which satisfies $q(F) \leq s$ and $q(E, F) \geq d/2$. Set $r = \min\{s, d/2\}$.

Now fix $i \geq i_0$ and let S be a cross-set of D_i with $q(S) \leq r$. Since $\bar{S}$ meets ∂D_i, we infer from (28) that $q(E, S) \geq d/2$. In particular, E is contained in one of the components of $D_i \setminus S$. Let F designate the other component. Then

$$M[\Delta(E, F : D_i)] \leq M[\Delta(E, S : D_i)] \leq M[\Delta(E, S : \bar{R}^n)] < M$$

by the minorizing principle for the modulus and by (30). Because F is connected we conclude using (29) that $q(F) < \varepsilon$. Consequently, $q[D_i^*(S)] < \varepsilon$. Taking the supremum over all such S yields $\eta(r, D_i) \leq \varepsilon$. Since this is true for each $i \geq i_0$, we obtain $\eta^*(r) = \limsup_{i \to \infty} \eta(r, D_i) \leq \varepsilon$. Because η^* is nondecreasing, it is now clear that $\lim_{r \to 0^+} \eta^*(r) = 0$, as asserted. The proof of Theorem 4.5 is complete.

As a consequence of Theorem 4.5 we obtain the following reformulation of Theorem 3.1 in the context of collared domains.

THEOREM 4.6. *Let D be a collared domain and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges pointwise in D to a homeomorphism. Assume that each f_i admits an extension to a continuous mapping $\bar{f}_i$ of $\bar{D}$. Then $\langle f_i \rangle$ converges uniformly on D if and only if $\lim_{r \to 0^+} \eta^*(r) = 0$, where η^* corresponds to the sequence $\langle f_i(D) \rangle$ of image domains.*

Proof. The sufficiency follows from Theorem 4.4; only the necessity needs to be established. The uniform convergence of $\langle f_i \rangle$ on D clearly forces the uniform convergence of $\langle \bar{f}_i \rangle$ on $\bar{D}$. This, in turn, implies that $f = \lim_i f_i$ has an extension to a continuous mapping of $\bar{D}$. Using results in [5], we conclude that $f(D)$ is finitely connected on the boundary. Theorem 4.5 now applies to insure that $\lim_{r \to 0^+} \eta^*(r) = 0$.

In this section an attempt has been made to characterize the uniform convergence of a sequence $\langle f_i \rangle$ of quasiconformal mappings in terms of conditions such as (26) which depend solely on the intrinsic geometry of the image domains $f_i(D)$. The results of this attempt, as stated in Theorem 4.5, might at first sight be regarded as somewhat unsatisfying, owing to the restriction imposed on the limit domain $f(D)$. However, the next result shows that this restriction cannot be avoided and that Theorem 4.5 is, in the framework of the present section, the best kind of result one can hope to establish.

THEOREM 4.7. *Let D be a domain which is quasiconformally equivalent to a collared domain and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges uniformly on D to a homeomorphism f. Suppose that $f(D)$ fails to be finitely connected on the boundary. Then there exists a sequence $\langle g_i \rangle$ of quasiconformal mappings of D with uniformly bounded maximal dilatations such that $g_i(D) = f_i(D)$ and such that $\langle g_i \rangle$ converges to f pointwise in D – but not uniformly on D.*

Proof. As in previous proofs, we may assume that D is a collared domain. Because $f(D)$ is not finitely connected on the boundary, f cannot have an extension to a continuous mapping of $\bar{D}$ [5]. Consequently, there is a point b in ∂D such that the cluster set $C(f, b)$ of f at b contains at least two points.

Since D is collared, there is a neighborhood U of b and a homeomorphism h of $U \cap \bar{D}$ into $\bar{B}^n$ which maps $U \cap D$ quasiconformally onto B^n and which satisfies $h(b) = e_n$, where $e_n = (0, 0, \ldots, 1)$ [5]. Fix δ_0 in $(0, 1)$ so that

$$B^n(e_n, 2\delta_0) \cap S^{n-1} \subset h(U \cap \partial D).$$

For $0 < \delta \leq 2\delta_0$, denote by C_δ the open cone

$$C_\delta = \{rx : r > 0, x \in B^n(e_n, \delta) \cap S^{n-1}\}.$$

Fix a sequence $\langle a_i' \rangle$ in $S^{n-1} \cap C_{\delta_0}$ converging to e_n such that the mapping g, defined in B^n by $g(x) = (f \circ h^{-1})(x)$, has a radial limit b_i' at the point a_i'. This is possible because a quasiconformal mapping of B^n has a radial limit at each point of S^{n-1}, with the possible exception of a set of conformal capacity zero [7]. We may assume that $\langle b_i' \rangle$ converges to some point b'. Clearly b' belongs to $C(g, e_n) = C(f, b)$. Choose a point b'' in $C(f, b)$, $b' \neq b''$, and choose neighborhoods U' and U'' of b' and b'', respectively, with disjoint closures. We may assume that b_i' lies in U' for every i. Next fix, for each i, a number r_i in $(0, 1)$ such that

$$g(L_i') \subset U', \tag{31}$$

where

$$L_i' = \{ra_i' : r_i \leq r < 1\}.$$

Finally, select a sequence $\langle a_i'' \rangle$ in $C_{\delta_0} \cap B^n$ converging to e_n such that

$$|a_i''| \geq r_i, \qquad g(a_i'') \in U'', \qquad \lim_{i \to \infty} g(a_i'') = b''. \tag{32}$$

