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The Boolean algebra of spectra

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Introduction

Let $\mathbf{Ho}^s$ denote the stable homotopy category of CW-spectra (cf. [Adams 2]), and for $E \in \mathbf{Ho}^s$ let E_* be the associated homology theory. For $E, G \in \mathbf{Ho}^s$ we say E_* and G_* have the same acyclic spectra if the following equivalent conditions hold:

(i) For $X \in \mathbf{Ho}^s$, $E_*X = 0 \Leftrightarrow G_*X = 0$.

(ii) For $f: X \to Y \in \mathbf{Ho}^s$, $f_*: E_*X \approx E_*Y \Leftrightarrow f_*: G_*X \approx G_*Y$.

This gives a very coarse equivalence relation for spectra, and we let $\mathbf{A(Ho^s)}$ consist of all the equivalence classes $\langle E\rangle$ for $E \in \mathbf{Ho}^s$, where $\langle E\rangle$ is given by all $G \in \mathbf{Ho}^s$ such that E_* and G_* have the same acyclic spectra. We partially order $\mathbf{A(Ho^s)}$ by writing $\langle E\rangle \le \langle G\rangle$ when each G_*-acyclic spectrum is E_*-acyclic. Our purpose in this note is to study the algebraic structure of $\mathbf{A(Ho^s)}$ when it is equipped with the relation $\le$ and the operations $\vee$ and $\wedge$ induced from the usual wedge and smash product for spectra.

We say that $\langle E\rangle \in \mathbf{A(Ho^s)}$ has a *complement* $\langle E\rangle^c \in \mathbf{A(Ho^s)}$ if $\langle E\rangle \wedge \langle E\rangle^c = \langle 0\rangle$ and $\langle E\rangle \vee \langle E\rangle^c = \langle S\rangle$ where S is the sphere spectrum, and we note that $\langle E\rangle^c$ is unique when it exists. We let $\mathbf{BA(Ho^s)} \subset \mathbf{A(Ho^s)}$ consist of those $\langle E\rangle \in \mathbf{A(Ho^s)}$ with complements, and we observe that $\mathbf{BA(Ho^s)}$ is a Boolean algebra. We prove that $\langle E\rangle \in \mathbf{BA(Ho^s)}$ whenever E is a (possibly infinite) wedge of finite CW-spectra. It would be most interesting to determine the sublattice of $\mathbf{BA(Ho^s)}$ given by such $\langle E\rangle$. We show that $\langle S^0 \cup_\alpha e^n\rangle = \langle S^0\rangle$ for each $\alpha \in [S^{n-1}, S^0]$ with $n \ne 1$, and that $\langle DE\rangle = \langle E\rangle$ when E is a finite CW-spectrum and DE is its Spanier-Whitehead dual. This incidentally implies that $G_*E = 0 \Leftrightarrow G^*E = 0$, for $G, E \in \mathbf{Ho}^s$ with E finite. Some other members of $\mathbf{BA(Ho^s)}$ are $\langle K\rangle$ and $\langle SZ_{(J)}\rangle$ where K is the spectrum of complex K-theory and $SZ_{(J)}$ is the Moore spectrum associated with a subring $Z_{(J)} \subset Q$. Indeed, $\langle K\rangle$ and $\langle SZ_{(J)}\rangle$ are of the form $\langle E\rangle^c$ where E is an appropriate wedge of finite CW-spectra, though the proof for K will be postponed to [Bousfield 3].

We also introduce a distributive lattice $\mathbf{DL(Ho^s)}$ given by all $\langle E\rangle \in \mathbf{A(Ho^s)}$ with $\langle E\rangle \wedge \langle E\rangle = \langle E\rangle$, and we show that $\mathbf{BA(Ho^s)} \subset \mathbf{DL(Ho^s)} \subset \mathbf{A(Ho^s)}$ where both containments are proper. It turns out that $\langle E\rangle \in \mathbf{DL(Ho^s)}$ whenever E is a (possibly infinite) wedge of ring spectra and finite CW-spectra. In fact, most familiar spectra represent elements of $\mathbf{DL(Ho^s)}$.

The class $\mathbf{A(Ho^s)}$ has applications to the homological localization theory of spectra, cf. [Bousfield 3], [Ravenel]. In particular, the E_*-localization is equivalent to the G_*-localization iff $\langle E\rangle = \langle G\rangle$, and a determination of $\mathbf{A(Ho^s)}$ would provide an inventory of the possible homological localization functors.

Our results on the structure of $\mathbf{A(Ho^s)}$, $\mathbf{BA(Ho^s)}$, and $\mathbf{DL(Ho^s)}$ are established in §2. Some of our proofs involve $[E,]_*$-colocalizations of spectra, and we develop the required theory in §1.

