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Manifolds with a given homology and fundamental group

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Introduction

The main results of this paper are an existence and a classification theorem for manifolds having a given fundamental group and a given (twisted) homology type. More precisely, let $(X, \partial X)$ be a Poincaré pair of formal dimension n in the sense of [W, Chapter 2], with X connected, $\pi_1(X) = \pi$ and orientation character $\omega : \pi \to \mathbf{Z}/2\mathbf{Z}$. Suppose that ∂X is either empty or a closed CAT-manifold, where CAT denotes a category of manifolds among the following: differentiable (C^∞), piecewise linear (PL) or topological (TOP).

Let $\Phi : H \twoheadrightarrow \pi$ be an epimorphism of finitely presented group and let $\mu : \partial X \to BH = K(H, 1)$ be a lifting of the natural map $j : X \to B\pi$. One defines $\mathcal{S}^s_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$ as the set of equivalence classes of homotopy commutative diagrams of the following form:

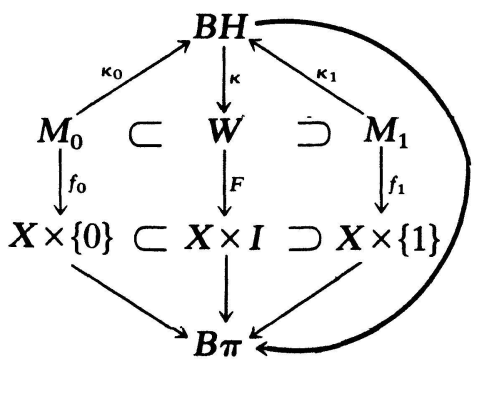
$$\begin{array}{ccc} M & \xrightarrow{\kappa} & BH \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Phi} \\ X & \xrightarrow{j} & B\pi \end{array}$$

where

(1) M is a compact manifold of dimension n with orientation character $f^*(\omega)$ and $\pi_1\kappa : \pi_1(M) \to H$ is an isomorphism.

(2) $f : (M, \partial M) \to (X, \partial X)$ is a map of degree one such that
- $f_* : H_*(M; \mathbf{Z}\pi) \to H_*(X; \mathbf{Z}\pi)$ (twisted coefficients) is an isomorphism.
- $f \mid \partial M : \partial M \to \partial X$ is a CAT-homeomorphism and $\mu \circ \pi_1(f \mid \partial M) = \pi_1(\kappa \mid \partial M)$.
- The torsion of f, which is well defined in $Wh(\pi)$ is equal to zero.

Such a diagram is denoted by (M, f, κ). Two diagrams (M_0, f_0, κ_0) and (M_1, f_1, κ_1) are called equivalent if there exists a cobordism (W, M_0, M_1) and a homotopy commutative diagram:



such that:

(a) $\pi_1\kappa$ is an isomorphism,

(b) F is a map of degree one, $F_*: H_*(W; \mathbf{Z}\pi) \to H_*(M\times I; \mathbf{Z}\pi)$ is an isomorphism and the torsion of F is equal to zero in $Wh(\pi)$. One asks also that $F \mid \partial W\text{-int}(M_0 \cup M_1)$ be a CAT-homeomorphism onto $\partial X \times I$.

By omitting the conditions on torsions in the above definitions, one gets another set denoted $\mathcal{S}^h_{\text{CAT}}(X \operatorname{rel} \partial X; \Phi)$.

For instance, for $e = s$ or h, $\mathcal{S}^e_{\text{CAT}}(X \operatorname{rel} \partial X; \mathrm{id}_\pi) = \mathcal{S}^e_{\text{CAT}}(X \operatorname{rel} \partial X)$ where the latter denotes as usual the homotopy CAT-structures on X (rel ∂X), as defined by Sullivan-Wall [W Chapter 10]. If $X = S^n$, $\mathcal{S}^e_{\text{CAT}}(X; \Phi)$ (abbreviation used when ∂X is empty) is the set of (H_*-and-π_1)-cobordism classes of homology spheres with fundamental group identified with H (see Section 6).

When $n \geq 5$ and $\ker \Phi$ is locally perfect (see Section 2), we establish a bijection from $\mathcal{S}^e_{\text{CAT}}(X \operatorname{rel} \partial X; \Phi)$ to a subset of $\mathcal{S}_{\text{CAT}}(X \operatorname{rel} \partial X) \times [X, BH^+]$ where $\iota: BH \to BH^+$ is the map obtained by the Quillen plus construction with respect to $\ker \Phi$. (It will be previously shown that $\ker \Phi$ is perfect if $\mathcal{S}^e_{\text{CAT}}(X \operatorname{rel} \partial X; \Phi)$ is not empty). This is the classification Theorem (Theorem 2.2) which will be proved in Section 4. The argument needs a variation of the results of [H2] which is made in Section 3.

In Section 5, we deduce from the classification Theorem an existence result for manifolds having a given fundamental group and a given twisted homology type. Many examples of new manifolds can be constructed in this way. For instance, we give a sufficient condition for a group to be the fundamental group of a knot whose infinite cyclic cover is acyclic (Its Alexander modules are thus all zero).

In view of the classification and existence theorem, the groups $\pi_i(BN^+)$ (N perfect) play an important role. Therefore, we give in the final Section 7 several computations of $\pi_i(BN^+)$ for some classical perfect groups N.

The classification Theorem is the result of several successive generalizations. In a first (unpublished) note [H1] the author announced the result for the case $X = S^n$ (See Section 6) but with the hypothesis that BN has finite skeleta (algebraically: N is of type (FP) in the sense of [B-E]). Later, P. Vogel [V2] generalized this case by removing the hypothesis (FP). Theorem 2.2 and 5.1 were announced in [H4] for ∂X empty and N finitely presented. Finally, the technique of [H-V] enabled the author to prove the results in the generality stated here (N locally perfect).

2. Basic constructions and statement of the classification theorem

We keep here the notations of the introduction.

LEMMA 2.0 *If* $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$ *is not empty, then* $N = \ker \Phi$ *is perfect.* $(i \in N = [N, N])$.

Proof. Let (M, f, κ) represent a class of $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$. Denote by $\tilde{X}$ the universal cover of X and by $\tilde{M}_N$ the cover of M with fundamental group N. Condition *b*) of the definition of $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$ implies that $\tilde{f}: \tilde{M}_N \to \tilde{X}$ induces an isomorphism on integral homology. Then N is perfect. ■

Observe that $f: M \to X$ can be identified with the Quillen plus map with respect to N.

