Some rational computations of the Waldhausen algebraic K theory.

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Some rational computations of the Waldhausen algebraic K theory

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Introduction

The purpose of this paper is to compute the rational part of the algebraic K-theory defined by Waldhausen [W] of the following type of topological (semisimplicial) rings R.:

(a) R. is an associative topological (semisimplicial) ring with unit and $\Pi_0(\mathrm{R.}) = \mathrm{Z}$ (Z denotes the ring of integers).

(b) There exists a ring homomorphism $\iota : \Pi_0(\mathrm{R.}) \to \mathrm{R.}$ so that $\pi\iota = \mathrm{id}$ where $\pi : \mathrm{R.} \to \Pi_0\ (\mathrm{R.})$[1] is the canonical projection.

(c) $\Pi_i(\mathrm{R.}) \otimes_{\mathrm{Z}} \mathrm{Q} = 0$[1] for $i \neq r \geq 1$ and $\Pi_r(\mathrm{R.}) \otimes_{\mathrm{Z}} \mathrm{Q} = \mathrm{Q}$, where Q is the ring of rationals.

These computations provide in particular the computation of the rational part of the Waldhausen's algebraic K-theory of a space X, in the case X has the rational homotopy type of a $\mathrm{K}(\mathrm{Z}, 2r)$[2] and implicitly the computation of the rational homotopy type of ${}^{Pl}\mathrm{Wh}(\mathrm{X})$, ${}^{Pl}\mathrm{Wh}\pm(\mathrm{X})$, ${}^{Diff}\mathrm{Wh}\ (\mathrm{X})\pm$ (see [B.L] for notations), for $\mathrm{K}(\mathrm{Z}, 2r)$.

In this paper we give the results only for ${}^{Diff}\mathrm{Wh}$, the problem of the computation of ${}^{Pl}\mathrm{Wh}(\mathrm{X})$, ${}^{Pl}\mathrm{Wh}\pm(\mathrm{X})$ will be contained in another paper on automorphisms of manifolds.

The methods of this paper allow the same computations for X of the rational homotopy type of a $\mathrm{K}(\mathrm{G}, 2r)$ but more "classical invariant theory" is necessary and the author has not yet worked it out.

The paper is organised as follows:

In section 1 we recall briefly the algebraic K-theory of Waldhausen for rings and for topological spaces and present the main results as a consequence of Theorem 3.1 of section 3. In section 2 we present the "invariant theory" necessary for the proof of Theorem 3.1 and in section 3 the proof of this theorem.

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[1] $\Pi_0(\mathrm{R.})$ denotes the ring of connected components and $\Pi_i(\mathrm{R.})$ the homotopy groups of R. with respect to the base point "0".

[2] $\mathrm{K}(\mathrm{G}, s)$ denotes the Eilenberg–MacLane space corresponding to G and s.

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§1

Let Ring^t be the category of topological (semisimplicial) rings which are always assumed to be associative and with unit, and continuous (semisimplicial) ring homomorphisms which are assumed to be unit preserving. Let Top_* be the category of based pointed topological spaces (semisimplicial complexes) and based point preserving maps, Gr^t be the category of topological (semisimplicial) groups and continuous (semisimplicial) homomorphisms and $\overset{\infty}{\Omega}$ the subcategory of Top_* consisting of ∞-loopspaces and ∞-loop space maps.

Following Waldhausen [W] one defines the algebraic K-theory as a functor $\mathbb{K}:\mathrm{Ring}^t \rightsquigarrow \overset{\infty}{\Omega}$ which is a homotopy functor in the sense that if f_1, f_2 are two homotopic morphisms (homotopic by "morphisms") then $\mathbb{K}(f_1)$ and $\mathbb{K}(f_2)$ are homotopic in $\overset{\infty}{\Omega}$ and if $f:R. \to R.$ is k-connected then $\mathbb{K}(f)$ is $(k+1)$-connected. Since we are not interested in the ∞-loop space structure of $\mathbb{K}$ we regard $\mathbb{K}$ as a functor with values in Top_* whose definition is the following.

For any n let $\widetilde{\mathrm{GL}}(R., n)$ be the space[3] (semisimplicial complex) of $n\times n$ matrices $\{a_{ij}\}$, $a_{ij}\in R.$, with $\{\pi(a_{ij})\}$ invertible. The composition of matrices endows $\widetilde{\mathrm{GL}}(R., n)$ with a structure of associative H-space and the inclusion $\widetilde{\mathrm{GL}}(R., n) \to \widetilde{\mathrm{GL}}(R., n+1)$ defined by $A \to \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1\end{array}\right)$ is a morphism of associative H-spaces. We take $\widetilde{\mathrm{GL}}(R.) = \widetilde{\mathrm{GL}}(R., \infty) = \lim_{\to} \widetilde{\mathrm{GL}}(R., n)$ which is an associative H-space whose $\Pi_0(\widetilde{\mathrm{GL}}(R.)) = \mathrm{GL}(\pi_0(R.)) = \lim_{\to} \mathrm{GL}(\Pi_0(R.); n)$. Applying the "classifying space" functor to $\widetilde{\mathrm{GL}}(R.)$ one obtains $B\widetilde{\mathrm{GL}}(R.)$ whose $\Pi_1(B\widetilde{\mathrm{GL}}(R.)) = \mathrm{GL}(\Pi_0(R))$ has the commutator a perfect group [L]. Consequently one can apply the Kervaire–Quillen's "+"-construction and the resulted space (semisimplicial complex) will be denoted by $B\widetilde{\mathrm{GL}}(R.)_+$. We define $\mathbb{K}(R.) = Z \times B\mathrm{GL}(R.)_+$ where Z denotes the ring of integers. If $f:R. \to R'.$ is a morphism, it induces $\mathbb{K}(f):\mathbb{K}(R.) \to \mathbb{K}(R'.)$ with the properties we have mentioned.

