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A universal space for normal bundles of n -manifolds

E. H. BROWN, JR, and F. P. PETERSON¹

§1. Introduction

In [3] the authors gave a simple criterion for deciding whether a polynomial in Stiefel–Whitney classes is zero on the normal bundles of all smooth n -manifolds. The ideal of relations among Stiefel–Whitney classes for all n -manifolds, $I_n \subset H^*(BO)$ was defined by

$$I_n = \{w \in H^*(BO) \mid w(\nu_{M^n}) = 0 \text{ for all } M^n\}$$

where M^n denotes a smooth n -manifold and ν_M is its stable normal bundle. Let $\Phi: H^*(BO) \simeq H^*(MO)$ be the Thom isomorphism and for $w \in H^*(BO)$, define wSq^i to be $\Phi^{-1}(\chi(Sq^i)\Phi(w))$. It was shown that I_n consists of all \mathbb{Z}_2 -linear combinations of elements of the form wSq^i where $2i > n - |w|$ ($|w|$ = dimension of w).

In this paper we give a stronger version of this result, namely:

THEOREM 1. *There is a space BO/I_n and a map $\pi: BO/I_n \rightarrow BO$ such that*

(a) *If M is a smooth, compact n -manifold and $h: M \rightarrow BO$ classifies ν_M , then there is a map $\bar{h}: M \rightarrow BO/I_n$ such that $\pi\bar{h} \simeq h$.*

(b) *The following sequence is exact.*

$$0 \longrightarrow I_n \subset H^*(BO) \xrightarrow{\pi^*} H^*(BO/I_n) \longrightarrow 0.$$

Theorem 1 shows that BO/I_n is a universal space for normal bundles of n -manifolds in that stably, every such bundle is induced from the bundle over BO/I_n and BO/I_n is the space with the smallest cohomology having this property.

Our original result on I_n suggested the possibility of defining higher order characteristic classes, that is, one could form a space B over BO by killing the

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elements of I_n . Then an element of $H^*(B)$ might give a “new” characteristic class for n -manifolds. For example, with $n = 4$ or 5 , the relation

$$(Sq^2 + w_1 \cup Sq^1 + w_2 U)(v_3) = v_3 Sq^2 = (1Sq^3)Sq^2 = 0$$

where v_3 is the Wu class, gives a class in $H^4(B)$ which is not a polynomial in Stiefel–Whitney classes. Theorem 1 shows that on an n -manifold this “new” class will be a polynomial in Stiefel-Whitney classes modulo indeterminacy.

The spaces BO/I_n are also related to the conjecture that any smooth n -manifold immerses in $R^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the dyadic expansion of n . Since this conjecture is equivalent to the normal bundle map $h : M^n \rightarrow BO$ lifting to $BO_{n-\alpha(n)}$ ([9]), the following is a stronger form of the conjecture:

CONJECTURE. $\pi : BO/I_n \rightarrow BO$ lifts to $BO_{n-\alpha(n)}$.

Using our proof of Theorem 1, our results in [4] can be restated in the following way which gives some plausibility to the above conjecture.

THEOREM 2. *If ζ is the stable universal bundle over BO , MO is its Thom spectrum, MO/I_n is the Thom spectrum of $\pi^*\zeta$ and $MO(n-\alpha(n))$ is the Thom spectrum of the universal bundle over $BO_{n-\alpha(n)}$, then MO/I_n lifts to $MO(n-\alpha(n))$.*

This paper is organized as follows: In §2 we give a detailed outline of the proof of Theorem 1 setting forth most of the notation and describing the various technical problems arising in the construction of BO/I_n . Then in Sections 3, 4, 5, and 6 we prove the various lemmas stated in §2. Throughout the remainder of this paper n is a fixed positive integer.

§2. Outline of the Proof of Theorem 1

All cohomology will be with Z_2 coefficients, A will be the mod two Steenrod algebra and $\chi : A \rightarrow A$ will be the canonical antiautomorphism. The semi-tensor product of A and $H^*(BO)$ ([6]) will be denoted by $A(BO)$, that is, $A(BO) = A \otimes H^*(BO)$ with the algebra structure defined by

$$(a \otimes u)(b \otimes v) = \sum ab'_i \otimes (\chi(b''_i)u)v$$

where $b \rightarrow \sum b'_i \otimes b''_i$ under the diagonal of A . We denote $a \otimes u$ by $a \circ u$.

By a spectrum Y , we will mean a collection of spaces Y_q and maps $g_q: SY_q \rightarrow Y_{q+1}$. If X and Y are spectra, a map $f: X \rightarrow Y$ of degree p will be a collection of homotopy classes $f_q \in [X_q, Y_{q+p}]$ compatible with the maps g_q . If ξ is a real k -plane bundle, $T(\xi)$ will denote its Thom spectrum, i.e., $T(\xi)_q = S^{q-k}$ (Thom space of ξ). Thus the Thom class is in $H^0(T(\xi))$. If ξ is a vector bundle over B , $\Phi: H^*(B) \approx H^*(T(\xi))$ will be the Thom isomorphism. We make $H^*(T(\xi))$ into an $A(BO)$ module as follows: Let $h: B \rightarrow BO$ classify ξ . If $u \in H^*(T(\xi))$, $w \in H^*(BO)$ and $a \in A$, $(a \circ w)u = a(h^*(w)u)$. One easily checks that $\Phi(I_n) \subset H^*(MO)$ is an $A(BO)$ submodule.

We begin by constructing an A -free, acyclic resolution of $\Phi(I_n)$. In [3] the following was proved:

THEOREM 2.1. *If $\{u_i\}$ is an A basis for $H^*(MO)$, then $\Phi(I_n)$ is the A module generated by*

$$\{\chi(Sq^j)u_i \mid 2j > n - |u_i|\}.$$

For a partition $\omega = \{j_1, j_2, \dots, j_l\}$ let $s_\omega \in H^*(BO)$ be the usual class ([17]) associated with the symmetric function $\sum t_1^{j_1} t_2^{j_2} \cdots t_l^{j_l}$. For each partition ω let ω' be the partition consisting of odd integers j , one for each $j2^r \in \omega$. Let

$$u_\omega = \prod_r s_{\omega_r}^{2^r}$$

Since

$$u_\omega = s_\omega + \sum s_{\omega'}$$

where ω' has fewer entries than ω and $\{s_\omega\}$ is a basis for $H^*(BO)$, $\{u_\omega\}$ is also a basis for $H^*(BO)$. Also $\{\Phi(u_\omega) \mid 2^i - 1 \notin \omega\}$ is an A basis for $H^*(MO)$ since $\{\Phi(s_\omega) \mid 2^i - 1 \notin \omega\}$ is.

In [2] an A -free acyclic resolution of $A/A\{\chi(Sq^i) \mid i > h\}$ was constructed. Combining these resolutions with 2.1 and the $\Phi(u_\omega)$ basis, we obtain the following resolution of $\Phi(I_n)$.

Let Λ be the graded free associative algebra over Z_2 with unit generated by λ_i , $i = 0, \pm 1, \pm 2, \dots$, $|\lambda_i| = i$, modulo the relations: If $2i < j$

$$\lambda_i \lambda_j = \sum \binom{s-1}{2s-(j-2i)} \lambda_{i+s} \lambda_{j-s}.$$

If $I = (i_1, i_2, \dots, i_l)$, let $\lambda_I = \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_l}$, $l(I) = l$, $t(I) = i_l$, and $\lambda_{(\)} = 1$. We define I

to be admissible if $2i_i \geq i_{i+1}$. As we will see in §3, $\{\lambda_I \mid I \text{ admissible}\}$ is a Z_2 basis for Λ . Let $\{\lambda^I \mid I \text{ admissible}\}$ be the dual basis of $\Lambda^* = \text{Hom}(\Lambda, Z_2)$.

Let U_l be the vector space over Z_2 with basis the symbols $\lambda^I u_\omega$ where I is admissible, $2^i - 1 \notin \omega$, $l(I) = l$ and $2(t(I) + 1) > n - |u_\omega|$. Grade U_l by $|\lambda^I u_\omega| = |\lambda^I| + |u_\omega|$. Let $d : A \otimes U_l \rightarrow A \otimes U_{l-1}$ be the A linear map defined by

$$d(1 \otimes \lambda^I u_\omega) = \sum \lambda^I (\lambda_j \lambda_J) \chi(Sq^j) \otimes \lambda^J u_\omega$$

where the sum ranges over all j and admissible J . Note by 2.2, if $\lambda^I (\lambda_j \lambda_J) \neq 0$, $t(J) \geq t(I)$ and hence d is well defined. Let $\eta : A \otimes U_0 \rightarrow H^*(MO)$ be given by $\eta(a \otimes \lambda^I u_\omega) = a\Phi(u_\omega)$.

PROPOSITION 2.3. *The following sequence is exact:*

$$\longrightarrow A \otimes U_l \xrightarrow{d} A \otimes U_{l-1} \longrightarrow \cdots \longrightarrow A \otimes U_0$$

and

$$\Phi(I_n) = \eta(\text{image}(d : A \otimes U_1 \rightarrow A \otimes U_0))$$

We prove 2.3 in §3.

For a graded vector space V over Z_2 , let $K(V)$ denote the Eilenberg-MacLane spectrum such that $\pi_*(K(V)) = V^*$ and $H^*(K(V)) = A \otimes V$.

PROPOSITION 2.4. *There is a sequence of Ω -spectra X_l , $l = 0, 1, 2, \dots$ and maps $\alpha_l : X_{l-1} \rightarrow K(U_l)$ of degree $+1$ such that*

- (i) $X_0 = K(U_0)$
- (ii) X_l is the fibration over X_{l-1} induced by α_l from the contractible fibration over $K(U_l)$.
- (iii) If $i : K(U_l) \rightarrow X_l$ is the inclusion of the fibre of $X_l \rightarrow X_{l-1}$, $(\alpha_{l+1} i)^* = d : A \otimes U_{l+1} \rightarrow A \otimes U_l$.
- (iv) If M is a smooth n -manifold, ν is its normal bundle, $g : MO \rightarrow K(U_0)$ realizes η and $h : T(\nu) \rightarrow MO$ comes from the classifying map of ν , then any lifting of $gh : T(\nu) \rightarrow X_0$ to X_{l-1} lifts to X_l .