DEFINITION OF $\varphi_1: \mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi) \to \mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X)$. *Let* (M, f, κ) represent a class of $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$. Since both π and H are finitely presented, N is the normal closure in H of finitely many elements. If $n \geq 5$, the Quillen plus construction with respect to N can be made by adding finitely many two and three cells to $M \times 1 \subset M \times I$, as in [H2 §3]. One thus obtains a cobordism (W, M, M') trivial on the boundary such that W and M' have the homotopy type of M^+ (simple homotopy type if $e = s$). We call W a plus cobordism from M (it is a semi-s-cobordism from M' in the sense of [H-V].) The map $f: M \to X$ extends to a map $\bar{f}: W \to X$ which restricts to $f': M' \to X$. This latter is a homotopy equivalence (simple if $e = s$) and defines a class of $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X)$. The reader will check easily that the class of f' depends only on the class of (M, f, κ) in $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$. This defines a map

$$\varphi_1: \mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi) \to \mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X).$$

DEFINITION OF $\varphi_2: \mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi) \to \{X; BH^+\}$. Let (M, f, κ) represent a class of $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$ and let $\bar{f}: W \to X$ be constructed from f as above. Let $\alpha: X \to W$ be a homotopy inverse of $\bar{f}$. The functoriality of the plus construction with respect to N provides a map $\kappa^+: W \to BH^+$, unique up to homotopy, such that $\kappa^+ \mid M = \iota_H \circ \kappa$ $(\iota_H: BH \to BH^+)$. By the universal property of the plus maps, the homotopy class of $\kappa^+ \circ \alpha$ depends only on the class of (M, f, κ) in $\mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi)$. Therefore, this defines a map

$$\varphi_2: \mathcal{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi) \to [X; BH^+]$$

where $[X; BH^+]$ denotes the set of homotopy classes of maps $g: X \to BH^+$ such that $g \mid \partial X = \iota_H \circ \mu$ (the homotopies being fixed on ∂X).

Let $\Phi^+: BH^+ \to B\pi$ be the map given by functoriality of the plus construction.

Define $\{X; BH^+\}$ as the subset of classes of $[X; BH^+]$ represented by $g: X \to BH^+$ such that

$$\pi_1 g = \pi_1(\Phi^+)^{-1} \circ \pi_1(j)$$

$\pi_2 g$ is surjective.

LEMMA 2.1. Im $\varphi_2 \subset \{X; BH^+\}$.

Proof. The fact that $\pi_1(\kappa^+ \circ \alpha) = \pi_1(\Phi^+)^{-1} \circ \pi_1(j)$ follows from the equation:

$$\Phi^+ \circ \kappa^+ \circ \alpha \simeq j \circ \bar{f} \circ \alpha \simeq j$$

The surjectivity of $\pi_2(\kappa^+ \circ \alpha)$ follows from the following identifications

$$\begin{array}{ccccccc}
\pi_2(X) & \simeq & \pi_2(\tilde{X}) & \xrightarrow{\ \simeq\ } & H_2(\tilde{X}) & \xleftarrow{\ \simeq\ } & H_2(\tilde{M}_N) \\
\downarrow{\scriptstyle \pi_2(\kappa^+ \circ \alpha)} & & \downarrow{\scriptstyle \widetilde{\pi_2(\kappa^+ \circ \alpha)}} & & \downarrow & & \downarrow{\scriptstyle H_2\tilde{\kappa}_N} \\
\pi_2(BH^+) & \xleftarrow{\ \simeq\ } & \pi_2(BN^+) & \xrightarrow{\ \simeq\ } & H_2(N) & = & H_2(N) \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array} \quad \blacksquare$$

A group G is called locally perfect if every finitely generated subgroup of G is contained in a finitely generated perfect subgroup of G. This implies that G is perfect. For various properties of locally perfect groups see [V2 §5] and [H-V].

CLASSIFICATION THEOREM 2.1. *Suppose that $N = \ker\theta$ is locally perfect and that the formal dimension of the Poincaré pair $(X, \partial X)$ is ≥ 5. Then the map*

$$\varphi = (\varphi_1, \varphi_2): S^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X; \Phi) \to S^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X) \times \{X; BH^+\}$$

is a bijection.

Remarks.

(1) $\mathscr{S}^e_{\mathrm{CAT}}(X \operatorname{rel} \partial X)$ can be studied by standard surgery techniques (Ex. [W Chapter 10]).

(2) $\{X; BH^+\}$ is a subset of $[X; BH^+]$. Here one may use obstruction theory (but BH^+ is not a simple space in general). For instance, if $H_*(N; \mathbf{Z})$ is finite for all $*$, then $\pi_i(BH^+)$ are finite for all $i \geq 2$ (see Section 7) and thus $\{X; BH^+\}$ is a finite set.

3. Manifolds structures on $\mathbf{Z}\pi$-Poincaré complexes which are not finite

In this section, we prove a variation of the [H2 Theorem 5.1.] which we need in order to prove Theorem 2.1.

Let $1 \to N \to H \to \pi \to 1$ be a short exact sequence of groups, where π and H are finitely presented and N is perfect. Let $\omega : \pi \to \mathbf{Z}/2\mathbf{Z}$ be a homomorphism. Let $(Z, \partial Z)$ be a CW-pair where Z is connected and ∂Z is a closed CAT-manifold of dimension $n-1$. Assume that $\pi_1(Z) = H$ and that $(Z^+, \partial Z)$ ($\iota : Z \to Z^+$, plus with respect to N) is a Poincaré pair in the sense of [W, §2]. (In particular Z^+ is equivalent to a finite complex and its simple homotopy type can be defined). Therefore, $(Z, \partial Z)$ is a $\mathbf{Z}\pi$-Poincaré pair [B2].

Let $(M^n, \partial M)$ be a CAT-manifold pair. A map $f : (M, \partial M) \to (Z, \partial Z)$ is called an e-$\mathbf{Z}\pi$-equivalence ($e = s$ or h) if:

(1) f is of degree one and $\pi_1 f$ is an isomorphism.
(2) $f^{-1}(\partial Z) = \partial M$ and $f \mid \partial M$ is a CAT-homeomorphism.
(3) $f_* : H_*(M; \mathbf{Z}\pi) \to H_*(Z; \mathbf{Z}\pi)$ is an isomorphism.
(4) If $e = s$, the torsion of $\iota \circ f : M \to Z^+$ is equal to zero in $Wh(\pi)$.