The "loop space" functor $\Omega:\mathrm{Top}_* \rightsquigarrow G^t_r$ in the semisimplicial case is the Kan's free group construction F and in topological case any "group type" construction

[3] With the obvious topology.

* Added in proofs: Similar results have been independently obtained by Hsiang and Staffeldt; more recently, the author and Hsiang and Staffeldt have obtained upper bounds for $\dim \Pi_i(\mathcal{H}(X)) \otimes Q$ for X 1-connected and with finite Betti numbers.

of the loop space for example the Milnor's construction [M_2] or $X \rightsquigarrow |F(\mathrm{Sing}\, X)|$ where Sing denotes the singular complex and $|\ldots|$ the "geometric realisation".

Let $\mathbf{Z}: G_r^t \rightsquigarrow \mathrm{Ring}^t$ be the functor which associates with any topological (semisimplicial) group G the ring $\mathbf{Z}(G)$ the topological (semisimplicial) analogous of the group ring; in the semisimplicial context it is actually the group ring, in topological context we can take $|\mathbf{Z}(\mathrm{Sing}\, G)|$ or any other functor from $G_r^t \rightsquigarrow \mathrm{Ring}^t$ which is essentially the infinite symmetric product. The composition $\mathcal{K} = \mathbb{K} \circ \mathbf{Z} \circ \Omega$ produces a functor defined on Top_* with values in $\overset{\infty}{\Omega}$ which is a homotopy functor and has the property that $f: X \to Y$ κ-connected implies $\mathcal{K}(f)$ κ-connected.

As we have mentioned, our purpose is to compute $\mathbb{K}(R.)$ for rings which satisfy the properties (a), (b), (c) mentioned in Introduction and in fact we prove the following theorem.

THEOREM 1.1. *If R. satisfies the conditions* (*a*), (*b*), (*c*) *then* $\mathbb{K}(R.)$ *has the rational homotopy of* $\mathbb{K}(Z) \times T_{r+1}$ *where* $T_r = \prod_{i=1}^{\infty} K(Z, 2si)$ *if* $r = 2s$ *and* $T_r = \prod_{i=1}^{\infty} K(Z; (2s+1)(2i-1))$ *if* $r = 2s+1$.

As a consequence one obtains.

THEOREM 1.2. *If X has the rational homotopy type of* $K(Z, 2r)$, $r > 0$, *then* $\mathcal{K}(X)$ *has the rational homotopy type of* $\mathbb{K}(Z) \times \prod_{i=1}^{\infty} K(Z; 2ri)$.

COROLLARY 1.3. *If X has the rational homotopy type of* $K(Z, 2r)$, ${}^{\mathrm{Diff}}\mathrm{Wh}(X)$, ${}^{\mathrm{Diff}}\mathrm{Wh}\pm(X)$ *have the rational homotopy type of* ${}^{\mathrm{Diff}}\mathrm{Wh}(pt)$ *respectively* ${}^{\mathrm{Diff}}\mathrm{Wh}\pm(pt)$.

Recall from [W] that ${}^{\mathrm{Diff}}\mathrm{Wh}(pt)$[4] has the rational homotopy type of $\prod_{i=1}^{\infty} K(Z; 4i)$ and from [F, H] that ${}^{\mathrm{Diff}}\mathrm{Wh}_+$ respectively ${}^{\mathrm{Diff}}\mathrm{Wh}_-$ have the rational homotopy type of ${}^{\mathrm{Diff}}\mathrm{Wh}(pt)$ respectively pt.

Proof of Theorem 1.1. Let $M_n(\mathring{R}.)$ respectively $\mathcal{M}_n(\mathring{R}.)$ be the space (semisimplicial complex) of $n \times n$ matrices with entries in $\mathring{R}.$, the connected component of "0", endowed with the composition law "+" respectively "$*$" given by $M * N = M + N + MN$.

Consider the diagram

$$(*)\quad \begin{array}{ccccc} \mathcal{M}_n(\mathring{R}.) & \xrightarrow{\sigma_n} & \widetilde{GL}(R., n) & \xrightarrow{\omega_n} & GL(\Pi_0(R.), n) \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_n} \\ \mathcal{M}_{n+1}(\mathring{R}.) & \xrightarrow{\sigma_{n+1}} & \widetilde{GL}(R., n+1) & \xrightarrow{\omega_{n+1}} & GL(\Pi_0(R.), n+1) \end{array}$$

[4] Our ${}^{\mathrm{Diff}}\mathrm{Wh}$ is the loop space of the one defined by Waldhausen in [W].

where $\sigma_n(M)=(M+I)$, $\tilde{\iota}_n$ and $\bar{\iota}_n$ are defined by $A \rightsquigarrow \begin{pmatrix} A & 0 \\ \hline 0 & 1 \end{pmatrix}$, i_n by $M \rightsquigarrow \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right)$ and ω_n by $\omega_n\{a_{ij}\}=\{\pi a_{ij}\}$.

Passing to the limit in diagram (*) one obtains the fibration

$$(**)\quad \mathcal{M}_\infty(R.) \xrightarrow{\sigma_\infty} \widetilde{GL}(R.) \xrightarrow{\omega_\infty} GL(\Pi_0(R.))$$

with all terms associative H-spaces and σ_∞ respectively ω_∞ homomorphisms. Applying the "classifying space" functor to (**) one obtains the fibration

$$(***)\quad B\mathcal{M}_\infty(\mathring{R}.) \xrightarrow{B\sigma_\infty} B\widetilde{GL}(R.) \xrightarrow{B\omega_\infty} BGL(\Pi_0(R.))$$

Assume now that the ring R. satisfies our hypothesis (b), hence there exists a morphism $\iota:\Pi_0(R.)\to R.$ so that $\Pi.\iota=\mathrm{id}$; ι induces the group homomorphism $\tilde{\iota}:GL(\Pi_0(R.)\to \widetilde{GL}(R.)$ and consequently we can define the representation ρ_∞ of $GL(\Pi_0(R.))$ on the H-space $M_\infty(\mathring{R}.)$ by $\rho_\infty(A;M)=\tilde{\iota}(A)\cdot M\cdot \tilde{\iota}(A)^{-1}$ for $A\in GL(\Pi_0(R.))$ $M\in\mathcal{M}_\infty(\mathring{R}.)$. Clearly

$$\rho_\infty(A;\ldots):\mathcal{M}_\infty(\mathring{R}.)\to\mathcal{M}_\infty(\mathring{R}.)$$

is an H-space isomorphism, consequently one can apply the classifying space functor to $\rho_\infty(A;\ldots)$ and obtain the action

$$B\rho_\infty:GL(\Pi_0(R.))\times B\mathcal{M}_\infty(\mathring{R}.)\to B\mathcal{M}_\infty(\mathring{R}.).$$

PROPOSITION 1.4. *The fibration* (***) *is the fibration over* $BGL(\Pi_0(R.))$ *associated with the action* $B\rho_\infty$.