Since the X_l 's are constructed from an acyclic complex,

$$\lim H^*(X_l) \approx \text{Coker}(d : A \otimes U_1 \rightarrow A \otimes U_0) \approx H^*(MO)/\Phi(I_n).$$

To construct BO/I_n we essentially construct a tower of spaces

$$\longrightarrow B_l \rightarrow B_{l-1} \rightarrow \cdots \rightarrow B_0 = BO$$

with fibres Eilenberg-MacLane spaces, such that if $T_l = T(\zeta_l)$ where $\zeta_l \rightarrow B_l$ is the pull back of the universal bundle over BO , then $T_l = X_l$ in dimensions $\leq n$. We can then, more or less, define $BO/I_n = \lim B_l$.

We recall how the cohomology of a Thom space of a vector bundle changes, in a stable range, when a cohomology class in the base is killed. Suppose $g : B \rightarrow BO$ is a map such that $g_* : \pi_q(B) \approx \pi_q(BO)$ for $2q \leq n$, V is a graded vector space with $V_q = 0$ for $2q \leq n$ and $p : B' \rightarrow B$ is the fibration induced by a map $\gamma : B \rightarrow K(V)_1$ ($K(V) = \{K(V)_q\}$). Let $T = T(g^*\zeta)$ and $T' = T(p^*g^*\zeta)$. Viewing $B' \subset B$ as the fibre of γ , γ factors as $B \xrightarrow{j} B/B' \xrightarrow{\gamma'} K(V)_1$. Let

$$\Psi : (A(BO) \otimes V)^q \rightarrow H^{q+1}(T/T')$$

be given by $\Psi(a \circ u \otimes v) = a(u\Phi((\gamma')^*(v_1)))$ where $v_1 \in H^*(K(V)_1)$ is the element corresponding to $v \in V$ and Φ is the relative Thom isomorphism. In §6 we show that Ψ is an isomorphism for $q \leq n$. (An equivalent form of this was proved in [1].) Combining this with the exact sequence of the pair (T, T') we obtain an exact sequence,

$$\rightarrow H^q(T) \rightarrow H^q(T') \rightarrow (A(BO) \otimes V)^q \rightarrow H^{q+1}(T) \rightarrow$$

for $q \leq n$.

The cohomology of X_l and X_{l-1} are related by the Serre exact sequence,

$$\rightarrow H^q(X_{l-1}) \rightarrow H^q(X_l) \rightarrow (A \otimes U_l)^q \rightarrow H^{q+1}(X_{l-1}) \rightarrow.$$

Thus if we have constructed B_{l-1} such that $T_{l-1} = X_{l-1}$ in dimensions $\leq n$ and we wish to construct B_l , we should take $B = B_{l-1}$ in the above and choose V_l so that $A(BO) \otimes V_l = A \otimes U_l$ as A modules. Our main algebraic result asserts that this is possible. Let

$$V_l = \{\lambda^l u_\omega \in U_l \mid \omega_r = \{ \} \text{ for } r \geq l\}$$

PROPOSITION 2.5. *There are A linear isomorphisms $\theta : A \otimes U_l \rightarrow A(BO) \otimes V_l$ and $A(BO)$ linear maps $d : A(BO) \otimes V_l \rightarrow A(BO) \otimes V_{l-1}$, $l > 1$ and $d : A(BO) \otimes V_1 \rightarrow H^*(MO)$ such that the following diagram is commutative:*

$$\begin{array}{ccccccc} \longrightarrow & A \otimes U_l & \xrightarrow{d} & A \otimes U_{l-1} & \longrightarrow & \cdots & \longrightarrow & A \otimes U_0 \\ & \downarrow \theta & & \downarrow \theta & & & & \downarrow r \\ \longrightarrow & A(BO) \otimes V_l & \xrightarrow{d} & A(BO) \otimes V_{l-1} & \longrightarrow & \cdots & \longrightarrow & H^*(MO). \end{array}$$

Furthermore, if $u \in V_l \subset U_l$, then $\theta(1 \otimes u) = 1 \otimes u$.

The construction of spaces B_l can now be made, modulo technical problems, using 2.5. Given B_{l-1} and $f_{l-1}: T_{l-1} \rightarrow X_{l-1}$, the k -invariant $\beta_l: B_{l-1} \rightarrow K(V_l)_1$ is defined by:

$$\Phi\beta_l^*(v_1) = f_{l-1}^*\alpha_l^*(v)$$

where $\alpha_l: X_{l-1} \rightarrow K(U_l)$ is the k -invariant for X_l , $v \in V$ and $v_1 \in H^*(K(V)_1)$ corresponds to v . If M is an n -manifold and $h: M \rightarrow BO$ classifies its normal bundle, 2.4(iv) shows that any lifting of h to B_{l-1} lifts to B_l . The $A(BO)$ linearity of d allows one (more or less) to construct $f_l: T_l \rightarrow X_l$. Actually, this straightforward procedure is marred by two technical details which we now describe.

Let $s = [n/2]$. To form B_1 from BO , one kills, among other things, the Wu class v_{s+1} , i.e. $d\lambda^s = \chi(Sq^{s+1})U = v_{s+1}U$, where the U is the Thom class. The map Ψ is zero on

$$\sum_{j>0} (Sq^j \circ v_{s+1-j}) \otimes \lambda^s \in (A(BO) \otimes V_1)^{2s+1}$$

As a result, there is a class $x \in H^{2s+1}(X_1)$ which goes to zero in $H^{2s+1}(T_1)$. The class x is killed in going from X_1 to X_2 . Hence if one were to follow the recipe given by 2.5, one would kill a class in B_1 which is already zero and thus produce a class in $H^{2s}(B_2)$ not coming from $H^{2s}(X_2)$. To avoid this, we omit a basis element from V_2 . This same phenomena occurs in dimension $2s + 2$ so we omit some more elements from V_2 and V_3 . Namely, let $\bar{V}_l \subset V_l$ be spanned by $\lambda^l u_\omega \in V_l$ except $\lambda^{0,0} w_s^2$, $\lambda^{0,-1} w_{s+1}^2$, $\lambda^{-1,-2} w_{s+2}^2$ and for s odd, $\lambda^{-1,-2,-4} w_1^4 w_s^2$ ($w_s = u_{(1,1,\dots,1)}$).

In §3 we define a certain $A(BO)$ linear map

$$r: A(BO) \otimes V_l \rightarrow A(BO) \otimes \bar{V}_l \tag{2.6}$$

such that $r|_{A(BO) \otimes \bar{V}_l}$ is the identity. We then use $r\theta$ in place of θ in our construction of B_l .

The second difficulty arises in the following fashion. Again suppose we have B_{l-1} and $f_{l-1}: T_{l-1} \rightarrow X_{l-1}$ and we construct B_l using \bar{V}_l instead of V_l as above. Let $g_l: T_{l-1}/T_l \rightarrow K(U_l)$ be the map such that $g_l^*(u) = \Psi r\theta(u)$ for $u \in U_l$. In order to construct $f_l: T_l \rightarrow X_l$ we need commutativity of the diagram

$$\begin{array}{ccc} T_{l-1} & \xrightarrow{j} & T_{l-1}/T_l \\ \downarrow f_{l-1} & & \downarrow g_l \\ X_{l-1} & \xrightarrow{\alpha_l} & K(U_l). \end{array}$$

We can only prove that this diagram commutes in dimensions $\leq 2s + 1$. To correct for this we relabel B_l above, B'_l and we form B_l from B'_l by killing the obstructions to commutativity as follows:

Define $\Delta = \Delta(f_{l-1}): U_l \rightarrow H^*(T_{l-1})$ by

$$\Delta(u) = f_{l-1}^* \alpha_l^* u - \sum x_i f_{l-1}^* \alpha_l^* u_i$$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$, $u_i \in \bar{V}_l$. Then

$$\begin{aligned} j^* g_l^*(u) &= j^* \Psi r\theta(u) = j^* \Psi \left(\sum x_i u_i \right) = \sum x_i j^* \Phi((\beta'_l)^*(u_i)) \\ &= \sum x_i \Phi(\beta_l^*(u_i)) = \sum x_i f_{l-1}^* \alpha_l^*(u_i) = \Delta(u) + f_{l-1}^* \alpha_l^*(u) \end{aligned}$$

Thus Δ is the deviation from commutativity of our diagram above. Let $W_l = U_l / \ker \Delta$. We kill $\Phi^{-1}(\Delta(W))$ in B'_l to form B_l .

To recapitulate, we inductively construct a sequence of spaces B_l , stable vector bundles ζ_l over B_l and maps $f_l: T_l = T(\zeta_l) \rightarrow X_l$ such that $\Delta(f_l) = 0$. We take $B_0 = BO$, $\zeta_0 = \zeta$ the universal bundle and f_0 the map such that $f_0^*(u_\omega) = \Phi(u_\omega)$ for $u_\omega \in U_0$. ($X_0 = K(U_0)$.) Referring to 2.5, $f_0^* = \eta$, $\alpha_1^* = d$ and $\Delta(f_0) = \eta d - d\theta = 0$. Suppose B_{l-1} , ζ_{l-1} and f_{l-1} have been defined and $\Delta(f_{l-1}) = 0$. Let $p': B'_l \rightarrow B_{l-1}$ be the fibration induced by $\beta_l: B_{l-1} \rightarrow K(\bar{V}_l)_1$ where β_l is defined by

$$\Phi(\beta_l^*(v_1)) = f_{l-1}^* \alpha_l^*(v)$$

for $v \in \bar{V}_l \subset U_l$ and $v_1 \in H^*(K(\bar{V}_l)_1)$ the element corresponding to v . Let $\zeta'_l = (p')^* \zeta_{l-1}$ and $T'_l = T(\zeta'_l)$.