Two e-$\mathbf{Z}\pi$-equivalences $f_i : (M_i, \partial M_i) \to (Z, \partial Z)$ are called equivalent if there exists a CAT-cobordism (W, M_1, M_2) and a map

$$F : (W, M_1, M_2) \to (Z \times I, Z \times \{0\}, Z \times \{1\})$$

such that:

(1) $\partial W = M_1 \cup M_2 \cup$ (an s-cobordism W_0 between ∂M_1 and ∂M_2).
(2) $F \mid M_i = f_i$, $F \mid W_0 : W_0 \to \partial Z \times I$ as a CAT-homeomorphism.
(3) $\pi_1 F$ is an isomorphism and $F_* : H_*(W; \mathbf{Z}\pi) \to H_*(Z \times I; \mathbf{Z}\pi)$ is an isomorphism.
(4) The torsion of F is equal to zero if $e = s$.

The set of equivalence classes of e-$\mathbf{Z}\pi$-equivalences from CAT-manifold pairs to $(Z, \partial Z)$ is denoted by $\mathcal{S}^e_{\text{CAT}}(Z \operatorname{rel} \partial Z; \mathbf{Z}\pi)$. If ∂Z is empty, this coincides with the definition of $\mathcal{S}^e_{\text{CAT}}(Z; \mathbf{Z}\pi)$ used in [H2, §5], and if $H = \pi$, one has $\mathcal{S}^e_{\text{CAT}}(Z \operatorname{rel} \partial Z)$.

There is a map

$$\lambda : \mathcal{S}^e_{\text{CAT}}(Z \operatorname{rel} \partial Z; \mathbf{Z}\pi) \to \mathcal{S}^e_{\text{CAT}}(Z^+ \operatorname{rel} \partial Z)$$

which is defined using a plus cobordism, as for φ_1 of §2.

THEOREM 3.1. *Suppose that N is locally perfect and acts trivially on $\pi_2(Z)$. Then λ is a bijection.*

This theorem was proven in [H2, Theorem 5.1] without the hypothesis that N acts trivially on $\pi_2(Z)$ but under the assumption that Z is a finite complex (or at least has a finite $[n-1/2]$-skeleton). Theorem 5.1 of [H2] is stated for ∂Z empty but the proof holds clearly in the relative case.

The proof given here follows the same idea as in [H-V proof of Theorem 2.1 and 3.1].

Proof. Let K_0 be a finite complex obtained by attaching 1 and 2 cells to ∂Z such that one has a commutative diagram

$$\begin{array}{ccc} \partial Z & \subset & Z \\ \searrow & & \nearrow \alpha_0 \\ & K_0 & \end{array}$$

with $\pi_1\alpha_0$ an isomorphism. By [H-V, Theorem 3.1] there exists a finite complex K_1 containing K_0 and a factorization

$$\begin{array}{ccc} K_0 & \subset & K_i \\ \alpha_0 \searrow & & \swarrow \alpha_1 \\ Z & \xrightarrow{\iota} & Z^+ \end{array}$$

such that $\iota \circ \alpha_1$ is a plus map. Observe that $\pi_1\alpha_1$ is onto. Since both $\pi_1(K_1)$ and $\pi_1(Z)$ are finitely presented, one can attach 2-cells to K_1 to obtain a finite complex K_2 and a factorization $\alpha_2 : K_2 \to Z$ of α_1 such that $\pi_1\alpha_2$ is an isomorphism.

Since $H_*(Z, K_1; \mathbf{Z}\pi) = 0$, one has $H_*(Z; K_2; \mathbf{Z}\pi) = 0$ for $* \neq 3$ and $H_3(Z, K_2; \mathbf{Z}\pi) \simeq H_2(K_2, K_1; \mathbf{Z}\pi)$ is the free $\mathbf{Z}\pi$-module generated by the two cells of K_2-K_1. This unique non-zero relative homology group can be killed by adding 3-cells to K_2 if and only if the Hurewicz homomorphism $\pi_3(Z, K_2) \to H_3(Z; K_2; \mathbf{Z}\pi)$ is onto. The universal coefficient spectral sequence for the complex $C_*(Z, K_1; \mathbf{Z}H)$ gives the exact sequence:

$$H_3(Z, K_2; \mathbf{Z}H) \to H_3(Z, K_2; \mathbf{Z}\pi) \to \mathrm{Tor}_1^{\mathbf{Z}H}(H_2(Z, K_2; \mathbf{Z}H); \mathbf{Z}\pi) \to 0.$$

On the other hand, one has

$$\mathrm{Tor}_1^{\mathbf{Z}H}(H_2(Z, K_2; \mathbf{Z}H)\mathbf{Z}\pi) \overset{(*)}{\simeq} \mathrm{Tor}_1^{\mathbf{Z}N}(H_2(Z, K_2; \mathbf{Z}H); \mathbf{Z})$$

$$\simeq H_1(N; H_2(Z; K_2; \mathbf{Z}H))$$

where the isomorphism $(*)$ is given by [C-E, Theorem 3.1]. Since $\pi_2(Z)$ is a trivial $\mathbf{Z}N$-module and since $H_2(Z, K_2; \mathbf{Z}H) \simeq \pi_2(Z, K_2)$ is a quotient of $\pi_2(Z)$, the group N acts trivially on $H_2(Z, K_2; \mathbf{Z}H)$ and then $H_1(N; H_2(Z, K_2; \mathbf{Z}H)) = 0$. Thus one has an epimorphism

$$\pi_3(Z, K_2) \twoheadrightarrow H_3(Z, K_2; \mathbf{Z}H) \twoheadrightarrow H_3(Z, K_2; \mathbf{Z}\pi).$$

Hence there exists a finite complex K_3 with a factorization

$$\begin{array}{ccccc} & & K_3 & & \\ & \nearrow & & \searrow^{\alpha_3} & \\ K_2 & & \xrightarrow{\alpha_2} & & Z \xrightarrow{\iota} Z^+ \end{array}$$

such that $\pi_1\alpha_3$ is an isomorphism and $\iota \circ \alpha_3$ is a plus map with respect to N. By adding more 2 and 3-cells to K_3, one may suppose that $0 = \tau(\iota \circ \alpha_3) \in Wh\pi$.

Since H and π are finitely presented, the condition that N is locally perfect is equivalent to the condition that N is the normal closure in H of a finitely generated perfect subgroup. Therefore, Theorem 5.1 of [H2] (or rather its relative version) says that

$$\lambda_3 : \mathcal{S}^e_{\mathrm{CAT}}(K_3 \operatorname{rel} \partial Z; \mathbf{Z}\pi) \to \mathcal{S}^e_{\mathrm{CAT}}(Z^+ \operatorname{rel} \partial Z)$$

is a bijection. Since λ_3 factors through λ, one deduces that λ is surjective.