Proof of Proposition 1.4. Let us recall the definition of the semidirect product of $\mathcal{M}_\infty(\mathring{R}.)\times_{\rho_\infty}GL(Z)$. This is the associative H-space structure defined on $\mathcal{M}_\infty(\mathring{R}.)\times GL(Z)$ by the following composition law

$$(M',A')\#(M,A)=(\{\rho_\infty(A^{-1};M)\}*M,A'\cdot A)$$

where $M,M'\in\mathcal{M}_\infty(\mathring{R}.)$ and $A,A'\in GL(Z)$. The natural projection $(M,A)\to A$ defines an homomorphism $p_2:\mathcal{M}_\infty(\mathring{R}.)\times_{\rho_\infty}GL(Z)\to GL(Z)$ whose kernel is exactly $\mathcal{M}_\infty(\mathring{R}.)$.

In order to prove Proposition 1.4 it is obviously enough to show that (**) is isomorphic to

$$\mathcal{M}_\infty(\mathring{R}.)\to\mathcal{M}_\infty(R.)\times_{\rho_\infty}GL(Z)\to GL(Z)$$

and this isomorphism is established by $\gamma : \widetilde{\mathrm{GL}}(\mathrm{R.}) \to \mathcal{M}_\infty(\mathring{\mathrm{R}}.) \times_{\rho_\infty} \mathrm{GL}(\mathrm{Z})$ defined by

$\gamma(\mathrm{A}) = (\sigma_\infty^{-1}\{\bar{\iota}(\omega(\mathrm{A})^{-1}) \cdot \mathrm{A} - \mathrm{I}\}, \omega(\mathrm{A}))$ which makes sense since

$\{\iota(\omega(\mathrm{A})^{-1}) \cdot \mathrm{A} - \mathrm{I}\} \in \sigma_\infty(\mathrm{M}_\infty(\mathrm{R.}))$

q.e.d.

Remark. The same proof shows that

$\mathrm{B}\mathcal{M}_n(\mathring{\mathrm{R}}.) \to \mathrm{B}\widetilde{\mathrm{GL}}(\mathrm{R.}, n) \to \mathrm{BGL}(\Pi_0(\mathrm{R.}), n))$ (with the hypothesis (b) on R.) is the fibration induced by the representation $\rho_n : \mathrm{GL}(\Pi_0(\mathrm{R.}), n) \times \mathcal{M}_n(\mathring{\mathrm{R}}.) \to \mathcal{M}_n(\mathring{\mathrm{R}}.)$ defined by the same formula.

Let us observe that if conditions (a), (b), (c) are satisfied then

$$\Pi_i(\mathrm{B}\mathcal{M}_\infty(\mathring{\mathrm{R}}.)) \otimes_{\mathrm{Z}} \mathrm{Q} = \Pi_{i-1}(\mathcal{M}_\infty(\mathring{\mathrm{R}}.)) \otimes_{\mathrm{Z}} \mathrm{Q} = \mathrm{M}_\infty(\Pi_{i-1}(\mathrm{R.}) \otimes_{\mathrm{Z}} \mathrm{Q}) = \begin{cases} 0 & \text{if } i \neq r+1 \\ \mathrm{M}_\infty(\mathrm{Q}) & \text{if } i = r+1 \end{cases}$$

and $\Pi_1(\mathrm{B}\mathcal{M}_\infty(\mathring{\mathrm{R}}.)) = \Pi_0(\mathcal{M}_\infty(\mathring{\mathrm{R}}.)) = 0$. Consequently the fibrewise "0-localisation" of the fibration (***) is the fibration

$\mathrm{K}(\mathrm{M}_\infty(\mathrm{Q}), r+1) \to \mathrm{E} \to \mathrm{BGL}(\mathrm{Z})$

associated with the action

$\bar{\rho}_\infty : \mathrm{GL}(\mathrm{Z}) \times \mathrm{K}(\mathrm{M}_\infty(\mathrm{Q}), r+1) \to \mathrm{K}(\mathrm{M}_\infty(\mathrm{Q}), r+1);$

this action is determined by the adjoint representations $\rho_\infty : \mathrm{GL}(\mathrm{Z}) \times \mathrm{M}_\infty(\mathrm{Q}) \to \mathrm{M}_\infty(\mathrm{Q})$ given by $\rho_\infty(\mathrm{A} : \mathrm{M}) = \mathrm{A.M.A}^{-1}$.

Warning. If (a) and (b) are satisfied and $\mathrm{B}\mathcal{M}_\infty(\mathring{\mathrm{R}}.)$ has trivial rational Postnicov invariants, we might be tempted to believe that the fibrewise "0-localisation" of the fibration (***) is the fibration with fibre $\prod_{s=2}^{\infty} (\mathrm{K}(\mathrm{G}_s \otimes \mathrm{Q}), s))$ associated with the action $\prod_{s=2}^{\infty} s_{p_\infty}$ where s_{p_∞} is the action induced by the representation $s_{p_\infty} : \mathrm{GL}(\mathrm{Z}) \times \mathrm{M}_\infty(\mathrm{G}_s \otimes \mathrm{Q}) \to \mathrm{M}_\infty(\mathrm{G}_s \otimes \mathrm{Q})$ defined by $s_{\rho_\infty}(\mathrm{A}, \mathrm{M}) = \mathrm{A} \cdot \mathrm{M} \cdot \mathrm{A}^{-1}$ this is not always the case.