Viewing $B'_l \subset B_{l-1}$ as the fibre of β_l , β_l factors through $\beta'_l: B_{l-1}/B'_l \rightarrow K(\bar{V}_l)_1$. Let $\Psi: A(BO) \otimes \bar{V}_l \rightarrow H^*(T_{l-1}/T'_l)$ be the $A(BO)$ linear map such that $\Psi(v) = \Phi((\beta'_l)^*(v_1))$ for $v \in \bar{V}_l$. Let θ be as in 2.5, r as in 2.6, and let $g'_l: T_{l-1}/T'_l \rightarrow K(U_l)$ be defined by $(g'_l)^*(u) = \Psi r\theta(u)$. Since $\Delta(f_{l-1}) = 0$, there is a map f'_l making a commutative diagram

$$\begin{array}{ccccccc} T_{l-1}/T'_l & \longrightarrow & T'_l & \longrightarrow & T_{l-1} & \longrightarrow & T_{l-1}/T'_l \\ \downarrow g'_l & & \downarrow f'_l & & \downarrow f_{l-1} & & \downarrow g'_l \\ K(U_l) & \xrightarrow{i} & X_l & \longrightarrow & X_{l-1} & \xrightarrow{\alpha_l} & K(U_l). \end{array}$$

Let $\Delta(f'_l): U_{l+1} \rightarrow H^*(T_l)$ be given by $\Delta(f'_l)(u) = (f'_l)^* \alpha_{l+1}^* u + \sum x_i (f'_l)^* \alpha_{l+1}^* u_i$ where $r\theta u = \sum x_i u_i$. Let $W_{l+1} = U_{l+1} / \ker \Delta(f'_l)$ and let $p: B_l \rightarrow B'_l$ be the fibration induced

by $\gamma_l : B_l \rightarrow K(W_{l+1})_1$ where $\Phi(\gamma_l^* u_1) = \Delta(f_l)(u)$ for $u \in W_{l+1}$. Finally let $\zeta_l = p^* \zeta'_l$ and $f_l = f'_l T(p)$. Then $\Delta(f_l) = T(p)^* \Delta(f'_l) = 0$ and the inductive step is complete.

In §5 we prove:

LEMMA 2.7. *If $l \geq 3$ and $q \leq n$, $f_l^* : H^q(X_l) \approx H^q(T(\zeta_l))$. Furthermore, if M is a smooth n -manifold and $h : M \rightarrow B_0 = BO$ classifies its normal bundle, then any lifting of h to B_{l-1} lifts to B_l .*

We next examine $H^*(B_l)$ for l large.

LEMMA 2.8. *If $l \geq n$, $V_l^q = U_l^q = 0$ for $q < n - 1$, $W_l^q = 0$ for $q \leq n$ and*

$$V_l^{n-1} = U_l^{n-1} = \{\lambda^{(0,0,\dots,0)} u_\omega \mid u_\omega \in U_0^{n-1}\}. \text{ Furthermore,}$$

$$\Phi(\beta_l^*(\lambda^{(0,0,\dots,0)} u_\omega)) = \delta_l \tilde{u}_\omega$$

$\tilde{u}_\omega \in H^*(T_{l-1}; Z_{2l})$, $u_\omega U \in H^*(T_{l-1})$ is the mod two reduction of \tilde{u}_ω and δ_l is the Bockstein associated with $Z_2 \rightarrow Z_{2l+1} \rightarrow Z_{2l}$.

Thus for $l \geq n$,

$$H^q(B_l) \approx H^q(BO)/I_n^q \quad q < n$$

$$H^n(B_l)/\Phi^{-1}\{\delta_{l+1} \tilde{u}_\omega\} \approx H^n(BO)/I_n^n$$

We form B_∞ from B_l , $l \geq n$, by killing classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_\omega) \in H^{n+1}(B_l; Z_\tau)$ where Z_τ denotes twisted integer coefficients, twisted by w_1 , $\Phi : H^*(B_l; Z_\tau) \approx H^*(T(\zeta_l); Z)$ is the Thom isomorphism and δ^l is the Bockstein associated with $Z \rightarrow Z \rightarrow Z_{2^l}$. Let \tilde{B}_l be the two sheeted cover of B_l defined by w_1 . The classes $\Phi^{-1}(\delta^{l+1} \tilde{u}_\omega)$ may be represented by Z_{2^l} -equivariant maps $x_\omega : \tilde{B}_l \rightarrow K(Z, n)$ where $K(Z, n)$ has the action defined by the nontrivial action of Z_2 on Z . Let \tilde{B}_∞ be the fibration over \tilde{B}_l induced by

$$x = \prod x_\omega : \tilde{B}_l \rightarrow \prod K(Z, n)$$

Since x is Z_2 -equivariant, Z_2 acts freely on \tilde{B}_∞ . Let $B_\infty = \tilde{B}_\infty/Z_2$. The map $B_\infty = \tilde{B}_\infty/Z_2 \rightarrow \tilde{B}_l/Z_2 = B_l$ has fibre $\prod K(Z, n)$. With Z_2 coefficients, $\pi_1(B_l)$ acts trivially on the cohomology of the fibre. The Serre spectral sequences, with Z_2 coefficients has its usual, nonlocal coefficient form and the usual argument shows

that in dimensions $\leq n$,

$$H^*(BO_\infty) = H^*(B_l) / \{\Phi^{-1}(\delta^{l+1}\tilde{u}_\omega)\}.$$

Thus for $q \leq n$

$$0 \rightarrow I_n^q \rightarrow H^q(BO) \rightarrow H^q(B_\infty) \rightarrow 0$$

is exact. Also if M is an n -manifold and $h: M \rightarrow B$ is covered by a bundle map $g: \nu \rightarrow \zeta$, $T(g)^*(\delta^{l+1}\tilde{u}_\omega) = \delta^{l+1}T(g^*)(\tilde{u}_l) = 0$ since the top homology class of $T(\nu)$ is spherical. Therefore, h lifts to B_∞ .

Finally, assume B_∞ is a CW complex and let

$$BO/I_n = B_\infty^n \cup e_1^{n+1} \cup e_2^{n+1} \cdots e_m^{n+1}$$

where e_i^{n+1} is attached by $f_i | S^n$, $f_i: (D^{n+1}, S^n) \rightarrow (B^{n+1}, B^n)$ and $[f_i] \in \pi_{n+1}(B_\infty^{n+1}, B_\infty^n)$ give a Z_2 -basis for the image of

$$\pi_{n+1}(B_\infty^{n+1}, B_\infty^n) \xrightarrow{\rho} H_{n+1}(B_\infty^{n+1}, B_\infty^n) \xrightarrow{\partial^*} H_n(B_\infty^n, B_\infty^{n-1})$$

The maps f_i give an extension of $B_\infty^n \subset B_\infty$, $f: BO/I_n \rightarrow B_\infty$ and

$$\begin{aligned} f^*: H^q(B_\infty) &\simeq H^q(BO/I_n) \quad \text{for } q \leq n \\ H^q(BO/I_n) &= H^q(BO)/I_n = 0 \quad \text{for } q > n \end{aligned}$$

Also any map of an n -manifold into B_∞ is homotopic to a map factoring through f . The proof of Theorem 1 is thus complete, modulo the lemmas and propositions of this section.

§3. Proofs of 2.3, 2.5, and 2.6

Let Λ_l^k be the Z_2 -subspace of Λ^* generated by λ^I with $l(I) = l$, $t(I) \geq k$, and I admissible. Let

$$d: A \otimes \Lambda_l^k \rightarrow A \otimes \Lambda_{l-1}^k$$

be defined by

$$d(1 \otimes \lambda^I) = \sum \lambda^I(\lambda_j \lambda_j) \chi(Sq^{j+1}) \otimes \lambda^J \tag{3.1}$$

where the sum is over all j and admissible J . Proposition 2.3 follows from 2.1 and 3.2(ii) below:

PROPOSITION 3.2.

- (i) $\{\lambda_I \mid I \text{ admissible}\}$ is a Z_2 -basis for Λ .
- (ii) The following is exact:

$$\longrightarrow A \otimes \Lambda_i^k \xrightarrow{d} A \otimes \Lambda_{i-1}^k \longrightarrow \cdots \longrightarrow A \otimes A_0^k \xrightarrow{\epsilon} A/A\{\chi(Sq^i) \mid i > k\}$$

where $\epsilon(a \otimes \lambda^{(i)}) = \{a\}$.

(iii) If I and J are admissible, $l(I) = l$, $l(J) = l - 1$, and $I_l = (1, 2, 4, \dots, 2^{l-1})$, then $\lambda^{I+rI_l}(\lambda_{j+r} \lambda_{J+2rI_{l-1}}) = \lambda^I(\lambda_j \lambda_J)$.

Proof. For any sequence $T = (t_1, t_2, \dots, t_l)$ and integer r , let $h^r(\lambda_T) = \lambda_{T+rI_l}$. Extending linearly, h^r gives a well defined map $h^r : \Lambda \rightarrow \Lambda$ since for any element of Λ of the form $\alpha = \lambda_{I_1} \beta \lambda_{I_2}$ where β is a relation for Λ as in 2.2, $h^r(\alpha)$ also has this form. Since $h^r h^{-r}$ is the identity, h^r is an isomorphism for all r . Furthermore, $h^r(\lambda_I)$ is admissible if and only if λ_I is admissible.

