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On division by inner factors

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§1. Introduction

Let G denote a Banach space of function $f(z)$ analytic in the open disc $|z|<1$, and suppose that G is contained within the Hardy space H^1. If $h \in H^1$ we may write $h(z)=0(z).\ I(z)$ where $0(z)$ is the outer factor of h, $0(z) \in H^1$ and $I(z)$ is the inner factor, which is a function in H^∞ for which $\lim_{r\to 1} f(re^{i\theta})$ exists and is of modulus one almost everywhere on $|z|=1$. Such an inner factor may be written as

$$I(z)=B(z)\cdot(S(z),$$

where $B(z)$ is the Blaschke product formed from the zeros of $h(z)$, and $S(z)$ is a singular inner factor of the form

$$S(z)=\exp\left\{-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ix}+z}{e^{ix}-z}\,d\mu(x)\right\}. \tag{1}$$

Here, $\mu(x)$ is a monotonic increasing function on $(0, 2\pi]$ which is singular with respect to Lebesgue measure on $(0, 2\pi]$. For the details of such a factorization and other properties of H^p spaces we refer to [3].

DEFINITION 1. A Banach space G, contained within H^1 is said to have the f-property if, given any $h \in G$ and any inner function I which divides h (i.e. $hI^{-1} \in H^1$) then $hI^{-1} \in G$.

This notation was introduced by Havin [6], reporting on earlier work by numerous authors [2], [9], [10], [11]; see also [15] and the lecture by Kahane [8]. In addition to the obvious spaces $H^p(p>1)$ many other spaces, such as various Lipschitz spaces, have the f-property.

If $\phi(z) \in H^\infty$ and $h(z) \in H^2$ we may write

$$\phi(e^{i\theta})h(e^{i\theta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

and define the projection operator $P(\bar{\Phi}h)$ by

$$P(\bar{\phi}h) \sim \sum_{0}^{\infty} a_n e^{in\theta}.$$

(i.e. P is the Toeplitz operator associated with $\bar{\phi} \in L^\infty$). Clearly

$$P(\bar{\phi}h)(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{\bar{\phi}(w)h(w)}{w-z}\, dw, \qquad |z|<1.$$

DEFINITION 2. A Banach space G contained in H^1 is said to have the $K-$ property if given any $h \in G$ and $\phi \in H^\infty$ then $P(\bar{\phi}h) \in G$.

This property appears to have been considered first by Korenblum [9]. Its usefulness consists in the fact that if ϕ is an inner function dividing $h \in G$ then $P(\bar{\phi}h) = h\phi^{-1}$. Thus a space having the K-property also possesses the f-property.

The first example of a space not possessing the f-property and hence not possessing the K-property was given by Gurarii [5]. The space in question is l^1 (called $W^\dagger$ by Gurarii), defined by

$$l^1 = \left\{f : f(z) = \sum_0^\infty f_n z^n, \|f\| = \sum_0^\infty |f_n| < \infty\right\}.$$

The particular function considered is

$$h(z) = \frac{\pi}{2}\Gamma^2(\tfrac{3}{8}) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\Gamma(k+\frac{3}{8})}{(k+\frac{3}{8})\Gamma(k+\frac{5}{8})} z^{(8k+3)^2}.$$

This function clearly belongs to l^1 but if we write

$$f(z) = \exp\left\{\gamma \frac{1+z}{1-z}\right\} \cdot h(z),$$

where $\gamma = 2^{-9}\pi^2$ then $f \in H^1$ but $f \notin l^1$, ([5] p. 29).

The main difficulty in Gurarii's proof is in detecting the presence of the "jump" inner factor $\exp\{-\gamma\, 1+z/1-z\}$ in $h(z)$. Gurarii does not assert that the resulting function $f(z)$ is outer—there might be other inner factors. It seems very difficult in general to detect the presence of singular inner factors of the form (1) when the representing function $\mu(t)$ is continuous.