For the injectivity of λ, let

$$f_i : (M_i, \partial M_i) \to (Z, \partial Z) \qquad (i = 1 \text{ or } 2)$$

represent two classes of $\mathcal{S}^e_{\mathrm{CAT}}(Z \operatorname{rel} \partial Z; \mathbf{Z}\pi)$. Let (P_i, M_i, M_i') be two plus cobordisms with the corresponding extensions $\bar{f}_i : P_i \to Z^+$ of f_i. Suppose now that $\lambda(f_1) = \lambda(f_2)$ which implies the existence of a e-cobordism (W, M_1', M_2') and an e-$\mathbf{Z}\pi$-equivalence

$$F : (W, M_1', M_2') \to (Z^+ \times I, Z^+ \times 0, Z^+ \times 1).$$

The injectivity of λ follows from the already proven surjectivity applied to the situation:

$$\begin{array}{ccc} \partial \bar{W} = M_1 \coprod M_2 \cup \partial M_1 \times I & \longrightarrow & Z \\ \cap & & \downarrow \\ \bar{W} = P_1 \cup W \cup P_2 & \longrightarrow & Z^+ \end{array} \quad \blacksquare$$

4. Proof of the classification theorem

Let $X \to BH^+$ represent an element of $\{X; BH^+\}$. Consider $BH \to BH^+$ as a Serre fibration with fiber A and take the pull-back diagram:

$$\begin{array}{ccccc} & & Z & \xrightarrow{\beta} & BH \\ & \nearrow & \downarrow{\scriptstyle\gamma} & & \downarrow \\ \partial X & & & & \\ & \searrow & & & \\ & & X & \xrightarrow{\alpha} & BH^+ \end{array} \qquad (Z \text{ and } \gamma \text{ depend on } \alpha)$$

LEMMA 4.1.

(i) *$\pi_1\beta$ is an isomorphism and the following diagram*

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & BH \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\Phi} \\ X & \xrightarrow{j} & B\pi \end{array}$$

is homotopy commutative.

(ii) $H_*(\gamma; \mathbf{Z}\pi) = 0$ *(Then $(Z, \partial X)$ is a $\mathbf{Z}\pi$-Poincaré pair).*

(iii) *N acts trivially on $\pi_2(Z)$*

Proof. (i) one has the diagram

$$\begin{array}{ccccccccc} \pi_2(X) & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(Z) & \longrightarrow & \pi_1(X) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\pi_2\alpha} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\pi_1\beta} & & \downarrow{\scriptstyle\cong} & & \\ \pi_2(BH^+) & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(BH) & \longrightarrow & \pi_1(BH^+) & \longrightarrow & 0 \end{array}$$

$\pi_2\alpha$ is surjective since $\alpha \in \{X; BH^+\}$. Therefore $\pi_1\beta$ is an isomorphism. The fact that $\alpha \in \{X; BH^+\}$ also gives the second part of assertion (i).

(ii) Observe that A is also the fiber of $BN \to BN^+$. Since this last map is a homology isomorphism and BN^+ is simply-connected, A is acyclic. But A is the fiber of $\tilde{Z}_N \to \tilde{X}$ where $\tilde{Z}_N$ is the covering of Z with fundamental group N and $\tilde{X}$ the universal covering of X. Therefore, $H_*(\gamma; \mathbf{Z}\pi)$ is an isomorphism.

It follows that $Z \to X$ can be identified with the map $Z \to Z^+$.

(iii) The fiber A is the Dror-acyclic functor $A(BN)$ of BN (see [D1] for the definition of $A(BN)$). A is thus characterized by $\pi_1(A) = \tilde{N}$, where $\tilde{N}$ is the universal central extension of N [K2] and $\tilde{N}$ acts trivially on $\pi_i(A)$ for $i \geq 2$. Let

$P = \text{Im}\,(\pi_2(A) \to \pi_2(Z))$ and $Q = \text{Im}\,(\pi_2(Z) \to \pi_2(X))$. One has the exact sequence of $\mathbf{Z}N$-modules

$$0 \to P \to \pi_2(Z) \to Q \to 0$$

where P and Q are trivial $\mathbf{Z}N$-modules. Since N is perfect, (iii) follows, as in [D. Lemma 2.6], from the five lemma used in the diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & P & \longrightarrow & \pi_2(Z) & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \\ 0 = H_1(N, Q) & \longrightarrow & & & H_0(N, P) & \longrightarrow & H_0(N; \pi_2(Z)) & \longrightarrow & H_0(N; Q) & \longrightarrow & 0 \end{array}$$

We can now give the proof of the classification Theorem. Let

$$(f', \alpha) \in \mathcal{S}^e_{\text{CAT}}(X \text{ rel } \partial X) \times \{X; BH^+\}.$$

By Lemma 4.1, the map $Z \xrightarrow{\gamma} X$ satisfies the hypothesis of Theorem 3.1. Thus the map

$$\lambda : \mathcal{S}^e_{\text{CAT}}(Z \text{ rel } \partial X; \mathbf{Z}\pi) \to \mathcal{S}^e_{\text{CAT}}(X \text{ rel } \partial X)$$

is bijective and there is a class of $\mathcal{S}^e_{\text{CAT}}(Z \text{ rel } \partial X; \mathbf{Z}\pi)$ represented by $f : (M, \partial M) \to (Z, \partial X)$ such that $\lambda(f) = f'$. Then $\varphi(M, \gamma \circ f, \beta \circ f) = (f', \alpha)$ and φ is surjective.

Now if $\varphi(M, f, \kappa) = \varphi(\bar{M}, \bar{f}, \bar{\kappa}) = (f', \alpha)$, then κ and $\bar{\kappa}$ both factor through Z. The injectivity of φ then follows from the injectivity of λ_α.

5. Existence theorem

In this section, we will deduce the following result from the classification Theorem.