Let $\bar{\Lambda} \subset \Lambda$ be the subalgebra generated by $\lambda_0, \lambda_1, \lambda_2, \dots$. In [8] it is proved that $\{\lambda_I \mid I \text{ admissible}\}$ is a basis for $\bar{\Lambda}$. For any λ_I , $h^r(\lambda_I) \in \bar{\Lambda}$ for r sufficiently large. Thus $\{\lambda_I \mid I \text{ admissible}\}$ is a basis for Λ .

In [2], 3.2(ii) was proved for $k \geq 0$. From 2.2 one sees that $\lambda_{-1} \lambda_{-1} = 0$ and if $t(J) \geq 0$, $\lambda_{-1} \lambda_J$ is a sum involving $\lambda_{J'}$'s with $t(J') > 0$ and $\lambda_J \lambda_{-1}$. Suppose $J_1 = (j_1, \dots, j_m)$, $J_2 = (j_{m+1}, \dots, j_l)$ and $J = (j_1, \dots, j_l)$ are admissible with J_1 or J_2 possibly the empty sequence $()$. Define $\lambda^{J_1} \lambda^{J_2} = \lambda^J$. Suppose $j_m \geq 0$ and $j_{m+1} < -1$. Then 3.1 yields

$$d(\lambda^{J_1} \lambda^{-1} \lambda^{J_2}) = (d\lambda^{J_1}) \lambda^{-1} \lambda^{J_2} + \lambda^{J_1} \lambda^{J_2}$$

$$d(\lambda^{J_1} \lambda^{J_2}) = (d\lambda^{J_1}) \lambda^{J_2}.$$

Let

$$D(\lambda^{J_1} \lambda^{J_2}) = \lambda^{J_1} \lambda^{-1} \lambda^{J_2}, D(\lambda^{J_1} \lambda^{-1} \lambda^{J_2}) = 0.$$

Then for $k < 0$, $D : A \otimes \Lambda_i^k \rightarrow A \otimes \Lambda_{i+1}^k$ satisfies $dD + Dd = \text{identity}$. Therefore 3.2(ii) holds for $k < 0$.

Finally we prove 3.2(iii). Note that if I is admissible, $I + rI_l$ is admissible and if $(h^r)^*: \Lambda^* \rightarrow \Lambda^*$ is the dual of h^r , $h^r, (h^r)^*\lambda^I = \lambda^{I-rI_l}$. Therefore

$$\begin{aligned} \lambda^I(\lambda_j \lambda_J) &= (h^r)^*(\lambda^{I+rI_l})(\lambda_j \lambda_J) \\ &= \lambda^{I+rI_l}(h^r(\lambda_j \lambda_J)) = \lambda^{I+rI_l}(\lambda_{j+r} \lambda_{J+2rI_{l-1}}) \end{aligned}$$

Proof of 2.5. Let $C_l = A \otimes U_l$, $D_l = A(BO) \otimes V_l$, $l > 0$, and $D_0 = H^*(MO)$. Denote $a \otimes u \in C_l$ by au and $a \circ v \otimes w \in D_l$, $l > 0$, by $(a \circ v)w$. We filter C_l and D_l as follows: $F_q(C_l)$ is spanned by $a\lambda^I u_l$ with $|u_\omega| \leq q$ and $F_q(D_l)$, $l > 0$, is spanned by all $a \circ v \lambda^I u_l$ with $|u_\omega| + 2^l |v| \leq q$. $F_q(D_0)$ is spanned by all au_ω where $a \in A$, $u_\omega \in U_0 = \{u_\omega \mid 2^i - 1 \notin \omega\}$ and $|u_\omega| \leq q$.

The chain complex (C_l, d) is a direct sum of chain complexes of the form described in 3.2, indexed by the $u_\omega \in U_0$. Hence d is filtration preserving and:

(3.3) The following is exact.

$$\longrightarrow F_q(C_l) \xrightarrow{d} F_q(C_{l-1}) \longrightarrow \cdots \longrightarrow F_q(C_0)$$

Using induction on l we define A linear maps $\theta: C_l \rightarrow D_l$ and $A(BO)$ linear maps $d: D_l \rightarrow D_{l-1}$ such that

- (i) θ is an isomorphism and $\theta: C_0 \rightarrow D_0$ is given by $\theta(a \otimes u_\omega) = a\Phi(u_\omega) \in H^*(MO)$, $u_\omega \in U_0$.
- (ii) $d\theta = \theta d$
- (iii) If $u \in V_l \subset U_l$, $\theta(u) = u$
- (iv) $\theta(F_q(C_l)) = F_q(D_l)$
- (v) Suppose $\lambda^I u_\omega \in U_l$. Let α and β be the partitions

$$\alpha = \bigcup_{r < l} 2^r \omega_r, \quad \beta = \bigcup_{r \geq l} 2^{r-l} \omega_r$$

Note $u_\omega = u_\alpha u_\beta^{2^l}$. Then θ satisfies

$$\theta(\lambda^I u_\omega) = u_\beta \lambda^{I'} u_\alpha \pmod{F_{|u_\omega|-1}(D_l)}$$

where $I' = I + |u_\beta| I_l$.

Note that Proposition 2.5 consists of statements (i), (ii), and (iii) above.

For $l = 0$, θ is defined by (i) and $d = 0$ on D_0 .

Suppose θ and d have been defined on C_k and D_k $k < l$, and satisfy (i)–(v). Define $d = d_D: D_l \rightarrow D_{l-1}$ to be the $A(BO)$ linear map such that for $u \in V_l$,

$d_D(u) = \theta(d_C u)$. We next define $\theta : C_l \rightarrow D_l$. Suppose $\lambda^I u_\omega \in U_l$ and $u_\omega = u_\alpha u_\beta^{2^l}$ as in (v). If $u_\beta = 1$, $\lambda^I u_\omega \in V_l$ and we define $\theta(\lambda^I u_\omega) = \lambda^I u_\omega$. In this case (i)–(v) are satisfied. Suppose $u_\beta \neq 1$. Let

$$X = \theta(d(\lambda^I u_\omega)) + u_\beta \theta(d\lambda^I u_\alpha)$$

where $I' = I + |u_\beta| I_l$. By induction, $\theta d = d\theta$ on C_{l-1} and hence $\partial X = 0$. We show that $X \in F_{p-1}(D_l)$ where $p = |u_\omega|$. Decompose u_α into $u_{\alpha_1} u_{\alpha_2}^{2^{l-1}}$ as in (v).

$$\begin{aligned} \theta(d\lambda^I u_\omega) &= \sum \lambda^I(\lambda_j \lambda_K) \chi(Sq^{j+1}) \theta(\lambda^K u_\omega) \\ &= \sum \lambda^I(\lambda_j \lambda_K) (\chi(Sq^{j+1}) \circ u_{\alpha_2} u_\beta^2) \lambda^{K'} u_{\alpha_1} \text{ mod } F_{p-1} \end{aligned}$$

where $K' = K + |u_{\alpha_2} u_\beta^2| I_{l-1}$. On the other hand,

$$u_\beta \theta(d\lambda^I u_\alpha) = \sum \lambda^{I'}(\lambda_j \lambda_J) u_\beta \chi(Sq^{j+1}) \theta(\lambda^J u_\alpha)$$

In $A(BO)$,

$$u_\beta \chi(Sq^{j+1}) = \chi(Sq^{j-q+1}) \circ u_\beta^2 + \sum_{k < q} \chi(Sq^{j-k+1}) \circ Sq^k u_\beta$$

where $q = |u_\beta|$.

$$\theta(\lambda^J u_\alpha) = u_{\alpha_2} \lambda^{J'} u_{\alpha_1} \text{ mod } F_{|u_\alpha|-1}$$

where $J' = J + |u_{\alpha_2}| I_{l-1}$. If $u \lambda^I v$ has filtration less than $|u_\alpha| - 1$ and $k < q$, $Sq^k u_\beta u \lambda^I v$ has filtration less than $p = |u_\omega|$.

Hence

$$u_\beta \theta(d\lambda^I u_\alpha) = \sum_{j,J} \lambda^{I'}(\lambda_j \lambda_J) (\chi(Sq^{j-q+1}) \circ u_{\alpha_2} u_\beta^2) \lambda^{J'} u_{\alpha_1} \text{ mod } F_{p-1}$$

In the above sum, replace j by $j + q$ and J by $K + 2qI_{l-1}$. Then

$$u_\beta \theta(d\lambda^I u_\alpha) = \sum_{j,K} \lambda^{I'}(\lambda_{j+q} \lambda_{K+2qI_{l-1}}) \chi(Sq^{j+1}) \circ u_{\alpha_2} u_\beta^2 \lambda^{K'} u_{\alpha_1} \text{ mod } F_{p-1}$$

where $K' = K + |u_{\alpha_2} u_\beta^2| I_{l-1}$. But $I' = I + qI_l$ and hence by 3.2(iii),

$$\lambda^{I'}(\lambda_{j+q} \lambda_{K+2qI_{l-1}}) = \lambda^I(\lambda_j \lambda_K)$$

Hence $X \in F_{p-1}(D_l)$.

By (iv) there is a $Y \in F_{p-1}(C_{l-1})$ such that $\theta(Y) = X$ and by (i) and (ii), $dY = 0$. Hence for $l > 1$, by 3.3, there is a $Z \in F_{p-1}(C_l)$ such that $dZ = Y$. We verify that there is such a Z for $l = 1$ by showing that when $l = 1$, $X \in \Phi(I_n)$. In this case

$$\begin{aligned} X &= \chi(Sq^{i+1})\Phi(u_\alpha u_\beta^2) + u_\beta \chi(Sq^{i+q+1})\Phi(u_\alpha) \\ &= \sum_{j < q} \chi(Sq^{i+q+1-j})\Phi((Sq^j u_\beta)u_\alpha) \end{aligned}$$

where $2(i+1) > n - q$, $q = |u_\beta|$. But then, $2(i+q-j+1) > n - |(Sq^j u_\beta)u_\alpha|$ and hence $X \in \Phi(I_n)$.