The proof of Theorem 1.1 follows now immediately from Theorem 3.1.

Proof of Theorem 1.2. If $\mathrm{X} = \mathrm{K}(\mathrm{Z}, 2r)$ then $\Omega\mathrm{X} = \mathrm{K}(\mathrm{Z}, 2r-1)$ and consequently $\mathbf{Z}\Omega(\mathrm{X})$ has as homotopy groups the homology groups of $\Omega(\mathrm{X})$ since

$\mathbf{Z}\Omega(X)$ is essentially the infinite symmetric product of ΩX. This makes clear that $\mathbf{Z}\Omega(X)$ satisfies (c) since X is 1-connected (a) is also satisfied and (b) is trivially satisfied since $Zpt = Z$. Consequently the theorem is true by Theorem 1.1 for $K(Z, 2r)$. The construction of the functor $\mathcal{H}$ implies immediately that if X and Y are rationally homotopy equivalent then $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are.

q.e.d

Proof of Corollary 1.3. In [W] Waldhausen defines two natural transformation $\mathcal{H}(\ldots) \to \mathcal{H}^s(\ldots)$ where $\mathcal{H}^s(\ldots)$ is the stabilized functor associated with $\mathcal{H}$, which is an unreduced homology theory, and $h(\ldots; \mathcal{H}(pt)) \to \mathcal{H}(\ldots)$ where $h(\ldots; \mathcal{H}(pt))$ is the reduced homology theory produced by the ∞-loop space $\mathcal{H}(pt)$.

The composition $h(\ldots; \mathcal{H}(pt)) \to \mathcal{H}^s(\ldots)$ is a natural transformation of homology theory and because $\mathcal{H}(pt) \to \mathcal{H}^s(pt)$, is rationally homotopy surjective, $h(X : \mathcal{H}(pt)) \to \mathcal{H}^s(X)$ is rationally homotopy surjective. On the other side $B^{\mathrm{Diff}}\mathrm{Wh}(X)$ is the fibre of $\mathcal{H}(X) \to \mathcal{H}^s(X)$. Consequently $\mathcal{H}(X)$ and $B^{\mathrm{Diff}}\mathrm{Wh}(X) \times \mathcal{H}^s(X)$ are rationally homotopy equivalent. (Waldhausen claims a much stronger fact namely $A(X)$ and $B^{\mathrm{Diff}}\mathrm{Wh}(X) \times A^s(X)$ are homotopy equivalent which will imply the mentioned rational homotopy equivalence).

§2

Let k be one of the fields Q, R, C of rational, real or complex numbers, and $M_n(k)$ be the Lie algebra of $GL(k, n)$, i.e. the Lie algebra of $n \times n$ matrices over k.

We denote by ${}^{k}\rho_n$ or ρ_n, using ${}^{k}\rho_n$ only when we are interested to explicit the field, the adjoint representation of $GL(k, n)$ on $M_n(k)$ defined by $\rho_n(A, M) = A \cdot M \cdot A^{-1}$ for $A = GL(k, n)$ and $M \in M_n(k)$, by ρ_n^* its dual representation and by $\Lambda^k \rho_n$, $S^k \rho_n$, $\Lambda^k \rho_n^*$, $S^k \rho_n^*$ the k-times exterior respectively symmetric power of ρ_n respectively ρ_n^*. Denote by $\mathrm{Inv}\,\xi$ the fixed point subspace of the representation ξ. The following theorem contains well known facts; since the present formulation is not easy to be found in literature we enclose the proof.

THEOREM 2.1. (1) *There exists an injective linear map* $s_n^k : H^k(U(n) : k) \to \Lambda^k M_n^*(k)$[5] *with* $s_n^k(H^k(U(n) : k)) = \mathrm{In\,v}\, \Lambda^k ({}^{k}\rho_n^*)$ *such that the following diagram is commutative.*

$$\begin{array}{ccc} H^k(U(n) : k) & \xrightarrow{s_n^k} & \Lambda^k M_n^*(k) \\ \Big\uparrow{\scriptstyle j_n^*} & & \Big\uparrow{\scriptstyle \Lambda^k i_n^*} \\ H^k(U(n+1) : k) & \xrightarrow{s_{n+1}^k} & \Lambda^k M_{n+1}^*(k) \end{array}$$

where j_n^* *is induced by the canonical inclusion* $\mathrm{U}(n) \xrightarrow{j_n} \mathrm{U}(n+1)$ *and* i_n *is the canonical Lie algebra inclusion* $i_n(\mathrm{M}) = \left|\begin{array}{c|c} \mathrm{M} & 0 \\ \hline 0 & 0 \end{array}\right|$.

(2) *There exists an injective linear map*

$$q_n^k : \mathrm{H}^{2k}(\mathrm{BU}(n) : \mathscr{k}) \to \overset{k}{\mathrm{S}}\, \mathrm{M}_n^*(\mathscr{k})$$

with

$$q_n^k(\mathrm{H}^{2k}(\mathrm{BU}(n) : \mathscr{k})) = \mathrm{Inv}\, \overset{k}{\mathrm{S}}\, {}^{\mathscr{k}}\rho_n^*$$

such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{H}^{2k}(\mathrm{BU}(n) : \mathscr{k}) & \xrightarrow{q_n{}^k} & \overset{k}{\mathrm{S}}\, \mathrm{M}_n^*(\mathscr{k}) \\ \uparrow{\scriptstyle 'j_n^*} & & \uparrow{\scriptstyle \mathrm{S}^k i_n^*} \\ \mathrm{H}^{2k}(\mathrm{BU}(n+1) : \mathscr{k}) & \xrightarrow{q_{n+1}{}^k} & \overset{k}{\mathrm{S}}\, \mathrm{M}_{n+1}^*(\mathscr{k}) \end{array}$$

where $'j_n^*$ *is induced by the canonical inclusion* $\mathrm{BU}(n) \xrightarrow{j_n} \mathrm{BU}(n+1)$.

