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## Reconstructing 1-coherent locally finite trees

CARSTEN THOMASSEN

*Abstract.* We prove a theorem implying the conjecture of J. A. Bondy and R. L. Hemminger that an infinite, locally finite tree containing no two-way infinite path is uniquely determined, up to isomorphism, from its collection of vertex-deleted subgraphs.

### Introduction and terminology

We say that two graphs  $G_1$  and  $G_2$  are *weakly hypomorphic* if there exist maps  $\varphi: V(G_1) \rightarrow V(G_2)$  and  $\psi: V(G_2) \rightarrow V(G_1)$  such that  $G_1 - x \simeq G_2 - \varphi(x)$  and  $G_2 - y \simeq G_1 - \psi(y)$  for each vertex  $x$  and  $y$  in  $V(G_1)$  and  $V(G_2)$ , respectively. In other words,  $G_1$  and  $G_2$  have the same isomorphism classes of vertex-deleted subgraphs. If  $\varphi$  and  $\psi$  can be chosen to be bijections, then  $G_1$  and  $G_2$  are *hypomorphic*. The reconstruction conjecture asserts that any two hypomorphic graphs are isomorphic. The conjecture is open for finite graphs and false for infinite graphs in general (see e.g. [5]). However, no counter-examples are known to the Harary–Schwenk–Scott conjecture [6] that any two hypomorphic, locally finite trees are isomorphic. As a first step towards a proof of this conjecture, Bondy and Hemminger [1] demonstrate the validity of the conjecture for  $m$ -coherent, locally finite trees for  $m \geq 2$  (a tree is  $m$ -coherent if it contains a set of  $m$ , but not  $m + 1$ , pairwise disjoint one-way infinite paths) and conjecture in an early version of [4] (the problem first appear in [3], but is mistakenly listed as being for 2-coherent locally finite trees) that any two hypomorphic 1-coherent locally finite trees are isomorphic. In this note we prove that the same conclusion holds under the weaker assumption that the trees are weakly hypomorphic.

If  $F$  is a forest and  $x, y$  are distinct vertices of  $F$ , then  $F[x, y]$  denotes the component of  $F - x$  containing  $y$  rooted at  $y$ . A *branch* of  $F$  at  $x$  is a rooted tree of the form  $F[x, y]$  where  $y$  is adjacent to  $x$ . When we speak of isomorphisms of branches we always mean root-preserving isomorphisms. The *height*  $h(x, T)$  of a vertex  $x$  in a tree  $T$  is the total number of vertices belonging to finite branches at  $x$ . If  $x$  has finite degree in  $T$ , then obviously  $h(x, T)$  is finite.

If  $T$  is a 1-coherent, locally finite tree, and  $P$  is a one-way infinite path of  $T$ , then the forest obtained from  $T$  by deleting all edges of  $P$  has only finite

components, by König's Lemma. Furthermore, if  $P'$  is any one-way infinite path of  $T$ , then  $P$  and  $P'$  have an infinite path in common.

### Isomorphism between weakly hypomorphic, 1-coherent, locally finite trees

**THEOREM.** *Let  $T_1$  be a 1-coherent, locally finite tree and let  $T_2$  be any graph which is weakly hypomorphic to  $T_1$ . Then  $T_1 \simeq T_2$ .*

*Proof.* Let  $\varphi: V(T_1) \rightarrow V(T_2)$  and  $\psi: V(T_2) \rightarrow V(T_1)$  be maps such that

$$T_1 - x \simeq T_2 - \varphi(x) \quad \text{and} \quad T_2 - y \simeq T_1 - \psi(y)$$

for each  $x$  and  $y$  in  $V(T_1)$  and  $V(T_2)$ , respectively. It is easily verified that  $T_2$  is a 1-coherent, locally finite tree. Let  $x$  be any endvertex of  $T_1$  and  $P_1: x_1x_2x_3 \cdots$  be a one-way infinite path of  $T_1 - x$ . Let  $P_2: y_1y_2y_3 \cdots$  be the image of  $P_1$  under some isomorphism of  $T_1 - x$  onto  $T_2 - \varphi(x)$ . With no loss of generality we can assume that  $x$  (resp.  $\varphi(x)$ ) is joined to a vertex in a branch of  $T_1 - x$  (resp.  $T_2 - \varphi(x)$ ) at  $x_1$  (resp.  $y_1$ ) for otherwise we consider paths of the form  $x_kx_{k+1} \cdots$  and  $y_ky_{k+1} \cdots$  instead of  $P_1$  and  $P_2$ , respectively.