It is shown in [7] that there is a number $K' > 1$, depending only on δ_0 and n, such that the following assertions are valid: for each pair of points z and w in C_{δ_0} with $|z| = |w|$ there exists a K'-quasiconformal mapping F of R^n onto itself (a "modified rotation") which satisfies $F(z) = w$, $|F(x)| = |x|$ for each x in R^n and $F(x) = x$ for all x outside $C_{2\delta_0}$. Furthermore, if $\langle \delta_i \rangle$ is a sequence in $(0, \delta_0)$ with $\delta_i \to 0$ and if z_i and w_i are points of C_{δ_i} with $|z_i| = |w_i|$, it is possible to choose mappings F_i of this type in such a way that $\lim_{i \to \infty} F_i(x) = x$ for each x in R^n.

Returning to the proof of Theorem 4.7, we use the preceding remarks to choose a sequence $\langle h_i \rangle$ of K'-quasiconformal mappings of B^n onto intself such that

$$h_i(a_i'') \in L_i', \qquad h_i(x) = x \tag{33}$$

for x in $B^n \setminus C_{2\delta_0}$ and

$$\lim_{i \to \infty} h_i(x) = x \tag{34}$$

for all x in B^n. Now define g_i in D by

$$g_i(x) = \begin{cases} (f_i \circ h^{-1} \circ h_i \circ h)(x), & \text{if } \; x \in D \cap U; \\ f_i(x), & \text{if } \; x \in D \setminus U. \end{cases} \tag{35}$$

Then g_i is a quasiconformal mapping with

$$K(g_i) \leq K' K(h)^2 \sup_i K(f_i)$$

and $g_i(D)=f_i(D)$. It is easily shown using (34) that $\langle g_i \rangle$ converges to f pointwise in D. On the other hand, write $b_i = h^{-1}(a_i'')$ and $L_i = h^{-1}(L_i')$ and observe that, for each i, $f(b_i)=(g\circ h)(b_i)=g(a_i'')\in U''$ by (32), while $g_i(b_i)\in f_i(L_i)$ by (33) and (35). Since, by (31), $f(L_i)=g(L_i')$ is contained in U', we conclude that

$$q[f(b_i), f(L_i)] \geq q(U', U'') > 0$$

for each i. Consequently,

$$q[f(b_i), g_i(b_i)] \geq q[f(b_i), f_i(L_i)] \geq \tfrac{1}{2} q(U', U'')$$

for large i, in view of the fact that $\langle f_i \rangle$ converges to f uniformly on D. It follows that $\langle g_i \rangle$ does not converge to f uniformly on D.

We conclude this section with a theorem which indicates the close relationship between extension problems and convergence problems.

THEOREM 4.8. *Let D be a collared domain and let f be a quasiconformal mapping of D onto a domain D'. Then f has an extension to a continuous mapping of $\bar{D}$ if and only if each sequence $\langle f_i \rangle$ of quasiconformal mappings of D onto D' with uniformly bounded maximal dilatations, which converges to f pointwise in D, converges to f uniformly on D.*

Proof. A continuous extension of f to $\bar{D}$ exists if and only if D' is finitely connected on the boundary [5]. Thus the sufficiency in Theorem 4.8 follows from Theorem 4.7, while the necessity is a consequence of Lemma 4.1 and Theorem 4.4.

5. Uniform convergence and the regularity of limit domains

We consider once again a domain D which is quasiconformally equivalent to a collared domain and a sequence $\langle f_i \rangle$ of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges pointwise in D to a homeomorphism f. Given the fact that each $f_i(D)$ has a "regular" boundary – say $f_i(D)$ is finitely connected on the boundary or even collared – one might optimistically expect to conclude that $f(D)$ exhibits corresponding boundary regularity. Such optimism is, in general, unwarranted, as simple examples demonstrate. There are, however, circumstances under which one might realistically hope to glean information concerning the boundary regularity of $f(D)$ from given data on the boundary regularity of the domains $f_i(D)$. It may happen that these domains "fit together" in a particularly felicitous configuration. The uniform convergence of $\langle f_i \rangle$ may also have an effect on the regularity of $f(D)$. In what follows, we will briefly investigate these notions.