Proof. Since ${}^{\mathrm{R}}\rho_n^*$ is a real form of ${}^{\mathrm{C}}\rho_n^*$ it is clear that the proof for $\mathscr{k} = \mathrm{R}$ implies the result for $\mathscr{k} = \mathrm{C}$.

Proof of (1). ($\mathscr{k} = \mathrm{R}, \mathrm{Q}$). Let η_n be the adjoint representation of $\mathrm{U}(n)$ on its Lie algebra; η_n^* is a real form of ${}^{\mathrm{R}}\rho_n^*$. Analogously let ${}^{\mathrm{Q}}\eta_n$ be the adjoint representation of the group

$${}^{\mathrm{Q}}\mathrm{U}(n) = \{\mathrm{A} = \{a_{tj} = \alpha_{tj} + i\beta_{tj}\} \mid \mathrm{A} \in \mathrm{U}(n),\ \alpha_{tj}, \beta_{tj} \in \mathrm{Q}\}$$

on the Q-Lie algebra

$$\mathrm{m} = \{\mathrm{M} = \{m_{tj} = \alpha_{tj} + i\beta_{tj}\} \mid m_{jt} + \bar{m}_{tj} = 0,\ \alpha_{tj}, \beta_{tj} \in \mathrm{Q}\}$$

given by ${}^{\mathrm{Q}}\eta_n(\mathrm{A}, \mathrm{M}) = \mathrm{A} \cdot \mathrm{M} \cdot \mathrm{A}^{-1}$.

[5] For a $\mathscr{k}$ vector we denote by V^* its dual.

Clearly it is enough to prove (1) for R η_n^* respectively ${}^{\mathrm{Q}}\eta_n^*$ in order to have it proved for ${}^{\mathrm{R}}\rho_n^*$ respectively ${}^{\mathrm{Q}}\rho_n^*$.

Let us recall that de Rham theory permits to associate with any closed differential form on a differentiable manifold a precise singular cohomology class with coefficients in R.

Therefore we have the linear map $t_n : \mathrm{Inv}\,(\Lambda^k\,\eta_n^*) \to \mathrm{H}^k(\mathrm{U}(n):\mathrm{R})$ constructed as follows; an element of $\mathrm{Inv}\,(\Lambda^k\,\eta_n^*)$ is regarded as a k-form on the Lie algebra of $\mathrm{U}(n)$ which by translation is extended to a k-differential form on the compact Lie group $\mathrm{U}(n)$; since the element we started with is in $\mathrm{Inv}\,(\Lambda^k\,\eta_n^*)$ the obtained differential form is biinvariant therefore closed.

It is well known (for any compact Lie group) that t_n is an isomorphism. Moreover $t_n/\Lambda^k\,{}^{\mathrm{Q}}\eta_n^*$ factors through $\mathrm{H}^k(\mathrm{U}(n):\mathrm{Q})$ since a form in $\mathrm{Inv}\,(\Lambda^k\,\eta_n^*)$ with rational coefficients (with respect to the canonical base) produces a cohomology class with rational periods on all integral cycles. Consequently we have the cummutative diagram

$$\begin{tikzcd}
\mathrm{Inv}\,\overset{k}{\Lambda}\mathrm{R}\eta_n^* \arrow[r, "t_n^k"] & \mathrm{H}^k(\mathrm{U}(n):\mathrm{R}) \\
\mathrm{Inv}\,\overset{k}{\Lambda}\,{}^{\mathrm{Q}}\eta_n^* \arrow[r, "{}^{\mathrm{Q}}t_n^k"] \arrow[u, "\bar{U}"'] & \mathrm{H}^k(\mathrm{U}(n):\mathrm{Q}) \arrow[u, "\bar{U}"']
\end{tikzcd}$$

which implies ${}^{\mathrm{Q}}t_n^k$ is an isomorphism.

We also observe that $\Lambda^k i_n^* : \Lambda^k \eta_{n+1}^* \to \Lambda^k \eta_n^*$ sends $\mathrm{Inv}\,\Lambda^k \eta_{n+1}^*$ into $\mathrm{Inv}\,\Lambda^k \eta_n^*$ where i_n is the canonical inclusion of the Lie algebra of $\mathrm{U}(n)$ into the Lie algebra of $\mathrm{U}(n+1)$, (analogously $\Lambda^{k\mathrm{Q}} i_n^*$ sends $\mathrm{Inv}\,\Lambda^{k\mathrm{Q}}\eta_{n+1}^*$ into $\mathrm{Inv}\,\Lambda^{k\mathrm{Q}}\eta_n^*$) and the following diagram is commutative

$$\begin{tikzcd}
\Lambda^k \eta_{n+1}^* \arrow[r] \arrow[d, "t_{n+1}^k"] & \Lambda^k \eta_n^* \arrow[dr, "t_n^k"] & \\
\mathrm{H}^k(\mathrm{U}(n+1); R) \arrow[rr] & & \mathrm{H}^k(\mathrm{U}(n); \mathrm{R})
\end{tikzcd}$$

since the correspondence "biinvariant forms" $\rightsquigarrow$ "cohomology" is a functorial isomorphism for the category of compact Lie groups. If we take $s_n^k = (t_n^k)^{-1}$ and ${}^{\mathrm{Q}}s_n^k = ({}^{\mathrm{Q}}t_n^k)^{-1}$, (1) is proved.

Proof of 2. ($\ell = \mathrm{Q}, \mathrm{R}$). Let us consider $c_n(\ell)$ the Lie subalgebra of $\mathrm{M}_n(\ell)$ consisting of the diagonal matrices and ${}^{\ell}\Theta_n$ the representation of the symmetric group on $c_n(\ell)$.