We now define $\theta(\lambda^I u_\omega)$ by induction on $|u_\omega| = \text{filtration degree of } \lambda^I u_\omega$. For $|u_\omega| = 0$, $\theta(\lambda^I 1) = \lambda^I 1$. If θ is defined on $F_{|u_\omega|-1}(C_l)$, let

$$\theta(\lambda^I u_\omega) = u_\beta \lambda^{I'} u_\alpha + \theta(Z)$$

where Z , α , β , and I' are as above. Then $d\theta(Z) = \theta(dZ) = \theta(Y) = X$ and

$$\begin{aligned} d\theta(\lambda^I u_\omega) &= d(u_\beta \lambda^{I'} u_\alpha) + d\theta(Z) \\ &= u_\beta \theta(d(\lambda^{I'} u_\alpha)) + X = \theta(d(\lambda^I u_\omega)) \end{aligned}$$

Note that elements of the form $u_\beta \lambda^{I'} u_\alpha$, as above, together with $F_{p-1}(D_l)$, span $F_p(D_l)$ over A . Thus $\theta : C_l \rightarrow D_l$ is an epimorphism. (It is at this point that we use λ^I where I has negative entries. For each $u_\beta \lambda^{I'} u_\alpha \in H^*(BO)V_l$ we need $\lambda^I u_\alpha u_\beta^{2^l} \in U_l$ such that $I' = I + |u_l| I_l$.) Elements of the form $\lambda^I u_\alpha u_\beta^{2^l}$ are an A basis for C_l and elements of the form $u_\beta \lambda^{I'} u_\alpha$ are an A basis for D_l . Hence $\theta : C_l \rightarrow D_l$ is an isomorphism and the proof of 2.5 is complete.

Proof of 2.6. Let $v_i \in H^*(BO)$ be the Wu classes, that is, $\Phi(v_i) = \chi(Sq^i)\Phi(1)$ where $\Phi : H^*(BO) \rightarrow H^*(MO)$ is the Thom isomorphism.

LEMMA 3.4.

$$v_i = \sum s_\omega$$

where the sum ranges over all ω with entries only of the form $2^j - 1$ and $|s_\omega| = i$.

Proof. We view $H^*(BO) \subset \mathbb{Z}_2[t_1, t_2, \dots]$, $|t_i| = 1$, and $t_1 t_2 \dots$ as the Thom class. Let $Sq = Sq^0 + Sq^1 + \dots$ and $v = v_0 + v_1 + \dots$. Then

$$\chi(Sq)t_i = \sum t_i^{2^j}$$

and

$$\begin{aligned} v(t_1, t_2, \dots)(t_1 t_2 \cdots) &= \chi(Sq)(t_1 t_2 \cdots) \\ &= \prod_i \left(\sum_j t_i^{2^j-1} \right) (t_1 t_2 \cdots) = \left(\sum_{\omega} s_{\omega} \right) (t_1 t_2 \cdots) \end{aligned}$$

where the sum ranges over ω with entries only of the form $2^j - 1$.

Let x_1 and $x_2 \in A(BO)$ be given by

$$x_1 = \sum_{j>0} Sq^j \circ v_{s+1-j}, \quad x_2 = \sum Sq^j \circ v_{s+2-j}$$

Recall $s = [n/2]$ and n is the dimension of the manifolds we are considering. Let $y_i^1 \in D_1$ be defined by

$$y_1^1 = x_1 \lambda^s, \quad y_2^1 = x_2 \lambda^s, \quad y_3^1 = v_{s+1} \lambda^{s+1} + v_{s+2} \lambda^s + x_2 \lambda^s$$

LEMMA 3.5. *There are elements $y_i^2 \in D_2$ such that $dy_i^2 = y_i^1$ and*

$$\begin{aligned} y_1^2 &= \lambda^{0,0} v_s^2 \bmod F_{2s-1} \\ y_2^2 &= \lambda^{0,-1} v_{s+1}^2 \bmod F_{2s+1} \\ y_3^2 &= \lambda^{-1,-2} v_{s+2} \bmod F_{2s+3} \end{aligned}$$

If s is odd, there is an element y_2^3 such that $y_2^3 = (Sq^1 + w_1)y_2^2$ and

$$y_2^3 = \lambda^{-1,-2,-4} w_1^4 v_{s+2}^2 \bmod F_{2s+7}$$

Proof. We first show that $dy_i^1 = 0$, $d: D_1 \rightarrow D_0 = H^*(MO)$. Let $U \in H^0(MO)$ be the Thom class.

$$\begin{aligned} dy_1^1 &= x_1 d\lambda^s = \sum Sq^j (v_{s+1-j} \chi(Sq^{s+1}) U) + v_{s+1} \chi(Sq^{s+1}) U \\ &= (Sq^{s+1} v_{s+1}) U + v_{s+1}^2 U = 0 \\ dy_2^1 &= \sum Sq^j (v_{s+2-j} \chi(Sq^{s+1}) U) \\ &= \sum Sq^j (v_{s+1} \chi(Sq^{s+2-j}) U) = (Sq^{s+2} v_{s+1}) U = 0 \\ dy_3^1 &= v_{s+1} \chi(Sq^{s+2}) U + v_{s+2} \chi(Sq^{s+1}) U + dy_2^1 = 0 \end{aligned}$$

We next show that y_1^2 exists. In $A \otimes \Lambda^*$ one may easily calculate $d\lambda^{0,0} = Sq^1 \lambda^0$.

Hence, by the arguments in the proof of 2.5,

$$\begin{aligned} d\lambda^{0,0}v_s^2 &= \theta(d\lambda^{0,0}v_s^2) = \theta(Sq^1\lambda^0v_s^2) \\ &= Sq^1 \circ v_s\lambda^s \text{ mod } F_{2s-1} \\ &= \sum_{j>0} Sq^j \circ v_{s+1-j}\lambda^s \text{ mod } F_{2s-1} = y_1^1 \text{ mod } F_{2s-1} \end{aligned}$$

Thus $u = d\lambda^{0,0}v_s^2 + y_1^1 \in F_{2s-1}$ and $du = 0$. Therefore there is a $z \in F_{2s-1}(D_2)$ such that $dz = u$. Let $y_1^2 = \lambda^{0,0}v_s^2 + z$. The existence of y_2^2 , y_3^2 , and y_3^3 are proven in an analogous fashion.

We now define $r : A(BO) \otimes V_l \rightarrow A(BO) \otimes \bar{V}_l$. For $l \neq 2$ and $l \neq 3$, s odd, $\bar{V}_l = V_l$ and r is the identity; $\bar{V}_l \subset V_l$ and $r|_{A(BO) \otimes \bar{V}_l}$ is the identity. \bar{V}_2 is formed from V_2 by omitting the basis elements $\lambda^{0,0}w_s^2$, $\lambda^{0,-1}w_{s+1}^2$ and $\lambda^{-1,-2}w_{s+2}^2$. By 3.4, v_i involves $w_i = s_{(1,1,\dots,1)}$ when v_i is expressed in the u_ω basis. Let

$$\begin{aligned} r(\lambda^{0,0}w_s^2) &= y_1^2 - \lambda^{0,0}w_s^2 \\ r(\lambda^{0,-1}w_{s+1}^2) &= y_2^2 - \lambda^{0,-1}w_{s+1}^2 \\ r(\lambda^{-1,-2}w_{s+2}^2) &= y_2^3 - \lambda^{-1,-2}w_{s+2}^2 \end{aligned}$$

We define r on $A(BO) \otimes V_3$ analogously. Then $r(y_i^2) = r(y_i^3) = 0$.

We conclude this section with an algebraic lemma about the y_j^i 's. Let $L_l \subset A(BO) \otimes V_l$ be defined as follows: $L_l = 0$ for $l = 0$, $l = 3$ and s even, and $l > 3$.

$$L_1 = A(BO)(\{y_i^1\} + S_1)$$

where $S_1 = \{v_3Sq^2\lambda^2\}$ when $s = 2$ and $S_1 = 0$ for $s \neq 2$.

$$L_2 = A(BO)(\{y_i^2\} + S)$$

where $S_2 = \{v_3\lambda^{1,2}\}$ when $s = 2$ and $S_2 = 0$, $s \neq 2$.

$$L_3 = A(BO)\{y_3^2\}$$

$$(d(v_3\lambda^{1,2}) = v_3Sq^2\lambda^2).$$

LEMMA 3.6. $d(L_l) \subset L_{l-1}$, $r(L_l) = 0$ for $l > 1$ and the sequence

$$\longrightarrow L_l \xrightarrow{d} L_{l-1} \longrightarrow \cdots \longrightarrow L_0$$

is exact at L_l^q for all l and $q \leq 2s + 2$.

Proof. The first part of 3.6 is clear from the definition of L_l . One easily checks that if $x \in A(BO)$, $|x| \leq 1$ and $d(xy_2^3) = 0$, then $x = 0$ and therefore $d : L_3^q \rightarrow L_2^{q+1}$ is an injection for $q \leq 2s + 2$. $d : L_2 \rightarrow L_1$ is clearly onto. To check exactness at L_2^q , $q \leq 2s + 2$ one must verify that if $y = x_1y_1^1 + x_2y_2^1 + x_3y_3^1 + x_4v_3Sq^2\lambda^2 = 0$, $x_i \in A(BO)$ and $|y| \leq 2s + 3$, then $x_1 = x_3 = x_4 = 0$ and $x_2 = 0$ or s is odd and $x_2 = Sq^1 + w_1$. This is a tedious but straightforward calculation, made somewhat simpler by the following observation. Let

$$F : A(BO) \otimes \{\lambda^s\} \rightarrow H^*(MO \wedge K(Z_2, N))$$

be given by

$$F(a \circ u\lambda^s) = a(u\chi(Sq^{s+1})U \otimes \iota_N)$$

Then

$$F(y_1^1) = v_{s+1}U \otimes \iota_N + U \otimes Sq^{s+1}\iota_N$$

$$F(y_2^1) = U \otimes Sq^{s+2}\iota_N$$

$$F(v_3Sq^2\lambda^2) = v_3^2U \otimes Sq^2\iota_N$$

We leave the details to the reader.

