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Area preserving twist homeomorphism of the annulus*

by JOHN N. MATHER

Let $f: A \to A$ be an orientation preserving and area preserving homeomorphism of the annulus. The homeomorphism f is said to be a *twist* homeomorphism if f admits a lifting $\tilde{F}: \tilde{A} \to \tilde{A}$ to the universal cover $\tilde{A}$ of A which moves one boundary component in one direction and the other boundary in the other direction. Such a lifting will be called *admissible.*

If x is a fixed point of f, and $\tilde{x}$ is a point of $\tilde{A}$ covering x, then $\tilde{f}(\tilde{x}) = \tilde{x} + k$ for some integer k. The integer k depends on $\tilde{f}$ but not on $\tilde{x}$. The integer k will be called the *Nielsen index* of x. We let F denote the set of fixed points of f of Nielsen index 0. A celebrated theorem of Birkhoff states:

THEOREM. (Birkhoff). *If f is an area preserving, orientation preserving, twist homeomorphism of A and $\tilde{f}$ is an admissible lifting, then F contains at least two points.*

This was proved (in a more general setting) in [2]. An earlier paper [1] of Birkhoff proves the existence of at least one point in F, and claims to prove the existence of a second point in F, but the proof of the existence of the second point was erroneous. A summary of Birkhoff's work on this problem, and a clear proof of Birkhoff's theorem may be found in [3]. The way Birkhoff's theorem is usually stated is that f has at least two fixed points, but the stronger condition we have stated is proved in [3] (by essentially Birkhoff's method).

We can now state the main theorem which we will prove in this paper.

THEOREM 1. *Let f be an area preserving, orientation preserving, twist homeomorphism of the annulus A. Let U be an open set lying in the interior of A and containing F. Then $\bigcup_{k=0}^{\infty} f^k(U)$ separates the two boundary components of A. In other words, the two boundary components of A lie in different components of $A - \bigcup_0^\infty f^k(U)$.*

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§1. Beginning of the proof of Theorem 1

First, we show that it is enough to prove that $\bigcup_{k=-\infty}^{\infty} f^k(U)$ separates the two boundary components of A.

LEMMA 1. *If $\bigcup_{k=-\infty}^{\infty} f^k(U)$ separates the two boundary components of A, then so does $\bigcup_{k=-N}^{N} f^k(U)$ for sufficiently large N.*

Proof. Let $C_n = A - \bigcup_{k=-N}^{N} f^k(U)$. If for any N, $\bigcup_{-N}^{N} f^k(U)$ does not separate, let B_N be the component of C_N which contains ∂A. Then $B_\infty = \bigcap_N B_N$ is connected, since it is the decreasing intersection of compact connected sets. This contradicts the hypothesis, and so proves the lemma. □

Now suppose $\bigcup_{k=-N}^{N} f^k(U)$ separates the boundary components of A. By applying f^N to $\bigcup_{k=-N}^{N} f^k(U)$, we see that $\bigcup_{k=0}^{2N} f^k(U)$ separates the boundary components of A, and thus $\bigcup_{k=0}^{\infty} f^k(U)$ does likewise.

Let W be the union of $\bigcup_{k=-\infty}^{\infty} f^k(U)$ with all points of A which are separated from both boundary components by $\bigcup_{k=-\infty}^{\infty} f^k(U)$. It is easily seen that W is an open set in A.

Moreover, if W separates the two components of the boundary of A, then $\bigcup_{k=-\infty}^{\infty} f^k(U)$ already does so. For, if not, the connected component D of $A - \bigcup_{k=-\infty}^{\infty} f^k(U)$ which contains one boundary component also contains the other. However, no points of D are in W, so W would not separate after all.

Thus, we have reduced the problem of showing that $\bigcup_{k=0}^{\infty} f^k(U)$ separates to the problem of showing that W separates. The topology of W is described by the following result.

LEMMA 2. *If W separates the two boundary components of A, then one component of W is homeomorphic to an open annulus, and all other components are homeomorphic to open disks. If W does not separate the two boundary components of A, then each component of W is homeomorphic to an open disk.*

Proof. Let D denote the complement of W in A. By Lefschetz duality, there is an isomorphism

$$H_1(W) \simeq \check{H}^1(A, D)$$

where the left side denotes singular homology with $\mathbf{Z}$ coefficients, and the right side denotes Čech cohomology with $\mathbf{Z}$ coefficients. (An exposition of Lefschetz duality is given in [6]. See Chapter 6, §2, Theorem 19. The cohomology appearing on the right side there is defined by means of a direct limit, but it is the same as Alexander cohomology by the tautness property of Alexander cohomology, cf.