We essentially use the notation and terminology of [Adams 2]. However, we let $\mathbf{Ho^s}$ be the category of CW-spectra and homotopy classes of maps of degree 0, cf. [Adams 2, p. 144]. Thus $\mathbf{Ho^s}$ is an additive category equipped with an equivalence $\sum : \mathbf{Ho^s} \to \mathbf{Ho^s}$ induced by the "shift" suspension $\sum$ of CW-spectra. We write $[X, Y]$ for the group of morphisms $X \to Y \in \mathbf{Ho^s}$, and write $[X, Y]_n$ for $[\sum^n X, Y]$ where $n \in Z$. By a cofibre sequence we mean a sequence in $\mathbf{Ho^s}$ equivalent to $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$ for some cellular map f of degree 0 between CW-spectra, cf. [Adams 2, p. 155]. Recall that $\mathbf{Ho^s}$ has arbitrary coproducts induced by the wedge $\vee$ for CW-spectra, and for $X, Y \in \mathbf{Ho^s}$ there is a natural smash product $X \wedge Y \in \mathbf{Ho^s}$ which is associative, commutative, and unitary (with the sphere spectrum S as unit) up to coherent natural equivalences, cf. [Adams 2, p. 158]. We call $E \in \mathbf{Ho^s}$ a ring spectrum if it is equipped with an associative (but not necessarily commutative) multiplication $E \wedge E \to E$ and a two sided unit $S \to E$ in $\mathbf{Ho^s}$. As usual, we let $X_*Y = \pi_* X \wedge Y \doteq [S, X \wedge Y]_*$ for $X, Y \in \mathbf{Ho^s}$.

§1. $[E,]_*$-colocalizations of spectra

In preparation for §2 and for [Bousfield 3], we now develop the $[E,]_*$-colocalization theory of spectra. Some of the concepts here have previously been developed by J. P. May (unpublished) and in [Bousfield 2].

For $E \in \mathbf{Ho^s}$, a map $f : A \to B \in \mathbf{Ho^s}$ is called an $[E,]_*$*-equivalence* if $f_* : [E, A]_* \approx [E, B]_*$, and a spectrum $C \in \mathbf{Ho^s}$ is called $[E,]_*$*-colocal* if $g_* : [C, X]_* \approx [C, Y]_*$ whenever $g : X \to Y$ is an $[E,]_*$-equivalence. It is easy to check:

(1.1) E is $[E,]_*$-colocal.

(1.2) If $\{X_\alpha\}$ is a set of $[E,]_*$-colocal spectra, then $\vee_\alpha X_\alpha$ is $[E,]_*$-colocal.

(1.3) If $W \to X \to Y$ is a cofibre sequence in $\mathbf{Ho^s}$ and any two of W, X, Y are $[E,]_*$-colocal, then so is the third.

(1.4) If X is $[E,]_*$-colocal, then so is $X \wedge Y$ for all $Y \in \mathbf{Ho^s}$.

A map $\varphi: X \to A \in \mathbf{Ho}^s$ is called an $[E, \,]_*$-*colocalization* of A if X is $[E, \,]_*$-colocal and φ is an $[E, \,]_*$-equivalence. Note that the $[E, \,]_*$-colocalizations of A are initial among the $[E, \,]_*$-equivalences with target A, and are terminal among the maps from $[E, \,]_*$-colocal spectra to A. $[E, \,]_*$-colocalizations are clearly unique up to equivalence and

PROPOSITION 1.5. *Each spectrum $A \in \mathbf{Ho}^s$ has an $[E, \,]_*$-colocalization.*

Proof. We inductively construct a transfinite sequence of inclusions of CW-spectra

$$A = B_0 \subset B_1 \subset \cdots \subset B_s \subset B_{s+1} \subset \cdots$$

where $B_\lambda = \bigcup_{s<\lambda} B_s$ for each limit ordinal λ and where $B_s \subset B_{s+1}$ is given by the push-out square

$$\begin{array}{ccc} \bigvee_{\alpha \in I} \bigvee_{f: M_\alpha \to B_s} M_\alpha & \longrightarrow & B_s \\ \downarrow & & \downarrow \\ \bigvee_{\alpha \in I} \bigvee_{f: M_\alpha \to B_s} \mathrm{Cone}(M_\alpha) & \longrightarrow & B_{s+1} \end{array}$$

in which $\{M_\alpha\}_{\alpha \in I}$ consists of all cofinal subspectra of the spectra $\sum^n E$ for $n \in Z$, and f ranges over all cellular functions $M_\alpha \to B_s$ of degree 0, cf. [Adams, p. 140, 154]. Now let σ be the number of stable cells in E and let γ be the first infinite ordinal of cardinality greater than σ. Then for each $\alpha \in I$, each cellular function $M_\alpha \to B_\gamma$ of degree 0 extends over $\mathrm{Cone}(M_\alpha)$ because the image of M_α is contained in B_s for some $s < \gamma$. Thus $[E, B_\gamma]_* = 0$. Since A is a closed subspectrum of B_γ (cf. [Adams 2, p. 154]), there is an associated cofibre sequence