§4. Proofs of 2.4 and 2.8

Let $\{A \otimes \Lambda_l^k, d\}$ be the chain complex described in Proposition 3.2.

PROPOSITION 4.1. *For each integer k , there are Ω -spectra $Y_l = Y_l(k)$ and maps $\rho_l = \rho_l(k) : Y_{l-1} \rightarrow K(\Lambda_l^k)$ of degree one, $l = 0, 1, 2, \dots$ such that*

(i) $Y_0 = K(\Lambda_0^k)$. Y_l is a fibration over Y_{l-1} induced by ρ_l from the contractible fibration over $K(\Lambda_l^k)$.

(ii) If $i : K(\Lambda_{l-1}^k) \rightarrow Y_{l-1}$ is the inclusion of the fibre,

$$(\rho_l i)^* = d : A \otimes \Lambda_l^k \rightarrow A \otimes \Lambda_{l-1}^k$$

where d is as in 3.2.

(iii) If M is a smooth, compact n -manifold and ν is its normal bundle, then

$$[T(\nu), Y_l]_p \rightarrow [T(\nu), Y_{l-1}]_p$$

is an epimorphism for $p < 2k + 2$.

(iv) Suppose $k = 0$. Let $I(l, 0) = (0, \dots, 0)$ have length l .

$$\rho_l^* \lambda^{I(l,0)} = \delta_l \tilde{\iota}$$

where $\iota \in H^0(Y_{l-1}; Z_{2l})$, $\tilde{\iota}$ reduced modulo two is the generator $\iota \in H^0(Y_{l-1}) \approx Z_2$ and δ_l is the Bockstein associated to $Z_2 \rightarrow Z_{2^{l+1}} \rightarrow Z_{2^l}$.

Proof. For $k \geq 0$, 4.1(i), (ii), and (iii) were proved in [5]. For $k < 0$, $\{A \otimes \Lambda_l^k, d\}$ is a free acyclic resolution of the zero A module so that the existence of Y_l and ρ_l easily follow by induction on l . If M is as in (iii), $v: T(\nu) \rightarrow Y_{l-1}$ has degree p , $p < 2k + 2$ and $k < 0$, then $|(\rho_l v)^*(\lambda^I)| > n$ and (iii) follows.

Finally we prove (iv). The formula for d in 3.1 shows that $d\lambda^{I(l,0)} = Sq^1 \lambda^{I(l-1,0)}$. The complex,

$$\longrightarrow A \otimes \{\lambda^{I(l,0)}\} \xrightarrow{d} A \otimes \{\lambda^{I(l-1,0)}\} \longrightarrow \dots \longrightarrow A \otimes \{\lambda^{I(0,0)}\}$$

is realized by the tower

$$\rightarrow K(Z_{2^l}) \rightarrow K(Z_{2^{l-1}}) \rightarrow \dots \rightarrow K(Z_2)$$

with k -invariants, $\delta_l: K(Z_{2^l}) \rightarrow K(Z_2)$. Except for $\lambda^{I(l,0)}$, the generators of Λ_l^0 have dimension > 0 and hence kill classes of dimension > 1 . Thus $Y_l = K(Z_{2^{l+1}})$ in dimensions ≤ 1 . Therefore (iv) holds.

Proof of 2.4: We wish to realize the complex $\{A \otimes U_l, d\}$ by a tower of spectra, X_i . Let $Y_l(k)$ and $\rho_l(k)$ be as in 4.1. For a spectrum Z , let SZ denote the shift suspension, i.e., $(SZ)_q = Z_{q+1}$. Define X_i and $\alpha_i: X_{i-1} \rightarrow K(Y_i)$ by

$$X_i = \prod_{u_\omega \in U_0} S^{|u_\omega|} Y_i([(n - |u_\omega|)/2])$$

$$\alpha_i = \prod S^{|u_\omega|} \rho_i([(n - |u_\omega|)/2])$$

The map α_i takes X_{i-1} into $K(U_i)$ since

$$\prod S^k K(\Lambda_i^k) = K(U_i)$$

where k ranges over $[(n - |u_\omega|)/2], |u_\omega| \in U_0$. Proposition 2.4 now follows directly from 4.1.

Proof of 2.8: Using induction on l , one easily proves that if I is admissible and $l = l(I)$,

$$|\lambda^I| \geq 2t(I) \left(1 - \frac{1}{2^l}\right)$$

Suppose $l \geq n$ and $\lambda^I u_\omega \in U_l$. Then $2(t(I) + 1) > n - |u_\omega|$. Therefore

$$|\lambda^I u_\omega| \geq 2t(I) \left(1 - \frac{1}{2^l}\right) + |u_\omega| \geq n - 1 - \frac{n - |u_\omega| - 1}{2^l} > n - 2$$

Also if $|u_\omega| > n - 1$, $|\lambda^I u_\omega| > n - 1$. If $|u_\omega| < n - 1$, $t(I) \geq 1$ and hence $|\lambda^I| \geq l \geq n$. Therefore $U_l^q = 0$ for $q < n - 1$ and $U_l^{n-1} = \{\lambda^{I(l,0)} u_\omega \mid u_\omega \in U_0^{n-1}\}$ since $\lambda^{I(l,0)}$ is the only λ^I with $t(I) \geq 0$ and $|\lambda^I| = 0$. If $r > l$ and $\omega_r \neq \{\}$, $|u_\omega| \geq |u_{\omega_r}^{2^r}| \geq 2^r > n$. Hence $V_l^q = U_l^q$ for $q \leq n - 1$.

By the definition of $\beta_l : B_{l-1} \rightarrow K(V_l)$,

$$\Phi(\beta_l^*(\lambda^{I(l,0)} u_\omega)) = f_{l-1}^* \alpha_l^*(\lambda^{I(l,0)} u_\omega)$$

By 4.1(iv) $\alpha_l^*(\lambda^{I(l,0)} u_\omega) = \delta_l \tilde{t}$ where $\tilde{t} \in H^*(X_{l-1}; Z_{2^l})$ comes from the factor of X_{l-1} , $Y([n - |u_\omega|/2])$. Since the diagram

$$\begin{array}{ccc} T_{l-1} & \xrightarrow{f_{l-1}} & X_{l-1} \\ \downarrow p_1 & & \downarrow p_2 \\ T_0 & \xrightarrow{f_0} & X_0 \end{array}$$

commutes, $\tilde{u} = f_{0-1}^* \tilde{t}$ reduced modulo two is $p_1^* f_0^* u_\omega = p_1^* u_\omega U_0 = u_\omega U_{l-1}$, where U_l is the Thom class of T_l and the proof of 2.8 is complete.

§5. Proof of 2.7

If G_1 and G_2 are graded groups and $h : G_1 \rightarrow G_2$ is a homomorphism of degree i , we will say that h is k connected if $h : G_1^q \rightarrow G_2^{q+i}$ is an epimorphism for $q < k$ and a monomorphism if $q \leq k$. We will say that a sequence of graded groups and homomorphisms,

$$\cdots \rightarrow G_l \rightarrow G_{l-1} \rightarrow \cdots$$

is k -exact if

$$G_{l+1}^{q-i} \rightarrow G_l^q \rightarrow G_{l-1}^{q+j}$$

is exact for all l and $q \leq k$.

In §3 we constructed isomorphisms $\theta : A \otimes U_l \rightarrow A(BO) \otimes V_l$ and a subcomplex $\{L_l, d\} \subset \{A(BO) \otimes V_l, d\}$ such that

$$\longrightarrow L_l \xrightarrow{d} L_{l-1} \xrightarrow{d} \cdots \longrightarrow L_0 = 0$$

is $2s+2$ exact, $s = [n/2]$. In §4 we constructed a tower of fibrations $\rightarrow X_l \rightarrow X_{l-1} \rightarrow$ with k -invariants $\alpha_l : X_{l-1} \rightarrow K(U_l)$ associated to the complex $\{A \otimes U_l, d\}$. Let

$$\bar{H}^*(K(U_l)) = H^*(K(U_l)) / \theta^{-1}(L_l)$$

$$\bar{H}^*(X_l) = H^*(X_l) / \alpha_{l-1}^* \theta^{-1}(L_{l-1})$$

LEMMA 5.1: *The maps*

$$K(U_l) \xrightarrow{i} X_l \xrightarrow{p} X_{l-1} \xrightarrow{\alpha_l} K(U_l)$$

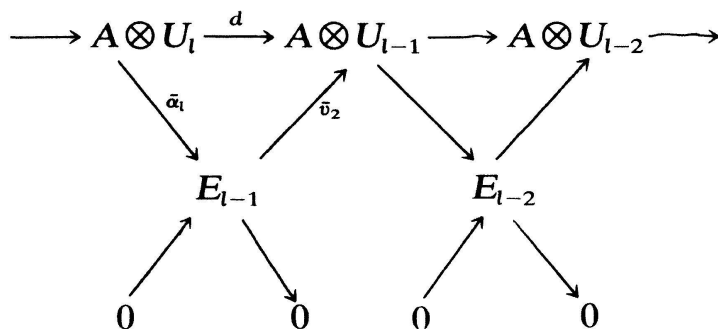
induce a $2s+2$ -exact sequence

$$\rightarrow \bar{H}^*(K(U)) \rightarrow \bar{H}^*(X_{l-1}) \rightarrow \bar{H}^*(X_l) \rightarrow$$

Proof: Let E_l be the kernel of

$$H^*(X_l) \rightarrow \lim_{k \rightarrow \infty} H^*(X_k)$$

Then $H^*(X_l) \approx H^*(MO) / \Phi(I_n) \oplus E_l$ and E_l and $A \otimes U_l$ are related by the diagram



where the $\bar{\alpha}_l$ and \bar{i}_l are defined by α_l^* and i_l^* and each pair of composable arrows is exact. Dividing $A \otimes U_l$ and E_{l-1} by $\theta^{-1}(L_l)$ and $\bar{\alpha}_l \theta^{-1}(L_{l-1})$, respectively, produces the same type of diagram with exactness replaced by $2s+2$ -exactness. The desired result then follows.