5.1 EXISTENCE THEOREM. *Let M^n be a closed* CAT-*manifold of dimension $n \geq 5$, with $\pi_1(M) = \pi$. Let $1 \to N \to H \xrightarrow{\Phi} \pi \to 1$ be an extension of π such that, H is finitely presented, N is locally perfect and $H_2(N; \mathbf{Z}) = 0$ (trivial action). Let $\mu : \pi_1(\partial M) \to H$ be a lifting of $\pi_1(\partial M) \to \pi$.*

Assume that one of the following conditions is realized:

(a) $\Phi^+ : BH^+ \to B\pi$ *admits a homotopy section* $s : B\pi \to BH^+$ *such that the following diagram is homotopy commutative*

$$\begin{array}{ccccc} BH & & \xrightarrow{\ \Phi.\ } & & BH^+ \\ \uparrow{\scriptstyle \mu} & & & & \uparrow{\scriptstyle s} \\ \partial M & \longrightarrow & M & \longrightarrow & B\pi \end{array}$$

(b) $H^i(B\pi, \partial M; \pi_{i-1}(BH^+)) = 0$ *for* $4 \leqslant i \leqslant n$ ($\pi_{i-1}(BH^+)$ *is a* $\mathbf{Z}\pi$*-module since* $\pi_1(BH^+) \xrightarrow{\pi_1\Phi+} \pi$ *is an isomorphism*).

(c) $H^i(M, \partial M; \pi_{i-1}(BH^+)) = 0$ *for* $i \geqslant 4$ ($\pi = \pi_1(M)$ *whence* $\pi_{i-1}(BH^+)$ *is a* $\mathbf{Z}\pi_1(M)$*-module*).

Then there exists a compact CAT*-manifold* V^n *with* $\partial V = \partial M$ *and a map* $f : V \to M$ *such that*

(1) $\pi_1(V) = H$, $\pi_1 f = \Phi$ *and* $\pi_2(\partial M) \to \pi_1(V)$ *is equal to* μ.
(2) $\omega_1(V) = f^*\omega_1(M)$ *where* ω_1 *is the first Stieffel–Whitney class.*
(3) $f_* : H_*(V; \mathbf{Z}\pi) \to H_*(M, \mathbf{Z}\pi)$ *is an isomorphism and* $0 = \tau(f) \in Wh\pi$.

Remarks and Examples

(1) If M and ∂M are simply connected, condition (a) is realized. When $M = S^n$, this gives the theorem of Kervaire [K1]. The manifold V which will be constructed by our proof will be $M \# \Sigma$ where Σ is the homology sphere with fundamental group N constructed in [K1].

(2) Condition (a) is automatically satisfied when ∂M is empty and the cohomology dimension of π is $\leqslant 3$. It is also fulfilled when π and $H_i(N; \mathbf{Z})$ are finite for all i and the orders of π and of $H_i(N_i; \mathbf{Z})$ are relatively prime (and ∂M empty). Indeed, the order of π and the order of $\pi_i(BN^+) \simeq \pi_i(BH^+)$ ($i \geqslant 2$) are then relatively prime and thus, by transfer, $H^*(\pi; \pi_{*-1}(BH^+)) = 0$ for $* \geqslant 3$.

EXAMPLE. $M^n = L^n_{p,q}$ a lens space with p prime to 120 and $N = \Delta$, the binary icosaedral group.

(3) Condition (a) is satisfied if $H \to \pi$ has a section $s : \pi \to H$ such that

$$\begin{array}{ccc} & & BH \\ & {\scriptstyle \mu}\nearrow & \uparrow{\scriptstyle s} \\ \partial H & \longrightarrow & B\pi \end{array}$$

is homotopy commutative.

(4) The condition $H_1(N)=0$ is necessary to obtain properties (1) and (3). The condition $H_2(N)=0$ is necessary when $\pi_2(M)=0$.

COROLLARY 5.2. *Let $1\to N\to H\to \mathbf{Z}\to 1$ be a short exact sequence of groups where H is finitely presented and is the normal closure of one element, N is locally perfect and satisfies $H_2(N;\mathbf{Z})=0$ (trivial action). Then, for any $n\geq 5$, there is a smooth knot $\eta: S^{n-2}\hookrightarrow S^n$ such that $\pi_1(S^n-\eta(S^{n-2}))=H$ and such that the infinite cyclic cover of $S^n-\eta(S^{n-2})$ is acyclic.*

EXAMPLE. Let us consider the universal central extension [K2]:

$$0\to H_2(S;\mathbf{Z})\to \tilde{S}\to S\to 1$$

of a finitely presented simple group S. One can take for H the semi-direct product of $\mathbf{Z}$ with $\tilde{S}$ for any $\mathbf{Z}$-action on $\tilde{S}$. Indeed, in view of Corollary 5.2, it suffices to prove that H is normally generated by one element ($\tilde{S}$ is finitely presented and $H_1(\tilde{S};\mathbf{Z})=H_2(\tilde{S};\mathbf{Z})=0$). Since S is simple, $H_2(S;\mathbf{Z})$ is the whole center of $\tilde{S}$ and the $\mathbf{Z}$-action on $\tilde{S}$ induces a $\mathbf{Z}$-action on S. Choose $a\in S$ such that $a^{-1}xa\neq x'$ for a least one $x\in S$, where x' is the image of x under the action of a generator t of $\mathbf{Z}$. Call $\tilde{a}$ a lifting of a in $\tilde{S}$. Then $\tilde{a}t^{-1}$ generates normally H. Indeed the relation $\tilde{a}=t$ induces non trivial relations in S and as S is simple, the perfect group $\tilde{S}/\{\tilde{a}^{-1}y\tilde{a}=y',\ y\in\tilde{S}\}$ must be a quotient of $H_2(S;\mathbf{Z})$, then must be trivial. Thus $\tilde{a}=t$ implies $y=1$ for all $y\in\tilde{S}$ and $t=1$.

Proof of Corollary 5.2. This comes from the existence theorem for $(M,\partial M)=(S^1\times D^{n-1}, S^1\times S^{n-2})$, the complement of the trivial knot, and the lifting $\mu:\pi_1(S^1\times S^{n-2})\simeq\mathbf{Z}\to H$ sends $1\in\mathbf{Z}$ onto a normal generator of H. The lifting μ gives rise to a section of Φ and then a section of Φ^+. Therefore condition (a) holds and the manifold pair $(V^n, S^1\times S^{n-2})$ given by the existence theorem is the complement of the required knot. ■

Proof of Theorem 5.1. The hypothesis $H_2(N;\mathbf{Z})=0$ implies that $\pi_2(BN^+)=\pi_2(BH^+)=0$. Therefore, using the classification theorem, a map $f: V\to M$ satisfying (1) to (3) will exist if and only if there is a lifting α:

$$\begin{array}{ccc} \partial M & \xrightarrow{\mu} & BH^+ \\ \cap & \overset{\alpha}{\nearrow} & \uparrow{\scriptstyle \Phi^+} \\ M & \xrightarrow{j} & B\pi \end{array}$$

of the classifying map j. Indeed, such an α belongs to $\{M, BH^+\}$ and f may be deduced from $(V, f, \delta) = \varphi^{-1}(\mathrm{id}_M, \alpha)$.