$S^k M_n^*(k)$ respectively $S^k c_n^*(k)$ can be identified to the vector space of the degree k homogeneous polynomials on $M_n(k)$ respectively on $c_n(k)$; let ${}_n\pi^\kappa : S^k M_n^*(k) \to S^k c_n^*(k)$ be the linear map defined by "restriction to $c_n(k)$." Clearly we have the commutative diagram

$$\begin{array}{ccc} \overset{k}{S} M_n^*(k) & \xleftarrow{S i_n^*} & \overset{k}{S} M_{n+1}^*(k) \\ \downarrow {\scriptstyle {}_n\pi^k} & & \downarrow {\scriptstyle {}_{n+1}\pi^k} \\ \overset{k}{S} c_n^*(k) & \xleftarrow{S i_n^*} & \overset{k}{S} c_{n+1}^*(k) \end{array}$$

where $S^k \; i_n^*(\mathrm{Inv}\, S^k \rho_{n+1}^*) \subset \mathrm{Inv}\, S^k \rho_n^*$ and $S^k i_n^*(\mathrm{Inv}\, S^k \Theta_{n+1}^*) = \mathrm{Inv}\, S^k (\Theta_n^*)$, ${}_r\pi^k(\mathrm{Inv}\, S^k \rho_r^*) \subset \mathrm{Inv}\, S^k \Theta_r^*$; $\mathrm{Inv}\, S^k \Theta_n^*$ is the fixed point subspace of $S^k \Theta_n^*$. Let ${}_r\bar{\pi}^k$: Inv $(S^k \rho_r^*) \to \mathrm{Inv}\, S^k \Theta_r^*$ be the same map as ${}_r\pi^K$ with the target restricted to $\mathrm{Inv}\, S^k \Theta_r^*$. We will prove that ${}_r\bar{\pi}^k$ is surjective checking that ${}_r\bar{\pi}^* = \oplus_{k=0}^{\infty} {}_r\bar{\pi}^k$, ${}_r\bar{\pi}^*:\mathrm{Inv}\, S\rho_r^* \to \mathrm{Inv}\, S\Theta_r^*$ where $S \cdots = \oplus_k S^k \cdots$ is. For this purpose we define

$$\mu_r : M_r(k) \to k^r \qquad \mu_r = (\mu_r^1, \ldots, \mu_r^r) \quad \text{with} \quad \mu_r^i : M_n(k) \to k$$

by $\mu_r^i(M) =$ the i-th coefficient of the characteristic polynomial of M. μ_r induces $\mu^* : P(k^r) \to \mathrm{Inv}\, S\rho_r^*$, $P(k^r)$ is the space of polynomials defined on k^r, and ${}_r\pi^* \cdot \mu^*$ is an isomorphism, hence ${}_r\pi^*$ is surjective. To check that ${}^k_r\pi^k$ is injective it suffices to show that ${}^C_r\pi^k$ is, since ${}^Q_r\pi^k$ and ${}^R_r\pi^k$ are restrictions of ${}^C_r\pi^k$; ${}^C_r\pi^k$ is injective because there exists an open dense set in $M_r(C)$ consisting of matrices which are conjugate to diagonal matrices. Consequently we have

$$\begin{array}{ccc} \mathrm{Inv}\, S^k \Theta_n & \xleftarrow{{}_n\pi^k} & \mathrm{Inv}\, \overset{k}{S} \rho_n^* \\ \uparrow {\scriptstyle S^k i_n^*} & & \uparrow {\scriptstyle S^k i_n^*} \\ \mathrm{Inv}\, \overset{k}{S} \Theta_{n+1} & \xleftarrow{{}_{n+1}\pi^k} & \mathrm{Inv}\, \overset{k}{S} \rho_{n+1}^* \end{array}$$

By A. Borel's theorem we know that for any k we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Inv}\, \overset{k}{S} \Theta_n^* & \xleftarrow{S i_n^*} & \mathrm{Inv}\, \overset{k}{S} \Theta_{n+1}^* \\ \downarrow {\scriptstyle l_n^k} & & \downarrow {\scriptstyle l_{n+1}^k} \\ H^{2k}(BU(n) : k) & \xleftarrow{i_n'} & H^{2k}(BU(n+1) : k) \end{array}$$

with l_n^k isomorphisms; consequently if we take $q_n^k = ({}_n\bar{\pi}^k) \cdot (l_n^k)^{-1}$ (2) is proved q.e.d.

Passing to duals we obtain the commutative diagrams

$$\begin{array}{ccc}
\overset{k}{\Lambda} M_n(k) & \xrightarrow{(s_n{}^k)^*} & H_k(U(n):k) \\
\big\downarrow{\scriptstyle \Lambda^k i_n} & & \big\downarrow{\scriptstyle i_{n*}} \\
\overset{k}{\Lambda} M_{n+1}(k) & \xrightarrow{(s^k_{n+1})^*} & H_k(U(n+1):k)
\end{array}$$

and

$$\begin{array}{ccc}
\overset{k}{S} M_n(k) & \xrightarrow{(q_n{}^k)^*} & H_{2k}(BU(n):k) \\
\big\downarrow{\scriptstyle S^k i_n} & & \big\downarrow{\scriptstyle ('j_n)_*} \\
\overset{k}{S} M_{n+1}(k) & \xrightarrow{(q^k_{n+1})^*} & H_{2k}(BU(n+1):k)
\end{array}$$

which induce $(s_\infty^k)^*: \Lambda^k M_\infty(k) \to H_k(U(\infty):k)$ and $(q_\infty^k)^*: S^k M_\infty(k) \to H_{2k}(BU(\infty); k)$

$(s_n^k)^*$, $(q_n^k)^*$ restricted to $\operatorname{Inv} \Lambda^k \rho_n$ respectively $\operatorname{Inv} S^k \rho_n$ are isomorphisms therefore $(s_\infty^k)^*$ and $(q_\infty^k)^*$ are, since $\operatorname{Inv} \Lambda \overset{k*}{\rho_\infty}(k) = \lim_n \operatorname{Inv}(\Lambda^k \rho_n^*(k))$ and $\operatorname{Inv} S^k(\rho_\infty^*(k)) = \lim_n \operatorname{Inv}(S^k \rho_n^*(k))$.