THEOREM 5.1. *Let D be a domain which is quasiconformally equivalent to a collared domain and let $\langle f_i \rangle$ be a sequence of quasiconformal mappings of D with uniformly bounded maximal dilatations which converges pointwise in D to a homeomorphism f. Assume that $\overline{f(D)} \subset f_i(D)$ for each i and that $f(D)$ is finitely connected on the boundary. Then $f(D)$ is collared.*

Proof. As in earlier proofs, we may assume that D is collared. Then f has a continuous extension to $\bar{D}$ [5]. Since $f(D) \subset f_i(D)$ for each i, [10, Theorem 4] implies that the sequence $\langle f_i^{-1} \rangle$ of inverse mappings converges to f^{-1} uniformly on $f(D)$. Because $\overline{f(D)} \subset f_i(D)$, each of the mappings f_i^{-1} is uniformly continuous on $f(D)$. Therefore f^{-1} is uniformly continuous on $f(D)$. Consequently, f^{-1} can be extended to a continuous mapping of $\overline{f(D)}$, that is, f extends to a homeomorphism of $\bar{D}$. This implies that D is collared.

Theorem 5.1 allows us to strengthen Lemma 4.2 in the present context.

COROLLARY 5.2. *Let D, $\langle f_i \rangle$ and f be as in Theorem 5.1. Suppose that $\overline{f(D)} \subset f_i(D)$ for each i and that $\lim_{r \to 0^+} \eta_*(r) = 0$, where η_* corresponds to the sequence $\langle f_i(D) \rangle$ of image domains. Then $f(D)$ is collared.*

We also note the following consequence of Theorem 5.1.

COROLLARY 5.3. *Let D, $\langle f_i \rangle$ and f be as in Theorem 5.1. Assume that each of the domains $f_i(D)$ is finitely connected on the boundary, that $\langle f_i \rangle$ converges uniformly on D and that $\overline{f(D)} \subset f_i(D)$ for each i. Then $f(D)$ is collared.*

It may happen that, in Theorem 5.1, each of the domains $f_i(D)$ is collared. In this case the condition $\overline{f(D)} \subset f_i(D)$ can be weakened to $f(D) \subset f_i(D)$ without affecting the conclusion. Moreover, the proof carries over verbatim. This is true because each f_i now extends to a homeomorphism on $\bar{D}$ and f_i^{-1} is still uniformly continuous on $f(D)$. We then obtain the following analogue of Corollary 5.3.

COROLLARY 5.4. *Let D, $\langle f_i \rangle$ and f be as in Theorem 5.1. Assume that each of the domains $f_i(D)$ is collared, that $\langle f_i \rangle$ converges uniformly on D and that $f(D) \subset f_i(D)$ for each i. Then $f(D)$ is collared.*

It is not difficult to construct examples which show that the conclusion of Corollary 5.4 may fail to hold if either the assumption on uniform convergence or the condition $f(D) \subset f_i(D)$ is omitted.

We conclude this paper with the following reformulation of Theorem 4.5 in the present setting.

COROLLARY 5.5. *Let D, $\langle f_i \rangle$ and f be as in Theorem* 5.1. *Assume that each of the domains $f_i(D)$ is collared and that $f(D) \subset f_i(D)$ for each i. Then $\langle f_i \rangle$ converges uniformly on D if and only if $f(D)$ is collared and* $\lim_{r \to 0^+} \eta^*(r) = 0$, *where η^* corresponds to the sequence $\langle f_i(D) \rangle$ of image domains.*

REFERENCES

[1] COURANT, R., *Über eine Eigenschaft der Abbildungsfunktionen bei konformer Abbildung*, Göttinger Nachr. (1922), 69–70.

[2] GAIER, D., *Über die konforme Abbildung veränderlicher Gebiete*, Math. Z. *64* (1965), 385–424.

[3] GEHRING, F. W., *The Carathéodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. AI *336/11* (1963), 1–21.

[4] MARTIO, O., *Equicontinuity theorem with an application to variational integrals*, Duke Math. J. *42* (1975), 569–581.

[5] NÄKKI, R., *Continuous boundary extension of quasiconformal mappings*, Ann. Acad. Sci. Fenn. AI *511* (1972), 1–10.

[6] ——, *Extension of Loewner's capacity theorem*, Trans. Amer. Math. Soc. *180* (1973), 229–236.

[7] ——, *Prime ends and quasiconformal mappings* To appear in J. d'Analyse Math.

[8] NÄKKI, R. and PALKA, B., *Uniform equicontinuity of quasiconformal mappings*, Proc. Amer. Math. Soc. *37* (1973). 427–433.

[9] PALKA, B., *Fréchet distance and the uniform convergence of quasiconformal mappings*, Duke Math. J. *39* (1972), 289–296.

[10] ——, *On the uniform convergence of quasiconformal mappings*, Trans. Amer. Math. Soc. *184* (1973), 137–152.

[11] SREBRO, U., *Conformal capacity and quasiconformal mappings in* $\bar{R}^n$, Israel J. Math. *9* (1971), 93–110.

[12] VÄISÄLÄ, J., *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. 229, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

[13] WARSCHAWSKI, S. E., *On the degree of variation in conformal mapping of variable regions*, Trans. Amer. Math. Soc. *69* (1950), 335–356.

[14] WILDER, R. L., *Topology of manifolds*, Amer. Math. Soc. Colloquium Ser. 32, Providence, 1949.

University of Jyväskylä
Jyväskylä, Finland

University of Texas
Austin, Texas
U.S.A.

Received September 21, 1978