$$\Sigma^{-1}(B_\gamma/A) \to A \to B_\gamma$$

in $\mathbf{Ho}^s$. The morphism $\sum^{-1}(B_\gamma/A) \to A$ is clearly an $[E, \,]_*$-equivalence, so it suffices to show $\sum^{-1}(B_\gamma/A)$ is $[E, \,]_*$-colocal. For this is suffices to show inductively that B_s/A is $[E, \,]_*$-colocal for all s. If B_s/A is $[E, \,]_*$-colocal, then so is B_{s+1}/A because there is a cofibre sequence

$$B_s/A \to B_{s+1}/A \to B_{s+1}/B_s \in \mathbf{Ho}^s$$

where B_{s+1}/B_s is equivalent to a wedge of iterated (de)-suspensions of E. If B_s/A

is $[E,\ ]_*$-colocal for all $s<\lambda$ where λ is a limit ordinal, then B_λ/A is $[E,\ ]_*$-colocal because there is a cofibre sequence

$$\bigvee_{s<\lambda} B_s/A \xrightarrow{1-g} \bigvee_{s<\lambda} B_s/A \to B_\lambda/A \in \mathbf{Ho}^s$$

where g is induced by the maps $B_s/A \to B_{s+1}/A$. This completes the induction and the proof 1.5.

For each $A \in \mathbf{Ho}^s$ let $\varphi: {}^E A \to A \in \mathbf{Ho}^s$ denote the $[E,\ ]_*$-colocalization given by $\Sigma^{-1}(B_\gamma/A) \to A$ above, and note that it is functorial and idempotent in the obvious sense. To clarify the nature of $[E,\ ]_*$-colocal spectra, we let *Class-E* denote the smallest class of spectra in $\mathbf{Ho}^s$ such that: (i) $E \in$ *Class-E*; (ii) if $\{X_\alpha\}$ is a set of spectra in *Class-E*, then $\vee_\alpha X_\alpha \in$ *Class-E*; and (iii) if $W \to X \to Y$ is a cofibre sequence in $\mathbf{Ho}^s$ and any two of W, X, Y are in *Class-E*, then so is the third.

PROPOSITION 1.6. *Class-E equals the class of* $[E,\ ]_*$*-colocal spectra in* $\mathbf{Ho}^s$.

Proof. *Class-E* is contained in the class of $[E,\ ]_*$-colocal spectra by (1.1)–(1.3). Conversely, if X is $[E,\ ]_*$-colocal, then $X \in$ *Class-E* because ${}^E X \simeq X$ and ${}^E X \in$ *Class-E* by the proof of 1.5.

We call a spectrum $W \in \mathbf{Ho}^s$ $[E,\ ]_*$-*trivial* if $[E, W]_* = 0$, and we note that $[V, W]_* = 0$ whenever V is $[E,\ ]_*$-colocal and W is $[E,\ ]_*$-trivial. Each spectrum A can be canonically built from $[E,\ ]_*$-colocal and $[E,\ ]_*$-trivial spectra as follows. Extend $\varphi: {}^E A \to A$ to the cofibre sequence

(1.7) ${}^E A \xrightarrow{\varphi} A \xrightarrow{\nu} A^E \in \mathbf{Ho}^s$

given by $\Sigma^{-1}(B_\gamma/A) \to A \to B_\gamma$ above, and observe that A^E is $[E,]_*$-trivial. Indeed, ν is clearly the $[E,]_*$-*trivialization* of A, i.e. ν is the initial example of a map from A to an $[E,]_*$-trivial spectrum. It is useful to observe:

(1.8) If $V \to X \to W$ is a cofibre sequence in $\mathbf{Ho}^s$ with V $[E,\ ]_*$-colocal and with W $[E,\ ]_*$-trivial, then $V \to X \to W$ is equivalent to the cofibre sequence

$${}^E X \xrightarrow{\varphi} X \xrightarrow{\nu} X^E$$

It is straightforward to check that the $[E,\ ]_*$-colocalization and $[E,\ ]_*$-trivialization functors on $\mathbf{Ho}^s$ commute with suspension and preserve cofibre sequences. In [Bousfield 3] we will show that for each $E \in \mathbf{Ho}^s$ there exists a spectrum $aE \in \mathbf{Ho}^s$ such that the E_*-localization and E_*-acyclization functors are respectively equivalent to the $[aE,\ ]_*$-trivialization and $[aE,\ ]_*$-colocalization functors on $\mathbf{Ho}^s$. Thus, many examples of trivialization and colocalization functors will be implicity studied in [Bousfield 3].