If  $T$  is a tree and  $x$  a vertex of  $T$ , then the degree of  $x$  in  $T$  equals the number of components of  $T - x$ . From this it follows that if one of  $T_1, T_2$  has maximum degree 2, then each of  $T_1, T_2$  is a one-way path, and hence  $T_1 \simeq T_2$  in this case. So we can assume that each of  $T_1, T_2$  contains a vertex of degree 3 or more.

If each of the vertices  $x_2, x_3, \dots$  has degree 2 in  $T_1$  (and hence each of  $y_2, y_3, \dots$  has degree 2 in  $T_2$ ), then we denote by  $k$  (resp.  $m$ ) the maximum distance in  $T_1$  (resp.  $T_2$ ) between an endvertex and a vertex of degree 3 or more and we put  $n = \max\{k, m\}$ . It is obvious that for each  $i \geq 1$ , no vertex of  $T_2$  other than  $y_i$  has height  $h(y_i, T_2)$ . Since

$$h(y_i, T_2) = h(x_i, T_1) = h(\varphi(x_i), T_2)$$

We conclude that  $\varphi(x_i) = y_i$  for each  $i \geq 1$ . Now let

$$\sigma: T_1 - x_{n+3} \rightarrow T_2 - y_{n+3}$$

be an isomorphism. By definition of  $n$ ,  $x_{n+2}$  (resp.  $y_{n+2}$ ) is the only vertex of the finite component of  $T_1 - x_{n+3}$  (resp.  $T_2 - y_{n+3}$ ) which has distance greater than  $n$  to a vertex of degree 3 or more. Hence

$$\sigma(x_{n+2}) = y_{n+2}$$

and  $\sigma$  can be extended to an isomorphism of  $T_1$  onto  $T_2$ .

So we assume that for some  $t \geq 2$ , the degree of  $x_t$  and  $y_t$  is at least 3. The union of finite branches of  $T_1$  at  $x_t$  (except  $T_1[x_t, x_{t-1}]$ ) is isomorphic to the union of finite branches of  $T_2$  at  $y_t$  (except  $T_2[y_t, y_{t-1}]$ ). We want to prove that

$$T_1[x_t, x_{t-1}] \simeq T_2[y_t, y_{t-1}].$$

This will imply  $T_1 \simeq T_2$ . If all finite branches at  $x_t$  are pairwise isomorphic and all finite branches at  $y_t$  are pairwise isomorphic, then obviously

$$T_1[x_t, x_{t-1}] \simeq T_2[y_t, y_{t-1}],$$

so we can assume, with no loss of generality, that there is a branch  $B = T_1[x_t, z_1]$  which is not isomorphic to  $T_1[x_t, x_{t-1}]$ .

Consider a vertex  $z_2$  of  $T_2$  such that  $T_2 - z_2 \simeq T_1 - z_1$ . Let  $s$  be the integer such that  $z_2$  and  $y_s$  belong to the same component of the forest obtained by deleting all edges of  $P_2$ . Suppose  $z_2$  is chosen such that  $s$  is minimal. Let

$$\pi : T_1 - z_1 \rightarrow T_2 - z_2$$

be an isomorphism, and let

$$P'_2 : y'_1 y'_2 y'_3 \cdots$$

be the image of  $P_1$  under  $\pi$ . Since  $T_2$  is 1-coherent, the intersection of  $P_2$  and  $P'_2$  is an infinite path  $y'_r y'_{r+1} \cdots = y_q y_{q+1} \cdots$ . For  $i$  sufficiently large,  $x_{r+i}$  and  $y'_{r+i}$  have the same height in  $T_1$  and  $T_2$ , respectively, so

$$h(y_{q+i}, T_2) = h(y'_{r+i}, T_2) = h(x_{r+i}, T_1) = h(y_{r+i}, T_2).$$

However,

$$h(y_1, T_2) < h(y_2, T_2) < \cdots,$$

so, for  $i$  sufficiently large,  $y_{q+i} = y_{r+i}$  and hence  $r = q$ . Suppose  $\pi$  is chosen such that  $r$  is minimal.