If Φ^+ admits a section s_p compatible with μ, one can take $\alpha = s \circ j$. Thus, Theorem 5.1 is proved for condition (a).

One can always define a section $s^{(2)}$ of φ^+ compatible with μ over ∂M union the two skeleton of $B\pi$, since $\pi_1\Phi^+$ is an isomorphism. The fact that $\pi_2(BH^+) = 0$, together with condition (b) show that there is no obstruction to extending $s^{(2)}$ in $s: B\pi \to BH^+$. Thus one gets condition (a) fulfilled. Finally, when condition (c) holds, one gets α as an extension of $s^{(2)} \circ j$ by obstruction theory (using again the hypothesis $\pi_2(BN^+) = 0$).

The proof of Theorem 5.1 is now complete. ■

6. Classification of homology spheres

Let us consider the set $\mathcal{S}^e_{\mathrm{CAT}}(X; \Phi)$ when $X = S^n$. One has $H = N$, $\Phi = 0$ and there is no difference between the cases $e = s$ or h. Thus, $\mathcal{S}^e_{\mathrm{CAT}}(S^n; \Phi)$ will be denoted by $\mathcal{S}_{\mathrm{CAT}}(S^n, H)$ throughout this section. By Theorems 2.1 and 5.1, or [K1], $\mathcal{S}_{\mathrm{CAT}}(S^n, H)$ is not empty if and only if H is finitely presented and $H_1(H) = H_2(H) = 0$.

Let $(M^n, f, \lambda) \in \mathcal{S}_{\mathrm{CAT}}(S^n, H)$. The manifold M^n is an oriented (integral) homology sphere. We shall omit in the notation the data of f which is here redondant; indeed, by obstruction theory, there is only one homotopy class of map $M \to S^n$ of degree one. Roughly speaking, $\mathcal{S}^e_{\mathrm{CAT}}(S^n, H)$ classifies the n-dimensional oriented CAT homology spheres with fundamental group identified to H, up to (H_*-and π_1)-cobordism. The bijection φ of Theorem 2.1 can be expressed in the following form

$$\varphi: \mathcal{S}_{\mathrm{CAT}}(S^n; H) \simeq \pi_n(BH^+)$$

when CAT = PL or TOP

$$\varphi: \mathcal{S}_{\mathrm{DIFF}}(S^n; H) \simeq \theta_n \oplus \pi_n(BH^+)$$

where θ_n is the Kervaire–Milnor group of homotopy spheres [KM].

The group law on $\theta_n \oplus \pi_n(BH^+)$ or $\pi_n(BH^+)$ can be geometrically interpreted in $\mathcal{S}_{\mathrm{CAT}}(S^n; H)$ in at least two ways:

(1) connected sum of maps followed by a fitting of the fundamental group like in the proof of 3.1. This gives the groups $\pi_n^H(BH)$ of [H3].

(2) The law of the groups $C_n(K)$ of [H1]. Recall that the elements of $C_n(K)$

are pairs (M, f) where M is a n-dimensional PL-homology sphere and $f: K \to M$ is an embedding from a fixed acyclic polyedron K ($2 \dim K + 2 \leq n$) into M such that $\pi_1 f$ is an isomorphism. The sum is a connected sum around a regular neighborhood of K. If we pose $\pi_1(K) = H$, the fundamental group of M is identified with H via $\pi_1 f$. One thus obtain an element of $\mathcal{S}_{\text{PL}}(S^n; H)$. Then, the groups $C_n(K)$ of [H1, Chapter 2] are isomorphic to $\pi_n(B\pi_1(K)^+)$. The isomorphism between $\mathcal{S}_{\text{PL}}(S^n; H)$ and $\pi_n(BH^+)$ was first established by the author [H1] when H is a finitely presented group of type $(\overline{FP})$ (See [B-E] for the original definition which is equivalent to BH has finite skeleta). If H is of type $(\overline{FP})$ one can deduce that the complex Z of § 4 has also finite skeleta and Theorem 5.1 of [H2] can be used in the proof of Theorem 2.1 instead of our Theorem 3.1. The first proof of the general case is due to P. Vogel [V2 Theorem 1.5] and uses a different principle.

Problem. Find a finitely presented perfect group which is not of type $(\overline{FP})$.

Finally, recall that a class of $\mathcal{S}_{\text{CAT}}(S^n, H)$ represented by $\kappa: \Sigma^n \to BH$ corresponds to zero in $\theta_n \oplus \pi_n(BH^+)$ (or $\pi_n(BH^+)$ if CAT = PL or TOP) if and only if there exists an acyclic compact CAT-manifold A^{n+1} with $\Sigma = \partial A$ and such that the inclusion of Σ into A induces an isomorphism on the fundamental groups. The argument of [H3 § 4] shows that $\varphi_2([\text{K}]) = 0$ in $\pi_n(BH^+)$. On the other hand, when CAT $= C^\infty$, a $\mathbf{C}^\infty$-plus cobordiam from Σ to a homotopy sphere Σ_0 union A^{n+1} (union over Σ) constitute a contractible C^∞-manifold with boundary Σ_0. Therefore $[\Sigma_0] = 0$ in Φ_n and thus $\varphi_2([\text{K}]) = 0$.

7. Computations of $\pi_n(BN^+)$

As we have seen in Section 6, the classification up to (H_*-and-π_1)-cobordism of homology spheres with fundamental group N reduces to the knowledge of $\pi_i(BN^+)$. This knowledge is also important in view of the existence and classification Theorems, for $\pi_i(BN^+) = \pi_i(BH^+)$ ($i \geq 2$) occurs as the obstruction coefficients in determining $\{X; BH^+\}$. In Subsection 7.1 below we give some general results and in Subsection 7.2 we make explicit computation for some classical cases. Other results, in connection with algebraic K-theory are given in [H3].

7.1. *General results*

Throughout this section, N is a perfect group and $H_*(N)$ means $H_*(N; \mathbf{Z})$ (trivial action).