§2. On the structure of $A(\mathbf{Ho}^s)$

We now examine the structure of the class $\mathbf{A}(\mathbf{Ho}^s)$ of "acyclicity types" of spectra, and we establish the results mentioned in the introduction concerning the distributive lattice $\mathbf{DL}(\mathbf{Ho}^s) \subset \mathbf{A}(\mathbf{Ho}^s)$ and the Boolean algebra $\mathbf{BA}(\mathbf{Ho}^s) \subset \mathbf{DL}(\mathbf{Ho}^s)$.

$\mathbf{A}(\mathbf{Ho}^s)$ has the following relations and operations:

(2.1) For $\langle X\rangle, \langle Y\rangle \in \mathbf{A}(\mathbf{Ho}^s)$, define $\langle X\rangle \le \langle Y\rangle$ if each Y_*-acyclic spectrum is X_*-acyclic. This is a partial order relation. Clearly $\langle 0\rangle$ is the smallest element of $\mathbf{A}(\mathbf{Ho}^s)$ and $\langle S\rangle$ is the largest. Note that if X is $[Y, \;]_*$-colocal (or equivalently, if $X \in \mathit{Class}\text{-}Y$), then $\langle X\rangle \le \langle Y\rangle$.

(2.2) For a set $\{\langle X_\alpha\rangle\}$ of elements in $\mathbf{A}(\mathbf{Ho}^s)$, define $\vee_\alpha \langle X_\alpha\rangle \in \mathbf{A}(\mathbf{Ho}^s)$ by $\vee_\alpha \langle X_\alpha\rangle = \langle \vee_\alpha X_\alpha\rangle$. Note that $\vee_\alpha\langle X_\alpha\rangle$ is the least upper bound of $\{\langle X_\alpha\rangle\}$ in $\mathbf{A}(\mathbf{Ho}^s)$, and $\vee$ is associative, commutative, and idempotent. Of course, $\langle 0\rangle \vee \langle X\rangle$ and $\langle S\rangle \vee \langle X\rangle = \langle S\rangle$.

(2.3) For $\langle X\rangle, \langle Y\rangle \in \mathbf{A}(\mathbf{Ho}^s)$ define $\langle X\rangle \wedge \langle Y\rangle \in \mathbf{A}(\mathbf{Ho}^s)$ by $\langle X\rangle \wedge \langle Y\rangle = \langle X \wedge Y\rangle$. This is well-defined: if $\langle X\rangle = \langle X_1\rangle$ and $\langle Y\rangle = \langle Y_1\rangle$, then clearly $\langle X \wedge Y\rangle = \langle X_1 \wedge Y\rangle = \langle X_1 \wedge Y_1\rangle$. Note that $\langle X\rangle \wedge \langle Y\rangle$ is a lower bound of $\langle X\rangle, \langle Y\rangle \in \mathbf{A}(\mathbf{Ho}^s)$, and that if $\langle X\rangle \le \langle X_1\rangle$ and $\langle Y\rangle \le \langle Y_1\rangle$ then $\langle X\rangle \wedge \langle Y\rangle \le \langle X_1\rangle \wedge \langle Y_1\rangle$. Clearly $\wedge$ is associative and commutative, with $\langle S\rangle \wedge \langle X\rangle = \langle X\rangle$ and $\langle 0\rangle \wedge \langle X\rangle = \langle 0\rangle$. Also the distributive law $\langle X\rangle \wedge (\vee_\alpha \langle Y_\alpha\rangle) = \vee_\alpha (\langle X\rangle \wedge \langle Y_\alpha\rangle)$ and absorption law $\langle X\rangle \vee (\langle X\rangle \wedge \langle Y\rangle) = \langle X\rangle$ hold.

(2.4) For each $\langle X\rangle \in \mathbf{A}(\mathbf{Ho}^s)$ there is an element $a\langle X\rangle \in \mathbf{A}(\mathbf{Ho}^s)$ such that $a\langle X\rangle$ is the greatest member of $\mathbf{A}(\mathbf{Ho}^s)$ with $\langle X\rangle \wedge a\langle X\rangle = \langle 0\rangle$. Moreover, $aa\langle X\rangle = \langle X\rangle$ for each $\langle X\rangle \in \mathbf{A}(\mathbf{Ho}^s)$, and $\langle X\rangle \le \langle Y\rangle$ if and only if $a\langle Y\rangle \le a\langle X\rangle$. This will be shown in [Bousfield 3], and we remark that $a\langle X\rangle = \langle aX\rangle$ where aX is the spectrum mentioned at the end of §1. It turns out that $\mathbf{DL}(\mathbf{Ho}^s)$ is not closed under $a(\;)$, although $a(\;)$ gives the complement in $\mathbf{BA}(\mathbf{Ho}^s)$. We won't use $a(\;)$ in this paper.