§2. Results

In this paper we present another example of a subspace X of H^1 which does not have the f-property. We construct a function $F \in X$ such that

$$F(z) = \Phi(z) \cdot S(z),$$

where $\Phi(z)$ is an outer function not in X and $S(z)$ is an inner function of the form (1) with $\mu(t)$ a continuous, monotonic, increasing function of t, which is singular with respect to Lebesgue measure on $(0, 2\pi]$. This direct construction avoids the difficulty of showing the presence of an inner factor.

We denote by B the Banach space of Bloch functions, i.e. functions $f(z)$ analytic in $|z|<1$ for which the norm

$$\|f\| = |f(0)| + \sup_{|z|<1} (1-|z|^2)\, |f^1(z)|$$

is finite. All polynomials belong to B and the closure of the polynomials in the Bloch norm is denoted by B_0. Alternatively, B_0 consists of those $f \in B$ for which

$$(1-|z|^2)f'(z) \to 0 \quad \text{as} \quad |z| \to 1-.$$

It is evident that $H^\infty \subset B$, but $H^\infty \not\subset B_0$—for these and other properties of Bloch functions we refer to [1]. The space X we wish to consider is

$$X = H^\infty \cap B_0.$$

It is easy to verify that if f and g belong to X so does $f.g$ so that, with a suitable choice of norm, X is a subalgebra of H^∞.

THEOREM 1. *X does not have the f-property (and so does not have the K-property).*

There is an interesting characterization of X which appears to be due, in the first instance, to M. Behrens. Let M denote the maximal ideal space of H^∞ and let $\hat{f}$ denote the transform of f in the standard Gelfand representation. Then $H^\infty \cap B_0$ consists precisely of all those $f \in H^\infty$ such that $\hat{f}$ is constant on all the non-trivial Gleason parts of M other than the image of the disc $|z|<1$. The parts are, of course, either single points or analytic discs, and the proof is a simple consequence of work of Hoffman [7]. The space X is thus a closed subalgebra of H^∞, a fact which can also be verified directly.

§3. Interference

To construct the function in X whose outer part (which belongs to H^∞ and so to B) does not belong to B_0, we require the following theorem of Shapiro [14 Theorem 4] regarding interference.

THEOREM A. *Let $\phi(x)$ be any increasing function on $[0, 2\pi]$ having continuous second derivative on the open interval $(0, 2\pi)$. Then there exists a function $f(x)$ continuous on $[0, 2\pi]$ such that*

$$\mu(x) = \phi(x) + f(x) \tag{2}$$

is increasing and singular,

$$|f(x+t)+f(x-t)-2f(x)| < Kt|\log t|^{-\frac{1}{2}}, \tag{3}$$

for some absolute constant K and all x, t satisfying $0<t<\frac{1}{2}$, $0 \leq x-t < x+t \leq 2\pi$.

Clearly $\mu(x)$ is continuous. As regards the function f, it is sufficient for our purposes to note that (3) implies that f belongs to the Zygmund class λ_*. Thus, if we define the function $F(z)$ by

$$F(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{ix}+z}{e^{ix}-z}\, df(x)\right\}, \tag{4}$$

then

$$F'(z) = 0(1-|z|^2)^{-1}, \qquad |z| \to 1-,$$

and so $F \in B_0$. (See [4] Theorem 1 for details.)

It is convenient to base the proof of Theorem 1 on another theorem, which seems of independent interest.

THEOREM 2. *There exists a monotonic increasing, infinitely differentiable function* $\phi(x)$ *defined for* $0<x<2\pi$ *such that the function*

$$\Phi(z)=\exp\left\{+\int_0^{2\pi}\frac{e^{ix}+z}{e^{ix}-z}\,d\phi(x)\right\} \tag{5}$$

belongs to H^∞, *but not to* B_0.