Chapter 6, §6, and Alexander cohomolgy is the same as Cech cohomology, Chapter 6, §8, Corollary8.)

It is easily seen that D is the union of the connected components of $A-\bigcup_{k=-\infty}^{\infty} f^k(U)$, which meet the boundary of A. Thus it has either two components (in the case W separates) or one component (in the case W doesn't). In the exact sequence

$$\tilde{\check{H}}^0(A) \to \tilde{\check{H}}^0(D) \to \check{H}^1(A, D) \to \check{H}^1(A) \to \check{H}^1(D),$$

we see that $\tilde{\check{H}}^0(A)=0$ and $\check{H}^1(A) \to \check{H}^1(D)$ is injective (because D contains the boundary components). Hence

$$\tilde{\check{H}}^0(D) \to \check{H}^1(A, D)$$

is an isomorphism, and $\check{H}^1(A, D)=\mathbf{Z}$ or 0 according to whether W separates or not. Thus $H_1(W)=\mathbf{Z}$ or 0 according to whether W separates or not. But it is well known that for any open connected set W_1 in the plane, W_1 is homeomorphic to an annulus if $H_1(W_1)=\mathbf{Z}$ and homeomorphic to a disk if $H_1(W_1)=0$. □

To prove theorem 1, we may assume that each component of U meets F, since in any case, we may replace U by a smaller open neighborhood of F having this property. In this case each component of W meets F. Assuming this is the case, we will complete the prooof by showing the *no* component of W is homeomorphic to a disk. Thus, according to Lemma 2, W will have just one component , homeomorphic to an annulus and separating the two boundary components of A.

Since F is open and closed in the fixed point set of f, we may define the Lefschetz index of F (with respect to f) just as we would define the Lefschetz index of an isolated fixed point of f. Specifically, let A^0 denote the interior of A and let $\nu \in H^2(A^0 \times A^0, A^0 \times A^0 - \Delta)$ denote the Thom Class, where Δ denotes the diagonal of $A^0 \times A^0$. Let G be a neighborhood of F in A^0 which contains no points of the fixed point set of f other than F. We have a mapping

$$(1, f):(G, G-F) \to (A^0 \times A^0, A^0 \times A^0 - \Delta)$$

Let $\mu \in H_2(G, G-F)$ be the orientation class. The Lefschetz index of F with respect to f is defined as

$$L_f(F)=\langle (1, f)^* \nu, \mu \rangle.$$

If F is finite this is just the sum of the Lefschetz indices of the points of F.

From the fact that f is a twist mapping, it follows easily that $L_f(F)=0$. Indeed, we may suppose that $\bar{G}$ contains no fixed points of f other than F. We let F^* be a compact neighborhood of F in G. We may compute $L_f(F)$ using $(G, G-F^*)$ in place of $(G, G-F)$. For a sufficiently small pertubation f' of f, we have

$$\langle(1,f)^*\nu, \mu\rangle = \langle(1,f')^*\nu, \mu\rangle$$

where $\mu \in H_2(G, G-F^*)$ denotes the orientation class. But the right side is $L_{f'}(F')$, where F' is the set of fixed points of f' of Nielsen class 0. Thus, we have shown that $L_f(F)$ is unchanged by small pertubations of f. However, the set of twist homeomorphisms has two components, each of which is easily seen to contain an f for which $L_f(F)=0$. Thus $L_f(F)=0$ for every twist homeomorphism f.

LEMMA 3. *Let W_1 be a connected component of W and let $F_1 = F \cap W_1$. Suppose $F_1 \neq \phi$. If W_1 is homeomorphic to an open disk, then $L_f(F_1)=1$. If W_1 is homeomorphic to an open annulus, then $L_f(F)=0$.*

Using this Lemma, we can finish the proof of Theorem 1 very easily. We have seen that we may suppose, without loss of generality, that each connected component of W meets F. In this case W has only finitely many connected components, $W = W_1 \cup \cdots \cup W_k$ and

$$L_f(F) = \sum_{c=1}^{k} L_f(F_c)$$

where $F_c = F \cap W_c$. But the left hand side is 0, and by Lemma 2 and 3 every summand on the right side is 0 or 1. Thus, every summand on the right side must be 0, which, according to Lemmas 2 and 3 is possible only if W is homeomorphic to an open annulus. Thus Theorem 1 will be proved, once Lemma 3 is proved.

We have actually proved more than Theorem 1: If every connected component of U meets F, then W is homeomorphic to an annulus.

§2. Caratheodory's theory of ends

We will develop that part of Caratheodory's theory of ends which we need for the proof of Lemma 3. We will state definitions and quote theorems, but for examples and proofs we will refer to Caratheodory's memoir [4] and a subsequent development of the theory due to Cartwright and Littlewood [5].