So far, $\mathbf{A}(\mathbf{Ho}^s)$ resembles a Boolean algebra with complement $a(\;)$, but the following lemma shows that $\wedge$ is not idempotent in $\mathbf{A}(\mathbf{Ho}^s)$.

LEMMA 2.5. *Let $X \in \mathbf{Ho}^s$ be a finite CW-spectrum with H_*X finite, and let $cX \in \mathbf{Ho}^s$ be the Brown-Comenetz dual of X. If $X \neq 0$, then $\langle cX\rangle \wedge \langle cX\rangle = \langle 0\rangle \neq \langle cX\rangle$.*

Proof. Using [Brown-Comenetz, 1.14] it is easy to show $H_*(cX; Z) = 0$, and thus $\langle H\rangle \wedge \langle cX\rangle = \langle 0\rangle$ where H is the spectrum for integral homology. Since $\pi_i cX$ is the Pontrjagin dual of $\pi_{-i}X$, it vanishes for sufficiently large i. Hence

$(cX)(n,\infty)\in \mathit{Class}\text{-}H$ for each n where $(cX)(n,\infty)$ is the $(n-1)$-connected section of cX. The cofibre sequence

$$\bigvee_{n\leq 0}(cX)(n,\infty)\to\bigvee_{n\leq 0}(cX)(n,\infty)\to cX\in \mathbf{Ho^s}$$

now shows $cX\in \mathit{Class}\text{-}H$, and thus $\langle cX\rangle\leq\langle H\rangle$. The lemma now follows since $\langle cX\rangle\wedge\langle cX\rangle\leq\langle H\rangle\wedge\langle cX\rangle=\langle 0\rangle$ and since $(cX)_*(S)\neq 0=0_*(S)$.

To avoid the pathological spectra revealed by 2.5, we introduce

2.6 The distributive lattice of spectra DL(Ho^s)

Let $\mathbf{DL(Ho^s)}$ consist of all $\langle X\rangle\in\mathbf{A(Ho^s)}$ with $\langle X\rangle\wedge\langle X\rangle=\langle X\rangle$. For instance, if E is a ring spectrum, then $\langle E\rangle\in\mathbf{DL(Ho^s)}$ because E is a retract of $E\wedge E$ in $\mathrm{Ho^s}$. Also, if E a Moore spectrum or a finite CW spectrum, then $\langle E\rangle\in\mathbf{DL(Ho^s)}$ by 2.9 and 2.13 below. Many other examples can be derived from the preceding, since $\mathbf{DL(Ho^s)}$ is closed under the operation $\vee$ (with any number of summands) and under $\wedge$; the proof for $\vee$ uses the equality $\langle X\rangle\vee(\langle X\rangle\wedge\langle Y\rangle)=\langle X\rangle$. With the operations $\vee$ and $\wedge$, $\mathbf{DL(Ho^s)}$ is clearly a distributive lattice with 0,1 as defined in the next paragraph.

We refer the reader to [Dwinger] or [Grätzer] for an exposition of distributive lattice theory, but for convenience we recall that a class L with binary operations $\vee$, $\wedge$ is a *distributive lattice with* 0,1 if:

(i) $x\wedge x=x$ and $x\vee x=x$ for $x\in L$.

(ii) $x\wedge y=y\wedge x$ and $x\vee y=y\vee x$ for $x, y\in L$.

(iii) $x\wedge(y\wedge z)=(x\wedge y)\wedge z$ and $x\vee(y\vee z)=(x\vee y)\vee z$ for $x, y, z\in L$.

(iv) $x\wedge(x\vee y)=x$ and $x\vee(x\wedge y)=x$ for $x, y\in L$.

(v) $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ and $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$ for $x, y, z\in L$.

(vi) There exist elements $0, 1\in L$ such that $0\vee x=x$ and $1\wedge x=x$ for all $x\in L$.

(Clearly, 0 and 1 are unique.) Now let L be a distributive lattice with 0,1. For $x, y\in L$ one writes $x\leq y$ if the equivalent conditions $x\wedge y=x$ and $x\vee y=y$ are satisfied. Then $\leq$ is a partial order relation on L, and $x\vee y$ (resp. $x\wedge y$) is the l.u.b. (resp. g.l.b.) of $x, y\in L$, cf. [Dwinger, p. 44] or [Grätzer, p. 6]. We also recall that L is called a *Boolean algebra* if for each $x\in L$ there exists $y\in L$ with $x\wedge y=0$ and $x\vee y=1$.