We now prove by contradiction that

$$r \leq \max \{t, s\}.$$

For suppose  $r > t$  and  $r > s$ . The union of branches of  $T_1$  at  $x_r$  (except  $T_1[x_r, x_{r-1}]$ )

is isomorphic to the union of branches of  $T_2$  at  $y_r$  (except  $T_2[y_r, y_{r-1}]$ ) and since  $\pi(x_r) = y_r$ , the union of branches of  $T_1 - z_1$  at  $x_r$  is isomorphic to the union of branches of  $T_2 - z_2$  at  $y_r$ . Combining these observations, we conclude that

$$(T_1 - z_1)[x_r, x_{r-1}] \cong (T_2 - z_2)[y_r, y_{r-1}]$$

and hence we can modify  $\pi$  so as to obtain an isomorphism  $T_1 - z_1 \rightarrow T_2 - z_2$  which takes  $x_{r-1}$  to  $y_{r-1}$ . This contradicts the minimality of  $r$ . Hence  $r \leq \max\{t, s\}$ .

We next prove that  $s \leq t$ . For suppose  $s > t$ . (Hence  $\pi(x_s) = y_s$ ). Since

$$h(y_s, T_2) = h(x_s, T_1) > h(z_1, T_1) = h(z_2, T_2),$$

we have  $z_2 \neq y_s$ , so  $z_2$  belongs to a finite branch  $B'$  of  $T_2$  at  $y_s$  (distinct from  $T_2[y_s, y_{s-1}]$ ). Since  $T_1 - x$  and  $T_2 - \varphi(x)$  have the same branches at  $x_s$  and  $y_s$ , respectively, and since deleting  $z_2$  from  $T_2$  results in a forest with fewer branches at  $y_s$  isomorphic to  $B'$ , it follows that  $\pi^{-1}$  maps  $T_2[y_s, y_{s-1}]$  onto a branch of  $T_1$  isomorphic to  $B'$ . But then  $T_2[y_s, y_{s-1}]$  contains a vertex  $z'_2$  s.t.  $T_2 - z_2 \cong T_2 - z'_2$ , a contradiction to the minimality of  $s$ .

Summarizing,  $s \leq t$  and  $\pi$  maps the path  $x_t x_{t+1} \cdots$  onto  $y_t y_{t+1} \cdots$ . In particular,  $x_t$  and  $y_t$  have the same degree in  $T_1 - z_1$  and  $T_2 - z_2$ , respectively. So  $z_2$  is adjacent to  $y_t$ . Since the number of branches at  $x_t$  isomorphic to  $B$  is one less in  $T_1 - z_1$  than in  $T_1$  and since  $T_1[x_t, x_{t-1}]$  is not isomorphic to  $B$ , it follows that  $T_2[y_t, y_{t-1}]$  is not isomorphic to  $B$  (again using the isomorphism between  $T_1 - x$  and  $T_2 - \varphi(x)$ ) and that  $z_2$  is the root of a branch at  $y_t$  isomorphic to  $B$ . In particular,  $z_2 \neq y_{t-1}$  and  $\pi$  maps  $T_1[x_t, x_{t-1}]$  onto a branch isomorphic to  $T_2[y_t, y_{t-1}]$ . Since  $T_1[x_t, x_{t-1}] \cong T_2[y_t, y_{t-1}]$  implies  $T_1 \cong T_2$ , the proof is complete.

### Further results and conjectures

We can prove that any two weakly hypomorphic, locally finite,  $m$ -coherent trees are isomorphic for  $m \geq 3$  and believe that with a little more effort the same can be proved for  $m = 2$ . Furthermore, we can prove that any two hypomorphic  $m$ -coherent trees are isomorphic for  $m \geq 3$  and Nešetřil [7, 8] proves it for  $m = 0$ . We conjecture that it also holds for  $m = 1, 2$ .

Halin (see [2]) conjectures that if  $G_1$  and  $G_2$  are infinite hypomorphic graphs, then  $G_i$  contains a subgraph isomorphic to  $G_{3-i}$  for  $i = 1, 2$ . We conjecture that the same conclusion holds under the weaker assumption that  $G_1$  and  $G_2$  are weakly hypomorphic.

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