Let G be a bounded open set in the plane. By a *cross-cut of G*, we will mean a simple arc which lies in G, except for its endpoints, which are in the boundary of

G. In the case the two endpoints coincide, so we have a simple closed curve with one point on the boundary of G, and otherwise in G, the curve will still be called a cross-cut, provided the two components it separates the plane into both contain points of the boundary of G.

A sequence $q_1, q_2, \ldots$ of cross-cuts will be called a *chain* if $q_i \cap q_j = \phi$ for $i \neq j$ (including endpoints), and each q_n separates G into two regions and q_{n+1} separates q_n from q_{n+2}. If $q_1, q_2, \ldots$ is a chain of cross-cuts, we let g_n denote that region of G, determined by q_n, which contains q_{n+1}. A sequence of open sets $g_1, g_2, \ldots$ obtained in this way will be called a *chain* of open sets. Note that $g_1 \supset g_2 \supset \cdots \supset g_n \supset \cdots$. We will say that such a chain of cross-cuts, or, equivalently chain of open sets determines an end E_g.

If E_g and E_h are ends, determined by chains $g_1, g_2, \ldots,$ and $h_1, h_2, \ldots$ of open sets we say E_g is contained in E_h and write $E_g \subset E_h$ if for each n, there is an $m(n)$ such that $g_{m(n)} \subset h_n$. If $E_g \subset E_h$ and $E_h \subset E_g$, we say E_g and E_h are the same end, and write $E_g = E_h$. If E_g is an end and for any end E_h such that $E_h \subset E_g$, we have $E_h = E_g$, then E_g is said to be a *prime* end. Let E_g be an end of G and let U be an open set in G. We will say U *contains* E_g if $U \supset g_n$ for some n, where $g_1 \supset g_2 \supset \cdots$ is a chain of open sets defining E_g.

Let $\hat{G}$ be G and all of its prime ends. We topologize $\hat{G}$ as follows. A subset U will be open if $U \cap G$ is open in the original topology of G, and for each prime end $E_g \in U$ there is an open set V in G such that:

(1) $V \subset U \cap G$,
(2) V contains E_g,
(3) If V contains a prime end E_h, then $E_h \in U$.

The principal result of Caratheodory's investigation leads to the following Proposition.

PROPOSITION (Caratheodory). *If G is simply connected, then $\hat{G}$ is homeomorphic to a closed disk.*

See [4], Satz XIII and the footnote to Satz XV. In fact Caratheodory formulated his results in terms of sequences, but what we have just stated is an easy consequence of Caratheodory's result. From the Proposition, we immediately obtain the following result.

COROLLARY. *If G is homeomorphic to an open annulus, then $\hat{G}$ is homeomorphic to a closed annulus.*

Now we suppose G is homeomorphic to an open disk or open annulus. We let

$h: G \to G$ be a homeomorphism which extends to a homeomorphism $\bar{h}: \bar{G} \to \bar{G}$. It is clear that h extends to a homeomorphism $\hat{h}: \hat{G} \to \hat{G}$. We will need the following result.

PROPOSITION (Cartwright and Littlewood). *If h is area preserving, and $\bar{h}$ has no fixed point on the boundary of G, then $\hat{h}$ has no fixed point on the boundary of $\hat{G}$.*

See Cartwright and Littlewood [5], Lemma 11. The hypothesis that h is area preserving is essential.

§3. End of the proof of Theorem 1

In §1, we have reduced the problem of proving Theorem 1 to proving Lemma 3. To prove Lemma 3, we first consider the case W_1 is homeomorphic to an open disk.

For each n, let $\pi_n : A_n \to A$ denote the n-fold covering of A. Thus, A_n is the annulus. Let $f_n : A_n \to A_n$ denote the mapping which covers f and is covered by $\tilde{f}$. Let W_n denote one component of $\pi_n^{-1} W_1$. The inverse image in A_n of a point of F is fixed under f_n, since its inverse image in $\tilde{A}$ is fixed under $\tilde{f}$. Since $F_1 \neq \phi$, there is a point in W_n which is fixed under f_n, and it follows that $f_n(W_n) = W_n$ and $f_n^{-1}(W_n) = W_n$, since $f(W) = W$ and $f^{-1}(W) = W$.

Since $F \subset W$, $F \cap \partial W_1 = \phi$. In other words, any fixed point of f on the boundary of W_1 must have non-zero Nielsen index. Let n be any integer which does not divide the Nielsen index of *any* fixed point on ∂W_1. Then no fixed point of f_n lies on ∂W_n.