For $\langle X\rangle, \langle Y\rangle\in\mathbf{DL(Ho^s)}$ we conclude that $\langle X\rangle\wedge\langle Y\rangle$ is the g.l.b. of $\langle X\rangle$ and $\langle Y\rangle$, where $\mathbf{DL(Ho^s)}$ has the partial ordering inherited from $\mathbf{A(Ho^s)}$. Of course, we previously observed that $\langle X\rangle\vee\langle Y\rangle$ is the l.u.b. of $\langle X\rangle$ and $\langle Y\rangle$. Thus the algebraic structure of $\mathbf{DL(Ho^s)}$ is contained in its partial ordering.

We call $\langle Y\rangle\in\mathbf{A(Ho^s)}$ the *complement* of $\langle X\rangle\in\mathbf{A(Ho^s)}$ if $\langle X\rangle\wedge\langle Y\rangle=\langle 0\rangle$ and $\langle X\rangle\vee\langle Y\rangle=\langle S\rangle$. Note that if $\langle Y_1\rangle$ is also the complement of $\langle X\rangle$, then $\langle Y\rangle=\langle Y_1\rangle$

because

$$\langle Y\rangle=\langle Y\rangle\wedge(\langle X\rangle\vee\langle Y_1\rangle)=\langle Y\rangle\wedge\langle Y_1\rangle=(\langle X\rangle\vee\langle Y\rangle)\wedge\langle Y_1\rangle=\langle Y_1\rangle.$$

If $\langle X\rangle$ has a complement $\langle Y\rangle$, then $\langle X\rangle\in\mathbf{DL(Ho^s)}$ because $\langle X\rangle=\langle X\rangle\wedge(\langle X\rangle\vee\langle Y\rangle)=\langle X\rangle\wedge\langle X\rangle$, but the members of $\mathbf{DL(Ho^s)}$ need not have complements.

LEMMA 2.7. *$\langle H\rangle\in\mathbf{DL(Ho^s)}$, but $\langle H\rangle$ does not have a complement.*

Proof. $\langle H\rangle\in\mathbf{DL(Ho^s)}$ since H is a ring spectrum. Suppose $\langle H\rangle$ has a complement $\langle L\rangle$. Let $\langle cX\rangle$ be as in 2.5, and recall that $\langle cX\rangle\neq\langle 0\rangle=\langle H\rangle\wedge\langle cX\rangle$ and $\langle cX\rangle\leq\langle H\rangle$. Thus

$$\langle cX\rangle=(\langle H\rangle\vee\langle L\rangle)\wedge\langle cX\rangle=\langle L\rangle\wedge\langle cX\rangle\leq\langle L\rangle\wedge\langle H\rangle=\langle 0\rangle$$

and this contradicts $\langle cX\rangle\neq\langle 0\rangle$. Therefore $\langle H\rangle$ cannot have a complement.

We now introduce

2.8 The Boolean algebra of spectra BA(Ho^s)

Let $\mathbf{BA(Ho^s)}$ consist of all $\langle X\rangle\in\mathbf{A(Ho^s)}$ such that $\langle X\rangle$ has a complement (written $\langle X\rangle^c$), and note that $\mathbf{BA(Ho^s)}\subset\mathbf{DL(Ho^s)}$. If E is a Moore spectrum or a (possibly infinite) wedge of finite CW spectra, then $\langle E\rangle\in\mathbf{BA(Ho^s)}$ by 2.9 and 2.13 below. Many other members of $\mathbf{BA(Ho^s)}$ can be derived from the preceding, since $\mathbf{BA(Ho^s)}$ is clearly closed under $(\ )^c$ and the binary operations $\vee$, $\wedge$; indeed, for $\langle X\rangle$, $\langle Y\rangle\in\mathbf{BA(Ho^s)}$

$$\langle X\rangle^{cc}=\langle X\rangle$$
$$(\langle X\rangle\vee\langle Y\rangle)^c=\langle X\rangle^c\wedge\langle Y\rangle^c$$
$$(\langle X\rangle\wedge\langle Y\rangle)^c=\langle X\rangle^c\vee\langle Y\rangle^c.$$

With these operations, $\mathbf{BA(Ho^s)}$ is clearly a Boolean algebra.