COROLLARY 2.2. *For any l and k*

(1) ${}_l m_\infty^k: H_l(GL(Z); \{\Lambda^k {}^k\rho_\infty\}) \to H_l(GL(Z); H_k(U(\infty):k)$ *and*

(2) ${}_l m_\infty^k: H_l(GL(Z): S^k {}^k\rho_\infty\}) — H_l(GL(Z); H_{2k}(BU(\infty):k))$[6] induced by $(s_\infty^k)^*$ respectively $(q_\infty^k)^*$ are isomorphisms.

Proof of Corollary 2.2. It is enough to prove the statement for $k = R$. Since the proof of (1) and (2) are the same we give only the proof of (1). We observe that ${}_l m_\infty^k = \lim_n {}_l m_n^k$ with ${}_l m_n^k: H_l(GL(Z; n), \{\Lambda^k {}^R\rho_n\}) \to H_l(GL(Z, n); H_n(U(n):R))$ induced by $(s_n^k)^*$ hence it suffices to check that ${}_l m_n^k$ is an isomorphism for n big enough, for instance $(n-1) \geq 4l$. Let us recall that if τ is an $GL(R, n)$ irreducible representation, it remains irreducible if restricted to $SL(R, n)$; by Theorem 1.1 [F, H] $H_l(SL(Z, n), \{\tau\}) = 0$ if $l \leq (n-1)/4$, hence $H_l(GL(Z, n):\{\tau\}) = 0$ for

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[6] $H \cdots (G; \{\tau\})$ denotes the homology of G with coefficients in the G-module defined by the representation τ, $H \cdots (G; N)$ the homology of G with coefficients in the trivial G-module N.

$l<(n-1)/4$ (applying Lindon's spectral sequence [M] ch XI Theorem 10.1). Since $\Lambda^k \rho_n$ decomposes as sum of irreducible representations and the trivial representation on $\mathrm{Inv}\, \Lambda^{k\,R}\rho_n = H_k(U(n):R)$ we conclude that ${}_l m_n^k$ is an isomorphism for $(n-1)\geq 4l$.

q.e.d.

Let V be a k-vector space, $\{V, r\}$ be the graded vector space with all but r-th components trivial (i.e. $\{V\,.\,r\}_i=0$ if $i\neq r$) and the r-th component isomorphic to V. We denote by $L(\{V, r\})$ the k-graded commutative[7] algebra generated by the graded vector space $\{V, r\}$. Clearly $L(\{V, s\})_s=0$ if $s\not\equiv 0 \pmod r$ and $L(\{V, r\})_{ir}=\Lambda^i V$, respectively $S^i V$ if r is odd respectively even. As algebra $L(\{V, r\})$ is isomorphic to an exterior respectively symmetric algebra if r is odd respectively even.

If $\rho: G\times V\to V$ is a representation of G on V, ρ induces the representation $L(\rho, r)$ of G on $L(\{V, r\})$; let $\mathrm{Inv}\, L(\rho, r)$ be subalgebra of $L(\rho, r)$ consisting of the invariant elements.

Clearly $\mathrm{Inv}\, L(\rho, r)$ is a k free algebra therefore $\mathrm{Inv}\, L(\rho, r)=L(W)$ where W is a k graded vector space. We are particularly interested to determine the graded vector space W in the case $\rho=\rho_\infty^*$. The result is contained in the following theorem:

THEOREM 2.3. $\mathrm{Inv}\, L({}^k\rho_n^*, r)=L(W)$ *where* W *is the following* k *graded vector space* $(n=1,2,3,\ldots,\infty)$

$$\dim_k (W_s)=\begin{cases} 0 & \text{if } s\equiv 0 \pmod r \\ \dim \Pi_i(U(n))\otimes k & \text{if } r \text{ is odd and } s=ri \\ \dim \Pi_{2i}(BU(n))\otimes k & \text{if } r \text{ is even and } s=ri \end{cases}$$

Proof. Choose a graded preserving linear injective map $\iota:\tilde{W}\to \mathrm{Inv}\,({}^k\rho_n^*, r)L(W)$ where $\tilde{W}_s=0$ if $s\not\equiv 0 \pmod r$ and $\tilde{W}_{ir}=\Pi_i(U(n)\otimes k)^*$ respectively $\Pi_{2i}(BU(n)\otimes k)^*$ if r is odd respectively even and $\iota_{ir}=s_n^i\cdot o_i'$ respectively $\iota_{ir}=q_n^i\cdot o_{2i}$ where o_i' is a right inverse of the Hourewicz-homomorphism $H^i(U(n):k)\to \Pi_i^*(U(n))\otimes k$. and o_{2i} a right inverse of the Hourewicz homomorphism $H^{2i}(BU(n):k)\to \Pi_{2i}^*(BU(n))\otimes k$ ι_{ir} is injective because o_i' and o_{2i} are injective. Since ι is injective and $\mathrm{Inv}\,({}^k\rho_n^*, r)$ is free ι extends to $L(\iota):L(\tilde{W})\to \mathrm{Inv}\,({}^k\rho_n^*, r)$ which is injective. To prove it is an isomorphism it suffices to check it

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[7] "Commutative" should be understood in "graded sense," namely $a\cdot b=(-1)^{\deg a\cdot \deg b} b\cdot a$ if a and b have pure degree.