PROPOSITION 7.1.1. *Suppose that $H_i(N) \in \mathscr{C}$ for $i \leq k$, where $\mathscr{C}$ is a perfect and weakly complete Serre class of abelian groups* [Hu p. 300]. *Then $\pi_i(BN^+) \in \mathscr{C}$ for $i \leq k$. In particular:*

(1) *$\pi_i(BN^+)$ is countable if N is countable.*

(2) *If $H_i(N)$ is finitely generated for $i \leq k$ then $\pi_i(BN^+)$ are finitely generated for $i \leq k$.*

(3) *If $H_i(N)$ is finite for $i \leq k$ then $\pi_i(BN^+)$ is finite for $i \leq k$.*

EXAMPLES.

(1) If N is finite, $H_i(N)$ is finite for all i. Thus $\pi_i(BN^+)$ is finite for all i and, by obstruction theory, $\{X; BH^+\}$ is a finite set. In particular there is finitely many (H_*-and-π_1)-cobordism classes of homology spheres of dimension $n \geq 5$ with a given finite fundamental group.

(2) If $H_i(N) = 0$ for all $i > 0$, then $\pi_i(BN^+) = 0$ for all i. Thus $\mathscr{S}^e_{\mathrm{CAT}}(X \text{ rel } \partial X; \Phi) \simeq \mathscr{S}^e_{\mathrm{CAT}}(X \text{ rel } \partial X)$.
Finitely presented acyclic groups exist, for instance the Highman's groups of presentation:

$$N_r = \{a_1, \ldots, a_r \mid a_1 a_2 a_1^{-1} a_2^{-2}, a_2 a_3 a_2^{-1} a_3^{-2}, \ldots, a_r a_1 a_r^{-1} a_1^{-2}\}$$

which are non-trivial when $r \geq 4$ [Hi]. The two dimensional complex determined by the above presentation is acyclic and is homotopy equivalent to BN_r (see [D-V]).

Proof of Proposition 7.1.1. If $\mathscr{C}$ is perfect and weakly complete, the Serre–Hurewicz isomorphism Theorem holds [Hu Theorem 1.8]. Then, Proposition 7.1.1 follows from $H_*(BN^+) = H_*(N)$ and $\pi_1(BN^+) = 1$. ■

PROPOSITION 7.1.2. *Let N_1 and N_2 be two perfect groups. Consider the maps*

$$t_\times^+ : B(N_1 \times N_2)^+ \to BN_1^+ \times BN_2^+$$

and

$$t_*^+ : BN_1^+ \vee BN_2^+ \to B(N_1 * N_2)^+$$

induced by

$$t_\times : B(N_1 \times N_2) \to BN_1 \times BN_2$$

and

$t_*: BN_1 \vee BN_2 \to B(N_1 * N_2)$. *Then* $t_\times^+$ *and* t_*^+ *are homotopy equivalences.*

Proof. Clearly t and t_* are homology equivalences. Then $t_\times^+$ and t_*^+ induce isomorphisms in homology and all the spaces are simply connected. ■

Remark. If N_1 and N_2 are finitely presented and if one represents the elements of $\pi_n(BN_i^+)$ by homology spheres with fundamental group identified with N_i (Section 6), then an element

$$(x, y) \in \pi_n(BN_1^+) \oplus \pi_n(BN_2^+) \subset \pi_n(B(N_1 * N_2)^+)$$

corresponds to the connected sum of the sphere representing x with the one representing y. The remaining part of $\pi_n(B(N_1 * N_2)^+)$ shows the existence of more sophisticated homology spheres with fundamental group $N_1 * N_2$.

PROPOSITION 7.1.3. *Let* $1 \to A \to H \to Q \to 1$ *be a short exact of groups with* H *and* Q *perfect and* A *abelian. Assume that* Q *acts trivially on* A. *Then*

$$BA \to BH^+ \to BQ^+$$

is a Serre fibration. In particular, $\pi_i(BH^+) \simeq \pi_i(BQ^+)$ *for* $i \geq 3$.

A similar result, with other hypotheses is due to J. Wagoner [W, lemma 3.1].

Proof. Call F the homotopy fiber of $BH^+ \to BQ^+$. One has the following commutative diagram:

$$\begin{array}{ccccc} BA & \longrightarrow & BH & \longrightarrow & BQ \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & BH^+ & \longrightarrow & BQ^+ \end{array}$$

in which the two right hand vertical arrows are homology isomorphisms. Our hypotheses permit us to use the comparison theorem and thus $BA \to F$ is a homology isomorphism. The space F is simple, since the total space BH^+ of the fibration is simply connected. The map $BA \to F$ is then a homology isomorphism between simple spaces; such a map is a homotopy equivalence [D3, 4.2].

7.2. *Some Computations*

7.2.1. *The binary icosaedral group* Δ

Recall that Δ admits the presentation $\{a, b \mid a^5 = b^3 = (ab)^2\}$ and contains 120 elements. Call F_{120} the homotopy theoretic fiber of a map from S^3 to itself of degree 120. If X is a space, ΩX denotes its loop space.

PROPOSITION. *The space $\Omega(B\Delta^+)$ is homotopy equivalent to F_{120}*

Remark. Considering the h-space structure on S^3 a map of degree 120 is given by $x \mapsto x^{120}$. Such a map induces the multiplication by 120 on all homotopy groups. Thus one has

(1) $\pi_1(B\Delta^+) = 1$
(2) $\pi_2(B\Delta^+) = 0$
(3) One has an exact sequence

$$0 \to \pi_i(S^3)/120\pi_i(S^3) \to \pi_i(B\Delta^+) \to \pi_{i-1}(S^3)_{120} \to 0$$

where $\pi_{i-1}(S^3)_{120}$ is the subgroup of $\pi_i(S^3)$ of elements whose order divides 120. In particular, $\pi_i(B\Delta^+)$ is a $\mathbf{Z}/120^2\,\mathbf{Z}$-module.