By the Cartwright and Littlewood result (§2), there is no fixed point of $\hat{f}_n$ on the boundary of $\hat{W}_n$, where $\hat{W}_n$ denotes the Caratheodory construction (§2). But $\hat{W}_n$ is a closed disk, so we have constructed a homeomorphism of a closed disk with no fixed points on the boundary. But the set of fixed points on the interior corresponds precisely to F_1. It is well known that the Lefschetz index of the set of points on the interior with respect f_n has Lefschetz index 1 when f_n has no fixed points on the boundary. Hence $L_f(F_1) = 1$, as asserted.

This is enough to prove Theorem 1. But to complete the proof of Lemma 3, we must consider the case W_1 is homeomorphic to an open annulus.

In case W_1 is an open annulus, it clearly separates the two boundary components of A. We let $W_n = \pi_n^{-1} W_1$; this is connected. We may choose n so that all the fixed points of $f_n : \bar{W}_n \to \bar{W}_n$ are in the interior of W_n, and so they correspond n to 1 to the points of F_1. Thus if we let F_n denote the set of fixed points of f_n in W_n we have

$$L_{f_n}(F_n) = nL_f(F_1).$$

But $\hat{f}_n : \hat{W}_n \to \hat{W}_n$ is a mapping of the closed annulus into itself, which does not have any fixed points on the boundary, by the Cartwright–Littlewood result (§2). Clearly

$$L_{f_n}(F_n) = L_{\hat{f}_n}(F_n) = 0,$$

so $L_f(F_1) = 0$.

§4. Remarks of the Proof

The only place in the proof where we have used the hypothesis that f is a twist homeomorphism is to assure $F \neq \phi$, and $F \cap \partial A = \phi$. Thus, our proof actually shows a more general result.

THEOREM 2. *Let f be an area preserving, orientation preserving homeomorphism of the annulus A and $\tilde{f}: \tilde{A} \to \tilde{A}$ any lifting of f. Suppose $F \neq \phi$ and $F \cap \partial A = \phi$. Let U be an open set lying in the interior of A and containing F. Then $\bigcup_{k=0}^{\infty} f^k(U)$ separates the two boundary components of A.*

§5. The visiting set

If X is a closed invariant set of a homeomorphism $f: A \to A$ we define its visiting set Vis (X) to be the set of all $x \in A$ such that if U is any neighborhood of x in A and V is any neighborhood of X in A then $f^n(U) \cap V \neq \phi$ for some integer n. The visiting set is clearly closed and invariant.

If x is a periodic point of f, and $\tilde{f}: \tilde{A} \to \tilde{A}$ is a lifting of f, then for any $\tilde{x}$ covering x, we have

$$\tilde{f}^q(\tilde{x}) = T^p(\tilde{x})$$

for some integers p and$q$ where T is a generator of the group of covering transformations of $\tilde{A}$. The rational number p/q depends on $\tilde{f}$ but not on $\tilde{x}$. We call p/q the rotation number of x (with respect to $\tilde{f}$).

THEOREM 3. *Let f be an orientation preserving, area preserving mapping of an annulus A onto itself, and let $\tilde{f}: \tilde{A} \to \tilde{A}$ be any lifting of f. Let $\alpha = p/q$ be a* *rational number and let* F_α *be the set of periodic points of rotation number* α *(with respect to* $\tilde{f}$*) and period q. If* $F_\alpha \neq \phi$ *and* $F_\alpha \cap \partial A = \phi$*, then* $\mathrm{Vis}\,(F_\alpha) \cap A^0$ *separates the two boundary components of A.*

Remark. Let a be the Poincare rotation number of $\tilde{f}$ on the lower boundary of A, and b the Poincare rotation number of $\tilde{f}$ restricted to the upper boundary of A. It is an easy consequence of the Birkhoff fixed point theorem that if α lies between a and b, then $F_\alpha \neq \phi$. Clearly, in this case, $F_\alpha \cap \partial A = \phi$.

Proof. The set F_α is the set of fixed points of f^q of Nielsen index 0 with respect to the covering transformation $T^{-p}\tilde{f}^q$. Therefore if V is any neighborhood of F_α in the interior of A, $\bigcup_{k=-\infty}^{\infty} f^{kq}(V)$ separates the two boundary components of A. Let $V_1, V_2, V_3, \ldots$ be a neighborhood basis of F_α, and let

$$X_n = \mathrm{closure}\,\Big(\bigcup_{k=-\infty}^{\infty} f^k(V_n)\Big).$$

Clearly

$$\mathrm{Vis}\,(F_\alpha) = \bigcap_{n=1}^{\infty} X_n.$$

However, since each $X_n \cap A^0$ separates the two boundary components, it follows easily that $\mathrm{Vis}\,(F_\alpha) \cap A^0$ does also.

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