As promised, we now prove

PROPOSITION 2.9. *If $E\in\mathbf{Ho^s}$ is a (possibly infinite) wedge of finite CW-spectra, then $\langle E\rangle\in\mathbf{BA(Ho^s)}$. Moreover, $\langle E\rangle=\langle {}^ES\rangle$ and $\langle E\rangle^c=\langle S^E\rangle$.*

Proof. Assume $E=\vee_\alpha B_\alpha$ where each B_α is a finite CW spectrum. A spectrum $Y\in\mathbf{Ho^s}$ is $[E,\ ]_*$-trivial iff $(DB_\alpha)\wedge Y\simeq 0\in\mathbf{Ho^s}$ for all α, where DB_α is the

Spanier-Whitehead dual of B_α. Thus if Y is $[E, \ ]_*$-trivial, then so is $X \wedge Y$ for all $X \in \mathbf{Ho^s}$. In particular, ${}^ES \wedge S^E$ is $[E, \ ]_*$-trivial as well as $[E, \ ]_*$-colocal (by 1.4), and thus ${}^ES \wedge S^E \simeq 0 \in \mathbf{Ho^s}$. Using the cofibering ${}^ES \to S \to S^E$ of (1.7), we conclude that $\langle {}^ES \rangle \vee \langle S^E \rangle = \langle S \rangle$, so $\langle {}^ES \rangle \in \mathbf{BA(Ho^s)}$ with $\langle {}^ES \rangle^c = \langle S^E \rangle$. It remains to show $\langle E \rangle = \langle {}^ES \rangle$. Applying (1.8) to the cofibre sequence

$$X \wedge {}^ES \to X \wedge S \to X \wedge S^E,$$

we find that $X \wedge {}^ES \simeq {}^EX$ and $X \wedge S^E \simeq X^E$ for all $X \in \mathbf{Ho^s}$. Since E is $[E, \ ]_*$-colocal, this implies $E \wedge {}^ES \simeq E$, and thus $\langle E \rangle \leq \langle {}^ES \rangle$. Since ${}^ES$ is $[E, \ ]_*$-colocal, we know $\langle {}^ES \rangle \leq \langle E \rangle$, and therefore $\langle E \rangle = \langle {}^ES \rangle$.

We remark that the above spectra ${}^ES$ and S^E satisfy the strong idempotency conditions ${}^ES \wedge {}^ES \simeq {}^ES$ and $S^E \wedge S^E \simeq S^E$. Indeed S^E is a commutative ring spectrum whose multiplication map $S^E \wedge S^E \to S^E$ is an equivalence. We next observe

PROPOSITION 2.10. *If* $E \in \mathbf{Ho^s}$ *is a finite CW spectrum, then* $\langle E \rangle = \langle DE \rangle$. *Consequently, for any* $G \in \mathbf{Ho^s}$, $G_*(E) = 0 \Leftrightarrow G^*(E) = 0$.

Proof. Since $[E, S^E]_* = 0$, we have $(DE) \wedge S^E \simeq 0$ and thus $(DE) \wedge {}^ES \simeq DE$. Since $\langle {}^ES \rangle = \langle E \rangle$, this implies $\langle DE \rangle \wedge \langle E \rangle = \langle DE \rangle$. Dually one shows $\langle E \rangle \wedge \langle DE \rangle = \langle E \rangle$, and therefore $\langle DE \rangle = \langle E \rangle$. The last statement is deduced using $G^*(E) = G_*(DE)$.

We next prove "triangle (in)equalities" for cofibre sequences. Call a map $f: A \to X \in \mathbf{Ho^s}$ *smash nilpotent* if the m-fold smash product

$$f \wedge \cdots \wedge f : A \wedge \cdots \wedge A \to X \wedge \cdots \wedge X \in \mathbf{Ho^s}$$

is the 0 map for some $m \geq 1$. Note that the smash nilpotent maps form a subgroup of $[A, X]$, and a composite fg is smash nilpotent if either f or g is. Moreover, if $i \neq 0$ then each map $S^i \to S^0 \in \mathbf{Ho^s}$ is a smash nilpotent by [Nishida].

PROPOSITION 2.11. *If* $A \xrightarrow{f} X \to B$ *is a cofibre sequence in* $\mathbf{Ho^s}$, *then*:

(i) $\langle A \rangle \leq \langle X \rangle \vee \langle B \rangle$, $\langle X \rangle \leq \langle A \rangle \vee \langle B \rangle$, and $\langle B \rangle \leq \langle A \rangle \vee \langle X \rangle$.

(ii) If $A, X \in \mathbf{DL(Ho^s)}$ and f is smash nilpotent, then $\langle B \rangle = \langle A \rangle \vee \langle X \rangle$.

Proof. Part (i) is obvious. For (ii) we assume $f \wedge \cdots \wedge f = 0$ in $\mathbf{Ho^s}$ and form a cofibre sequence

$$A \wedge \cdots \wedge A \xrightarrow{f \wedge \cdots \wedge f} X \wedge \cdots \wedge X \to C \in \mathbf{Ho^s}$$

Then

$$\langle C\rangle = \langle A\wedge\cdots\wedge A\rangle\vee\langle X\wedge\cdots\wedge X\rangle = \langle A\rangle\vee\langle X\rangle$$

and also $\langle C\rangle\le\langle B\rangle$ because C is $[B,\ ]_*$-colocal (i.e. $C\in$ Class-B). Thus $\langle A\rangle\vee\langle X\rangle\le\langle B\rangle$ and the opposite inequality is given by (i).