Proof of Theorem 1. Let $\phi(x)$ be as in Theorem 2 and $f(x)$ as in Theorem A. Then $\mu(x)=\phi(x)+f(x)$, or $f(x)=\mu(x)+(-\phi(x))$ with $\mu(x)$ singular and $-\phi(x)$ absolutely continuous with respect to Lebesgue measure. Thus

$$\begin{aligned} F(z)&=\exp\left\{-\int_0^{2\pi}\frac{e^{ix}+z}{e^{ix}-z}\,df(x)\right\}\\ &=\exp\left\{+\int_0^{2\pi}\frac{e^{ix}+z}{e^{ix}-z}\,d\phi(x)\right\}\cdot\exp\left\{-\int_0^{2\pi}\frac{e^{ix}+z}{e^{ix}-z}\,d\mu(x)\right\}\\ &=\Phi(z)\cdot S(z). \end{aligned}$$

Since $\Phi(z)$ and $S(z)$ belong to H^∞ so does $F(z)$, and $\Phi(z)$ is the outer part of $F(z)$. Thus $F(z)\in H^\infty\cap B_0=X$ by (3) but, by construction, $\Phi(z)$ does not belong to B_0. Thus X does not have the f-property.

§4. Proof of Theorem 2

We construct the required function $\phi(x)$ satisfying $\phi'(x)\leq 1$ for all x. This will imply that $|\Phi(z)|\leq e^{2\pi}$ for $|z|\leq 1$ and so $\Phi(z)\in H^\infty$. The standard example of a function in H^∞ but not in B_0 is the inner function $\exp\{-1+z/1-z\}$ which is of the form (1) with $\mu(x)$ discontinuous. Our construction is based on the modified function

$$\Phi(\delta,z)=\exp\left\{\int_{-\delta}^{\delta}\frac{e^{ix}+z}{e^{ix}-z}\,d(x+\delta)\right\}.$$

An integration yields

$$\Phi(\delta,z)=\exp\left\{-2\delta-2i\log\frac{e^{i\delta}-z}{e^{-i\delta}-z}\right\}.$$

We now consider z_0 real and equal to $1-\delta$. Then

$$(1-|z_0|^2)\Phi'(\delta, z_0)=\delta(2-\delta)\, e^{-2\delta} \exp\left[-2i \log\frac{e^{i\delta}-z_0}{e^{-i\delta}-z_0}\right]2i\left(\frac{1}{e^{i\delta}-z_0}-\frac{1}{e^{-i\delta}-z_0}\right)$$

An easy calculation now shows that

$$\lim_{\delta\to 0}(1-|z_0|^2)\Phi'(\delta, z_0)=-4\exp\left\{-2i\log\frac{1+i}{1-i}\right\}$$
$$=-4\, e^{\pi}.$$

We now define sequences $\{\delta_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ so that the function

$$\Phi(z)=\prod_{n=1}^{\infty}\Phi(\delta_n, ze^{i\theta_n})$$

has the required properties. We choose $\theta_1=0$ and $\theta_n=\pi(1-1/n)$, $n>1$. We then choose each δ_n so small so that, with $z_n=(1-\delta_n)\, e^{-i\theta_n}$,

$$(1-|z_n|^2)\phi^1(\delta_n, z_n)\approx -4\, e^{\pi}$$

but

$$|\Phi(\delta_n, z_m)|\approx 1 \qquad (m\neq n).$$

It is clear that this can be achieved simply by choosing the sequence $\{\delta_n\}$ sufficiently small. Thus for our function $\Phi(z)$ we have

$$\lim_{n\to\infty}(1-|z_n|^2)\,\Phi'(z_n)>0,$$

whereas $|z_n|\to 1$ as $n\to\infty$. Thus $\Phi(z)\notin B_0$.

If $\Phi(z)$ as constructed above has the representation (5), the representing function $\phi(x)$ will not be differentiable at the points $x=\theta_n+\delta_n$, $n=1,2,3,\ldots$. Nevertheless by adjusting the function $\phi(x)$ in intervals $[\theta_n-\delta_n-\epsilon_n, \theta_n-\delta_n+\epsilon_n]$ and $[\theta_n+\delta_n-\epsilon_n, \theta_n+\delta_n+\epsilon_n]$ where ϵ_n is now chosen very small compared with δ_n we may make the function $\Phi(x)$ as smooth as we please. Thus Theorem 2 is proved.