In §2 we defined maps

$$g'_l: K(U_l) \rightarrow T_{l-1}/T'_l$$

In §6 we prove:

LEMMA 5.2. *The map g'_l induces a $2s+2$ -connected map*

$$F_l: \bar{H}^*(K(U_l)) \rightarrow H^*(T_{l-1}/T'_l)$$

for $l \geq 1$.

Proof of 2.7: We first prove 2.7(ii). Suppose M is a smooth n -manifold, $h: M \rightarrow B_0 = BO$ classifies ν , the normal bundle of M and $\tilde{h}: M \rightarrow B_{l-1}$ is a lifting of h . Let $T(\tilde{h}): T(\nu) \rightarrow T_{l-1}$ denote the associated Thom space map. Then $f_{l-1}T(\tilde{h}): T(\nu) \rightarrow X_{l-1}$ is a lifting of $f_0T(h): T(\nu) \rightarrow X_0$ and hence by 2.4(iv), $f_{l-1}T(\tilde{h})$ lifts to X_l and therefore $\alpha_l f_{l-1}T(\tilde{h}) = 0$. Thus for $v \in \bar{V}_l$

$$\Phi h^* \beta_l^*(v_1) = T(\tilde{h})^* \Phi(\beta_l^*(v_1)) = T(\tilde{h})^* f_{l-1}^* \alpha_l^*(v) = 0$$

Thus $\beta_l \tilde{h} = 0$ and \tilde{h} lifts to $h': M \rightarrow B'_l$

If $u \in U_{l+1}$, $\bar{u} = \{u\} \in W_{l+1} = U_{l+1}/\ker \Delta$ and $\nu \theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$, then

$$\Phi((h')^* \gamma_l^*(\bar{u}_1)) = T(h')^* \Phi(\gamma_l^* \bar{u}_1) = T(h')^* \Delta(u).$$

Recall,

$$\Delta(u) = (f'_l)^* \alpha_{l+1}^* u - \sum x_i (f'_{l+1})^* \alpha^* u_i$$

But $T(h')^*$ is $A(BO)$ linear and $\alpha_{l+1} f'_l T(h') = 0$ as above. Thus $T(h')^* \Delta(u) = 0$ and hence $\gamma_l h' = 0$. Therefore h' lifts to B_l and the proof of 2.7(ii) is complete. We note for further reference:

LEMMA 5.3: $T(h')^* \Delta(u) = 0$ for $u \in U_{l+1}$.

LEMMA 5.4. If $\delta^*: H^*(T'_l) \rightarrow H^*(T_{l-1}/T'_l)$, $\delta^* \Delta(u) = 0$ for $u \in U_{l+1}$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} H^*(X_l) & \xrightarrow{i^*} & H^*(K(U_l)) \\ \downarrow f_l & & \downarrow (g_l) \\ H^*(T_l) & \xrightarrow{\alpha^*} & H^*(T_{l-1}/T_l) \end{array}$$

Recall, g'_0 realizes $\Psi r\theta$, $i^*\alpha_{l+1}^* = d$ and Ψ , r , and $d : A(BO) \otimes V_{l-1} \rightarrow A(BO) \otimes V_{l-1}$ are $A(BO)$ linear. Hence,

$$\begin{aligned} \delta^* \Delta(u) &= \delta^*((f_l)^* \alpha_{\alpha+1}^* u + \sum x_i (f_l)^* \alpha_{l+1}^* u_i) \\ &= (g_l)^* i^* \alpha_{l+1}^* u + \sum x_i (g_l)^* i^* \alpha_{l+1}^* u_i \\ &= \Psi r\theta du + \sum x_i \Psi r\theta du_i = \Psi r d\theta u + \sum \Psi r dx_i \theta(u_i) \end{aligned}$$

where $r\theta(u) = \sum x_i u_i$, $x_i \in A(BO)$ and $u_i \in V_{l+1}$. But for $v \in V_{l+1}$, $\theta(v) = v$. Thus

$$\sum x_i \theta(u_i) = \sum x_i u_i = r\theta u = \theta u + z$$

where $z \in L_{l+1}$. Furthermore $dz \in L_l$. Hence $\delta^* \Delta(u) = \Psi r dz = \Psi r \theta \theta^{-1} dz = (g_l)^* \theta^{-1} dz$.

But by 5.2, $\theta^{-1}(L_l)$ is the kernel of $(g_l)^*$.

We now prove that f_l induces a $2s+2$ -connected map $\bar{f}_l : \bar{H}(X_l) \rightarrow H^*(T_l)$ by induction on $l \geq 0$. We first show that \bar{f}_l is well defined.

$$\bar{H}^*(X_l) = H^*(X_l) / \alpha_{l+1}^*(\theta^{-1}(L_{l+1}))$$

From the commutative diagram:

$$\begin{array}{ccc} T_l & \xrightarrow{j} & T_l/T'_{l+1} \\ \downarrow f_l & & \downarrow g'_{l+1} \\ X_l & \xrightarrow{\alpha_{l+1}} & K(U_{l+1}) \end{array}$$

we see that

$$f_l^* \alpha_{l+1}^*(\theta^{-1}(L_{l+1})) = j^*(g'_{l+1})^*(\theta^{-1}(L_{l+1}))$$

By 5.2, $\theta^{-1}(L_{l+1})$ is in the kernel of $(g'_{l+1})^*$.

Since f_0^* is an isomorphism, $\bar{f}_0 = f_0^*$ and \bar{f}_0 is an isomorphism.

Suppose \bar{f}_{l-1} is $2s+2$ connected. If $u \in U_{l+1}$, $\Delta(u) \in H^q(T'_l)$ pulls back to $H^q(T_{l-1})$ since, by 5.4, $\delta^* \Delta(u) = 0$ and it pulls back to $H^q(X_{l-1})$ if $q < 2s+2$, that is, if $|u| < 2s+1$, $\Delta(u) = (f'_l)^* p^* x$ where $p: X_l \rightarrow X_{l-1}$. But since the X_l 's are constructed from an acyclic complex, image $p^* = \text{image } (H^*(X_0) \rightarrow H^*(X_l))$. Therefore image $(f'_l)^* p^* = \text{image } (H^*(T_0) \rightarrow H^*(T'_l)) = H^*(MO)/\Phi(I_n)$. But by 5.3, $\Delta(u)$ is zero on all n -manifolds. Hence $\Delta(u) = 0$ and we have shown that $W_{l+1}^q = (U_{l+1}/\ker \Delta)^q = 0$ for $q < 2s+1$. Therefore $H^q(B'_l) \rightarrow H^q(B_l)$ is an isomorphism for $q \leq 2s+2$ since B_l is a fibration over B'_{l-1} induced by $\gamma_l: B'_l \rightarrow K(W_{l+1})_1$. Then $H^q(T'_l/T_l) = H^q(B'_l, B_l) = 0$ for $q < 2s+2$ and hence

$$H^*(T_{l-1}/T'_l) \rightarrow H^*(T_{l-1}/T_l)$$

is $(2s+2)$ -connected. Let g_l be the composition

$$T_{l-1}/T_l \longrightarrow T_{l-1}/T'_l \xrightarrow{g_l} K(U_l)$$

and let $\bar{g}_l: \bar{H}^*(K(U_l)) \rightarrow H^*(T_{l-1}/T_l)$ be induced by g_l . Then \bar{g}_l is $(2s+2)$ -connected by 5.2. Consider the commutative diagram:

$$\begin{array}{ccccccc} \longrightarrow & \bar{H}^*(K(U_l)) & \longrightarrow & \bar{H}^*(X_{l-1}) & \longrightarrow & \bar{H}^*(X_l) & \longrightarrow & \bar{H}^*(K(U)) & \longrightarrow \\ & \downarrow \bar{g}_l & & \downarrow \bar{f}_{l-1} & & \downarrow \bar{f}_\sim & & \downarrow & \\ \longrightarrow & H^*(T_{l-1}/T_l) & \longrightarrow & H^*(T_{l-1}) & \longrightarrow & H^*(T_l) & \longrightarrow & H^*(T_{l-1}/T_l) & \longrightarrow \end{array}$$

A five lemma argument and the fact that \bar{f}_{l-1} and \bar{g}_l are $(2s+2)$ -connected shows that \bar{f}_l is $2s+2$ -connected.

Since $L_l = 0$ for $l > 3$, $\bar{H}^*(X_l) = H^*(X_l)$ for $l \geq 3$ and therefore $f_l^*: H^q(X_l) \rightarrow H^q(T_l)$ is an isomorphism for $q \leq n < 2s+2$. This completes the proof of 2.7.

§6. Proof of 5.2

LEMMA 6.1.

$$H^q(B_{l-1}) \rightarrow H^q(B'_l)$$

is an isomorphism for $l > 1$ and $q \leq s+1$. For $l = 1$ it is an epimorphism for $q \leq s+1$ and $v_{s+1}, w_1 v_{s+1}, Sq^1 v_{s+1}$ and v_{s+2} generate the kernel for $q \leq s+2$.