COROLLARY 2.12. *If* $n\neq 1$ *and* $\alpha: S^{n-1}\to S^0$ *in* **Ho**^{s}, *then* $\langle S^0\cup_\alpha e^n\rangle = \langle S\rangle$.

Because of Nishida's theorem, this follows from 2.11; or instead of using 2.11, we could have used the easy result that if $\Sigma^m A \xrightarrow{f} A \to B$ is a cofibre sequence in **Ho**^{s} with f nilpotent in $[A, A]_*$, then $\langle B\rangle = \langle A\rangle$. By combining 2.12 with R. Wood's result $K\simeq(S^0\cup_\eta e^2)\wedge KO$, we recover the result $\langle K\rangle = \langle KO\rangle$, cf. [Meier], [Ravenel].

Of course, 2.12 fails when $n=1$, and we now consider $\langle SG\rangle$ where G is an abelian group and $SG\in$ **Ho** is a Moore spectrum of type $(G,0)$ (i.e. $\pi_i SG=0$ for $i<0$, $H_0SG=G$, and $H_iSG=0$ for $i>0$). There is a short exact sequence

$$0\to G\otimes\pi_n X\to\pi_n(SG\wedge X)\to \operatorname{Tor}(G,\pi_{n-1}X)\to 0.$$

Thus

$$\langle S\rangle = \langle SQ\rangle\vee\bigvee_{p\text{ prime}}\langle SZ/p\rangle$$

$\langle SQ\rangle\wedge\langle SZ/p\rangle = \langle 0\rangle = \langle SZ/p\rangle\wedge\langle SZ/q\rangle$ for primes $p\neq q$.
It follows that **BA(Ho^s)** has a sub-Boolean algebra **MBA(Ho^s)** whose members are the wedges of subsets of

$$I=\{\langle SQ\rangle, \langle SZ/2\rangle, \langle SZ/3\rangle, \langle SZ/5\rangle, \ldots\}.$$

Moreover, there is an obvious Boolean algebra isomorphism between **MBA(Ho^s)** and the power set $P(I)$. Note that for any set J of primes

$$\langle SZ_{(J)}\rangle = \langle SQ\rangle\vee\bigvee_{p\in J}\langle SZ/p\rangle$$

where $Z_{(J)}$ is the localization of Z at J. More generally,

PROPOSITION 2.13. *For each abelian group* G, $\langle SG\rangle\in$ **MBA(Ho^s)**.

Proof. Let C be the class of all abelian groups A such that $G \otimes A = 0 = \mathrm{Tor}(G, A)$, and note that $(SG)_* X = 0$ if $\pi_i X \in C$ for all i. The result now follows easily from [Bousfield 1,2.3] since C is a "special" class.

We conclude by noting that **BA(Ho^s)** contains many elements outside **MBA(Ho^s)**. For p prime let

$$A(p): \Sigma^m SZ/p \to SZ/p \in \mathbf{Ho}^s$$

be the K_*-equivalence of [Adams 1, §12] where $m = 2p-2$ for p odd and $m = 8$ for $p = 2$. It is easy to check that the cofibre of $A(p)$ represents an element of **BA(Ho^s)** outside **MBA(Ho^s)**. In [Bousfield 3] we will show that $\langle K \rangle = \langle E \rangle^c$ where E is the wedge of the cofibres of all the $A(p)$.

REFERENCES

ADAMS, J. F. 1: *On the groups J(X)-IV.* Topology *5* (1966), 21–71.

ADAMS, J. F. 2: *Stable homotopy and generalized homology.* The University of Chicago Press, 1974.

BALBES, R. and DWINGER, P.: *Distributive Lattices.* The University of Missouri Press, 1974.

BOUSFIELD, A. K., 1: *Types of acyclicity.* J. Pure Appl. Algebra *4* (1974), 293–298.

BOUSFIELD, A. K., 2: *Constructions of factorization systems in categories.* J. Pure Appl. Algebra *9* (1977), 207–220.

BOUSFIELD, A. K., 3: *The localization of spectra with respect to homology.* (To appear).

BROWN E. H. and COMENETZ, M. *Pontrjain duality for generalized homology and cohomology theories.* Amer. J. of Math. *98* (1976), 1–27.

GRÄTZER, G., *Lattice Theory.* W. H. Freeman and Company, San Francisco, 1971.

MEIER, W., *Localization with respect to K-theory.* To appear.

NISHIDA, G., *The nilpotency of elements of the stable homotopy groups of spheres,* J. Math. Soc. Japan *25* (1973), 707–732.

RAVENEL, D., *Localization with respect to certain periodic homology theories.* Lecture given at Northwestern University, March 1977.

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