is an isomorphism in any degree or else it suffices to show $\dim_k L(\tilde{W})_i = \dim \operatorname{Inv}({}^k\rho_n^*, r)_i$. Or, if r is odd then

$$\dim_k L(\tilde{W})_s = \begin{cases} 0 \quad \text{if} \quad s \not\equiv 0 \pmod r \\ \operatorname{card}\left\{(\alpha_1, \alpha_2, \ldots, \alpha_p) \Big/ \begin{matrix} 1 \le \alpha_1 < \alpha_2 \cdots < \alpha_p < n \\ 2\alpha_1 + \cdots + 2\alpha_p - p = i, p < n \end{matrix}\right\} \\ = \dim H^i(U(n) : k) \quad \text{if} \quad s = ir \end{cases}$$

and if r is even

$$\dim_k L(\tilde{W})_s = \begin{cases} 0 \quad \text{if} \quad s \not\equiv 0 \pmod r \\ \operatorname{card}\left\{(\alpha_1 \alpha_2 \cdots \alpha_n) \Big/ \begin{matrix} 0 \le \alpha_i \le n \\ \alpha_1 + 2\alpha_2 + \cdots n\alpha_n = 2i \end{matrix}\right\} \\ = \dim H^{21}(BU(n) : k) \quad \text{if} \quad s = ir \end{cases}$$

For $n = \infty$ the result follows from the observation that for any fixed degree $\operatorname{Inv} L(\rho_n^*, r) \leftarrow \operatorname{Inv} L(\rho_{n+1}^*, r)$ is àn isomorphism if n is big enough. This happens because of Theorem 2.1 and the stability property for the cohomology of $U(n)$ and $BU(n)$.

§ 3

The restriction of the adjoint representation ρ_∞ of GL(Q) on $M_\infty(Q)$ to the subgroup GL(Z) defines the action $\bar{\rho}_\infty$ of GL(Z) on $K(M_\infty(Q), r)$ and therefore the fibration $K(V, r) \to E \xrightarrow{\pi} BGL(Z)$, $V = M_\infty(Q)$. If $r > 1$ then $\Pi_1(E) = \Pi_1(BGL(Z)) = GL(Z)$ whose commutator is a perfect normal subgroup, hence one can apply the Quillen "+" construction.

THEOREM 3.1. $E_+ = BGL(Z)_+ \times T_r$, π_+ *is the projection on* $BGL_\infty(Z)_+$ *where* T_r *has the homotopy type of* $\prod_{i=1}^\infty K(Q; (2s+1)(2i-1))$ *if* $r = 2s+1$ *and of* $\prod_{i=1}^\infty K(Q; 2si)$ *if* $r = 2s$.

Proof. The proof will be given in two steps. In step 1 we will produce an explicit construction of F_1, F_2, $f: F_1 \to F_2$, F_1, F_2 CW-complexes (semisimplicial complexes), f continuous (semisimplicial) map, together with a continuous (semisimplicial) action $\mu: GL(Z) \times F_1 \to F_1$ so that the following properties are satisfied

(a) F_1 is homotopy equivalent to $K(M_\infty(Q), r)$ and F_2 to T_r;

(b) The action μ induces on the r-th homotopy group of F_1 the representation ρ_∞.

(c) If F_2 is endowed with the trivial action of GL(Z) then f is equivariant.

(d) The minimal model (in the sense of Sullivan) [S] of F_1 is the commutative graded algebra $L(\rho_\infty^*, r)$ endowed with the differential 0, the minimal model of F_2 is the graded commutative algebra $\operatorname{Inv} L(\rho_\infty^*, r) = L(W)$ with differential 0 and the morphism induced by f is the inclusion $L(W) = \operatorname{Inv} L(\rho_\infty^*, r) \subset L(\rho_\infty^*, r)$.

To construct F_1, F_2, f, μ we use the "spatial realisation" functor $\langle\ \rangle$ of D. Sullivan [S][8] and take $F_1 = \langle L(M_\infty^*(Q), r),\ d=0\rangle$, $F_2 = \langle \operatorname{Inv} L(\rho_\infty^*, r),\ d=0\rangle$ $f = \langle$inclusion of $\operatorname{Inv} L(\rho_\infty^*, r)$ into $L(\rho_\infty^*, r) = L(M_\infty^*(Q), r)\rangle$ and $\mu_A : F_1 \to F_1$ for any $A \in GL(Z)$ is $\langle L(\rho_\infty^*(A) : M_\infty^*(Q) \to M_\infty^*(Q), r)\rangle$. (a), (b), (c) are trivially satisfied and (d) follows simply remarking that $(L(\rho_\infty^*, r), d=0)$ and $(\operatorname{Inv} L(\rho_\infty^*, r), d=0)$ are actually minimal models. We recall from Sullivan's theory of minimal models that a 1-connected space X has trivial rational Postnicov invariants iff the differential in the minimal model is trivial, hence F_2 is a product of Eilenberg MacLane's.

Theorem 2.3 gives the homotopy equivalence of F_2 and T_r.

Step 2. We consider the diagram

$$\begin{array}{ccccc}
F_2 & \longrightarrow & E_2 & \longrightarrow & BGL(Z) \\
\uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f_E} & & \uparrow{\scriptstyle id} \\
F_1 & \longrightarrow & E_1 & \longrightarrow & BGL(Z)
\end{array}$$

with horizontal lines the fibrations induced by the action and the trivial action of GL(Z) on F_2 and observe that $F_1 \to E_1 \to BGL(Z)$ is actually the fibration $K(V, r) \to E \to BGL(Z)$ while $F_2 \to E_2 \to BGL(Z)$ is the trivial fibration with F_2 homotopy equivalent to $T_r \cdot (f, f_E, id)$ induces a morphism of the spectral sequence (in homology) of the first fibration in the spectral sequence of the second and Corollary 2.2 claims that this morphism is an isomorphism for $E^2 \ldots$, hence f_E induces an isomorphism on integral homology and on Π_1 hence $f_{E_+} : E_{1_+} \to E_{2_+}$ is a homotopy equivalence; this proves Theorem 3.1. q.e.d.

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[8] Recall that the "spatial realisation" of a 1-connected a differential graded algebra A is the geometric realisation of the semisimplicial complex whose k-simplexes are *d.g.a.* maps from A to the Q-de Rham algebra of the standard simplex. The degenaracies and the face operators are obviously defined.

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