Proof. As we saw in the proof of 2.8, if $\lambda^I u_\omega \in V_l$, $|\lambda^I u_\omega| \cong (n-1) - (n - |u_\omega| - 1)/2^l$. Hence the lowest dimensional element in V_l is of the form λ^I with $t(I) = s$. For such an I , $|\lambda^I| \cong s + 2$ except for $l = 1$ or $l = 2$ and $s = 1$ and 2. The space B'_l is a fibration over B_{l-1} induced by $\beta_l : B_{l-1} \rightarrow K(V_l)_1$ and for $l > 1$, $K(V_l)_1$ is $s + 2$ connected except when $l = 2$ and $s = 1$ or 2. For $s = 1$ or 2, the lowest dimensional elements in V_2 are $\lambda^{1,1}$ and $\lambda^{1,2}$ respectively; $d\lambda^{1,1} \neq 0$ and $d\lambda^{1,2} \neq 0$ so these elements kill nonzero classes in B_1 . Thus for $l > 1$, $H^q(B_{l-1}) \approx H^q(B'_l)$ for $q \leq s + 1$.

Suppose $l = 1$. From 3.1 one sees that $d\lambda^i = \chi(Sq^{i+1})U = \Phi(v_{i+1})$ where U is the Thom class and v_{i+1} is the Wu class. Hence $\beta_1 : B_0 \rightarrow K(V_1)_1$ takes λ^i into v_{i+1} . One easily checks that $V_1^q = 0$ for $q < s$, $V_1^s = \{\lambda^s\}$ and $V_1^{s+1} = \{\lambda^{s+1}\}$. The remainder of 6.1 now follows by a simple Serre spectral sequence argument.

Let $K_l = K(V_l)_1$. Viewing $\beta_l : B_{l-1} \rightarrow K_l$ as a fibre map with fibre B'_l , consider the pair of fibrations p_1 and p_2 :

$$\begin{array}{ccc} (B_{l-1}, B'_l) & \xrightarrow{c} & (B_{l-1} \times K_l, B_{l-1} \times \{*\}) \\ & \searrow p_1 & \swarrow p_2 \\ & & (K_l, *) \end{array}$$

where p_1 is defined by β_l , p_2 is projection on the second factor and $c = id \times p$. Note c is a fibre preserving map so we may use it to compare the Serre spectral sequences of p_1 and p_2 .

LEMMA 6.2. For $l > 1$, $c^* : H^q(B_{l-1} \times K_l, B_{l-1} \times \{*\}) \rightarrow H^q(B_{l-1}, B'_l)$ is an isomorphism for $q \leq 2s + 3$. For $l = 1$, c^* is an epimorphism for $q \leq 2s + 2$ and for $q \leq 2s + 3$ the kernel is generated by

$$\begin{aligned} & v_{s+1} \otimes \lambda_1^s + 1 \otimes (\lambda_1^s)^2 \\ & v_{s+1} \otimes Sq^1 \lambda_1^s + 1 \otimes \lambda_1^s Sq^1 \lambda_1^s \\ & v_{s+1} \otimes \lambda_1^{s+1} + 1 \otimes \lambda_1^s \lambda_1^{s+1} \\ & w_1 v_{s+1} \otimes \lambda_1^s + w_1 \otimes (\lambda_1^s)^2 \\ & Sq^1 v_{s+1} \otimes \lambda_1^s + 1 \otimes \lambda_1^s Sq^1 \lambda_1^s \\ & v_{s+2} \otimes \lambda_1^s + 1 \otimes \lambda_1^s \lambda_1^{s+1} \end{aligned}$$

Proof. Let $E_r^{p,q}$ and $\bar{E}_r^{p,q}$ denote the Serre spectral sequences for p_1 and p_2 respectively.

$$E_2^{p,q} = H^p(K_l, *) \otimes H^q(B_{l-1})$$

$$\bar{E}_2^{p,q} = H^p(K_l, *) \otimes H^q(B')$$

As we saw above, for $l > 1$, K_l is $s + 2$ connected and $H^q(B_{l-1}) \approx H^q(B'_l)$ for $q \leq s + 1$. Therefore c induces an isomorphism at the E_2 level for $p + q \leq 2s + 3$ and the differentials are trivial for p_2 because it is a product fibration. This proves 6.2 for $l > 1$.

For $l = 1$, 6.2 is true at the E_2 level with the first summands in the above list of elements as a basis for the kernel; the second summands are of lower filtration. The same is true at the E_∞ level, so to complete the proof, we must show that these elements are in the kernel of c^* .

Under the map $H^*(B_0, B'_1) \rightarrow H^*(B_0)$, $c^*(1 \otimes \lambda_1^s)$ goes to v_{s+1} . Hence

$$c^*(v_{s+1} \otimes \lambda_1^s + 1 \otimes (\lambda_1^s)^2) = v_{s+1}c^*(1 \otimes \lambda_1^s) + c^*(1 \otimes \lambda_1^s)^2 = 0$$

(If $j: X \subset (X, A)$ and $x \in H^*(X, A)$, $x^2 = (j^*x)x$.) The same argument applies to the other five elements.

Let

$$\phi: (A(BO) \otimes \bar{V}_l)^q \rightarrow H^{q+1}(T_{l-1} \wedge K_l)$$

be defined by

$$\phi((a \otimes w)u) = a(wU \otimes u_1)$$

where U is the Thom class, $a \in A$, $w \in H^*(BO)$ and $u \in \bar{V}_l$.

LEMMA 6.3. *For $q \leq 2s + 1$, ϕ is an epimorphism. For $q \leq 2s + 2$ the kernel of ϕ is zero for $l > 1$ and $(l, s) \neq (2, 2)$, is $\{v_3\lambda^{1,2}\}$ for $(l, s) = (2, 2)$ and is $\{\sum Sq^i \circ v_{s+2-i}\lambda^s\}$ for $l = 1$.*

Proof. Let $\mu, \mu': A(BO) \rightarrow A(BO)$ be defined by

$$\begin{aligned} \mu(a \circ w) &= \sum a'_i \circ w\sigma(a''_i) \\ \mu'(a \circ w) &= \sum a'_i \circ w\chi(a''_i) \end{aligned}$$

(Recall, σa is defined by $(\sigma a)/U = \chi(a)(wU)$.)

Where $a \rightarrow \sum a'_i \otimes a''_i$ in the diagonal in A . Then $\mu\mu' = \mu'\mu = \text{identity}$ and thus μ is a Z_2 -isomorphism. Let $\phi' = \phi(\mu \otimes id)$. Then

$$\phi'((a \circ w)u) = \sum a'_i(\chi(a''_i)(wU) \otimes u_1) = wU \otimes au_1$$

Let λ^l be the lowest dimensional element in \bar{V}_l ; $|\lambda^l| > s$ for $l = 1$. The lowest

dimensional element in $H^*(T_{l-1} \wedge K_l)$ not in the image of ϕ' is $U \otimes (\lambda_1^l \cup Sq^i \lambda_1^l)$, an element of dimension $\geq 2s + 3$. Hence ϕ is an epimorphism for $q < 2s + 2$. The lowest dimensional elements in the kernel of ϕ' are $1 \circ v_{s+1} \lambda_1^l$ or $(Sq^m \circ 1) \lambda_1^l$ where $m = |\lambda_1^l| + 1$. For $l > 2$, $(l, s) \neq (2, 2)$, $\lambda_1^l > s + 1$ and hence these elements occur in dimensions $> 2s + 3$. For $(l, s) = (2, 2)$, $\phi(v_3 \lambda^{1,2}) = \phi'(v_3 \lambda^{1,2}) = 0$. For $l = 1$

$$0 = \phi'((Sq^{s+2} \circ 1) \lambda^s) = \phi\left(\left(\sum Sq^i \circ v_{s+2-i}\right) \lambda^s\right)$$

This proves the last part of 6.3.

Proof of 5.2: We must show that

$$(g_l')^* = \Psi r \theta : (A \otimes U_l)^q \rightarrow H^{q+1}(T_{l-1}/T_l')$$

is an epimorphism for $q \leq 2s + 2$ and $(L_l)^q$ is the kernel for $q \leq 2s + 2$. By 2.5, θ is an isomorphism. Let ϕ be the map in 6.3 and c the map in 6.2. Lifting c to the Thom space level we obtain a map

$$T(c) : T_{l-1}/T_l' \rightarrow T_{l-1} \wedge K_l$$

Furthermore $\Psi = T(c)^*$. Thus by 6.2 and 6.3, Ψ is an epimorphism for $q \leq 2s + 1$ and since r is an epimorphism, $(g_l')^*$ is an epimorphism for $q \leq 2s + 1$. For $l > 1$ and $(l, s) \neq (2, 2)$, $T(c)^*$ and ϕ are monomorphisms for $q \leq 2s + 2$ and L_l^q is the kernel of r . When $(l, s) = (2, 2)$ $r(L_l) = \{v_3 \lambda^{1,2}\}$. This completes the proof of 5.2 for $l > 1$.

Suppose $l = 1$. Then $r = \text{identity}$. We wish to show that $L_1 = \phi^{-1}(\ker T(c)^*)$. In 6.2 a basis for $\ker c^*$ was given for $q \leq 2s + 2$. Since image $\phi = \text{image } \phi'$ cannot involve cup products (except squares) in $H^*(K_l)$, the above basis shows that the following is a basis for image $\phi \cap \ker T(c)^*$:

$$\begin{aligned} &v_{s+1} U \otimes \lambda_1^s + U \otimes Sq^{s+1} \lambda_1^s \\ &w_1 v_{s+1} U \otimes \lambda_1^s + w_1 U \otimes Sq^{s+1} \lambda_1^s \\ &v_{s+1} U \otimes Sq^1 \lambda_1^s + (Sq^1 v_{s+1}) U \otimes \lambda_1^s \\ &v_{s+1} U \otimes \lambda_1^{s+2} + v_{s+2} U \otimes \lambda_1^s \end{aligned}$$

Thus a basis for $\phi^{-1}(\ker c^*)$ is ϕ^{-1} of these elements and $(\sum Sq^i \circ v_{s+2-i}) \lambda^s$ from the kernel of ϕ . A simple calculation shows that these elements form a basis for L_1^q , $q \leq 2s + 2$, completing the proof of 5.2.

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