§5. Analytic VMO

We denote by BMOA the Banach space of functions $f(z)$ analytic for $|z|<1$ for which the norm

$$\|f\| = |f(0)| + \sup_{|\xi|<1} \|f_\xi\|_2 < \infty.$$

Here

$$(\|f_\xi\|_2)^2 = \frac{1}{2\pi}\int_0^{2\pi} \left| f\left(\frac{z+\xi}{1+\bar{\xi}z}\right) - f(\xi)\right|^2 |dz|, \qquad z = e^{i\theta}.$$

The subspace of BMOA for which

$$\|f_\xi\|_2 \to 0 \quad \text{as} \quad |\xi| \to 1- \tag{6}$$

is denoted by VMOA. A function belongs to VMOA if and only if its boundary values on $|z|=1$ belong to the space of functions of vanishing mean oscillation as defined, for example in [13]. This characterization of VMOA was introduced in [12] where it was shown that $f \in \text{VMOA} \Rightarrow f \in B_0$.

From the above characterization two interesting facts follow immediately:

(i) No inner function in H^∞ can belong to VMOA. To see this we note that if f is inner then there is a sequence $\{\xi_n\}$ with $|\xi_n| \to 1$ as $n \to \infty$ such that $f(\xi_n) \to 0$ as $n \to \infty$. Since

$$\frac{1}{2\pi}\int_0^{2\pi} \left| f\left(\frac{z+\xi}{1+\bar{\xi}z}\right)\right|^2 d\theta = 1$$

for all ξ with $|\xi|<1$ it follows that (6) does not hold.

(ii) If $Y = H^\infty \cap \text{VMOA}$ then $f, g \in Y$ implies $f \cdot g \in Y$. Thus Y, with suitable choice of norm can be made into a Banach algebra. The space Y is the "analytic" part of the space QC of quasicontinuous functions introduced by Sarason [13]. Since it was observed by Sarason that $QC = (H^\infty + C) \cap (\overline{H^\infty + C})$, we see that $Y = H^\infty \cap (\overline{H^\infty + C})$. In particular we see that Y is again a closed subalgebra of H^∞, a fact which can also be verified directly.

If A is defined by

$$A = \{f(z) : f(z) \text{ analytic for } |z|<1, \text{continuous for } |z| \leq 1\}.$$

then we have that $A \subset Y \subset X \subset H^\infty$. It is easy to see that all of the above inclusions are strict. An example of a function in X but not in Y is a function of the form (4) where $f(x)$ is a monotonic singular function satisfying (3). That such functions exist is known from [14, Theorem 2]. An example of a function in Y but not in A is a function $f(z)$ mapping $|z|<1$ $1-1$ conformally onto a bounded domain which has a bad prime end. Clearly $f \notin A$ but since f is univalent and in B_0 it is in VMOA ([12] Satz 1).

It would be nice to find a characterization of $Y = H^\infty$ VMOA similar to that of X mentioned in §2. Even though no inner function can belong to Y there are functions in Y which have an inner factor—the function constructed by Gurarii is in A and has a jump inner factor.

In conclusion it seems worthwhile to point out the following theorem

THEOREM 3. *Let $F(z)$ be given by* (4) *where $f(x)$ satisfies* (3). *Then there is a constant K_1 so that*

$$|F'(z)| < \frac{K_1}{1-|z|^2}\left[\log\left(\frac{1}{1-|z|^2}\right)\right]^{-\frac{1}{2}}$$

for $|z|<1$.

For our purposes we needed only that $F'(z) = O(1-|z|^2)^{-1}$, $|z| \to 1-$. The proof of this theorem is a straightforward adaptation of the proofs of ([4] Theorem 1) and ([16] Theorem 7), and so is omitted.

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