The tables of [T] enables us to compute the order of $\pi_i(B\Delta^+)$

n	3	4	4	6	7	8	9	10	11	12	13	14
$\lvert\pi_n(B\Delta^+)\rvert$	120	2	4	24	24	4	6	45	30	8	96	1152

n	15	16	17	18	19	20	21	22
$\lvert\pi_n(B\Delta^+)\rvert$	192	24	180	900	360	576	2304	2304

Proof of the Proposition. The argument comes from [D2, proof of Proposition 9.1]. Let Σ_Δ be the Poincaré sphere of dimension 3 with $\pi_1(\Sigma_\Delta) = \Delta$ and universal cover S^3. Call U the homotopy fiber of $\Sigma_\Delta^+ \to B\Delta^+$. One has the homotopy commutative diagram:

$$\begin{array}{ccccc} S^3 & \longrightarrow & \Sigma_\Delta & \longrightarrow & B\Delta \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\ i\ } & \Sigma_\Delta^+ & \longrightarrow & B\Delta^+ \end{array}$$

The two right hand maps are homology isomorphisms. The space $B\Delta^+$ is simply connected and the perfect group Δ acts trivially on $\tilde{H}_*(S^3) = \mathbf{Z}$ since $\operatorname{Aut}\mathbf{Z}$ is abelian. By the comparison Theorem, the map $S^3 \to U$ is a homology isomorphism. Since $H_1(\Delta) = H_2(\Delta) = 0$, $B\Delta^+$ is 2-connected and U is 1-connected. Therefore, $S^3 \to U$ is a homotopy equivalence. Observe that $\Sigma_\Delta^+ \simeq S^3$ and, since the covering map $S^3 \to \Sigma_\Delta$ is of degree 120, the map i is of degree 120. Thus, $\Omega(B^+) \simeq$ fiber $(i) \simeq F_{120}$. ■

7.2.2. *Fundamental group of a 3-dimensional homomology sphere*

Let V be a 3-dimensional manifold such that $H_*(V;\mathbf{Z}) = H_*(S^3;\mathbf{Z})$. By the Kneser–Milnor unique decomposition Theorem [Mi], V can be written in a unique way as a connected sum

$$V = V_1 \# V_2 \# \cdots \# V_m$$

where V_i are prime manifolds. Suppose that $\pi_1(V_i)$ is infinite for $1 \leqslant i \leqslant k$ and finite for $k+1 \leqslant i \leqslant m$. Therefore, the space BN^+ for the perfect group $N = \pi_1(V)$ can be described as follows.

PROPOSITION. *BN^+ has the homotopy type of*

$$\underbrace{S^3 \vee \cdots \vee S^3}_{k \text{ copies}} \quad \bigvee \quad \underbrace{B\Delta^+ \vee \cdots \vee B\Delta^+}_{(m-k) \text{ copies}}$$

where Δ is the binary icosaedral group, see 7.2.1. *In particular, BN^+ is rationally equivalent to a bouquet of k copies of S^3.*

Proof. From [Mi], one deduces that the V_i's are of three possible type

(1) $\pi_1(V_i)$ is infinite and $V_i \simeq B\pi_1(V_i)$
(2) $V_i = S^1 \times S^2$
(3) $\pi_1(V_i)$ is finite.

Since V is a homology sphere, each V_i must be a homology sphere which excludes possibility 2). Thus $V_i = B\pi_1(V_i)$ for $i \leqslant k$ and $V_i^+ \simeq S^3$. If $\pi_1(V_i)$ is finite, then $\pi_1(V_i)$ must be isomorphic to Δ [K1 Theorem 2]. This proves the proposition, using Proposition 7.1.2. ■

Remark. The existence of 3-dimensional homology spheres V_i such that $V_i = B\pi_1(V_i)$ is classical. For instance, the ones obtained by gluing the complements of two non-trivial knots by automorphism of $S^1 \times S^1$ which exchanges the factors (classical Dehn's construction [De]).

Of course, the fundamental group of a 3-dimensional homology sphere is the fundamental group of a n-dimensional homology sphere for all $n \geqslant 5$ [Ke]. Take such a group N with $BN^+ = S^3 \vee S^3$. Since $\pi_i(S^3 \vee S^3)$ is infinite for i odd$\geqslant 3$, there is infinitely many (H_*-and-π_1)-cobordism classes of n-dimensional homology spheres with fundamental group N for all n odd≥ 5.

7.2.3. *Alternate groups*

Denote by A_n (respectively S_n) the alternate (respectively symmetric) group of permutations of n objects and $A_\infty = \lim_\rightarrow A_n$, $S_\infty = \lim_\rightarrow S_n$. By [P], one has an isomorphism of $\pi_i(BS_\infty^+)$ with π_i^s, the i^{th} stable homotopy group of spheres. Thus, the composition $A_n \to A_\infty \to S_\infty$ gives a homomorphism $\beta_i^n : \pi_i(BA_n^+) \to \pi_i^s$.

PROPOSITION A

(1) β_i^n *is an isomorphism when* $2 \leq i < (n-1)/3$ *or when* $2 \leq i < (n+1)/2$ *and* $n \equiv 2 (\text{mod } 3)$.

(2) $\beta_i^n \otimes \mathbf{Z}[\frac{1}{3}] : \pi_i(BA_n^+) \otimes \mathbf{Z}[\frac{1}{3}] \to \pi_i^s \otimes \mathbf{Z}[\frac{1}{3}]$
is an isomorphism for $2 \leq i < (n+1)/2$, *except if* $i = 3$ *and* $n = 6$.

(3) $\beta_i^{3i+\varepsilon}$ ($\varepsilon = 0$ *or* 1) *is an epimorphism with kernel isomorphic to* $\mathbf{Z}/3\mathbf{Z}$.

The precise determination of Ker $\beta_i^{3i+\varepsilon}$ was pointed out to me by the referee. The proof of Proposition A is given at the end of this section and uses Proposition B below.

Let $C = \{1, t\}$ be the group with two elements. If G is an abelian group, we use the notation by G^+ for G considered as a trivial $\mathbf{Z}\,C$-module and G^- when the C-action is $tx = -x$. Let F_p denote the field with p-elements.

PROPOSITION B (due to P. Vogel). *Let k be a finite field of characteristic $p \neq 2$. Then*

$$H_i(S_n; \mathbf{F}_p^-) = \begin{cases} 0 & \text{if } \; n \not\equiv 0 \;\text{ or }\; 1 \,(\text{mod}\, p). \\ 0 & \text{if } \; n = \lambda p \;\text{ or }\; \lambda p + 1 \;\text{ and }\; i < (p-2)\lambda \\ \mathbf{F}_p & \text{if } \; n = \lambda p \;\text{ or }\; \lambda p + 1 \;\text{ and }\; i = (p-2)\lambda \end{cases}$$

In particular $H_*(S_\infty; G^-) = 0$ for all abelian group G.

Proof of Proposition B. We use the notations of [V1 Chapter IV]. By [V1 Theorem 4], $\bigoplus_n H_*(S_n; \mathbf{F}_p^-)$ is the free commutative $\mathbf{F}_p^{(1)}$-algebra generated by the elements $a^{(1)}(j_1 \cdots j_r) \in H_{j_1+\cdots+j_r}(S_p; \mathbf{F}_p^-)$, where $(j_1, \ldots j_r)$ ranges over all 1-admissible sequences of positive integers [V1 p. 347]. If $(j_1, \ldots j_r)$ is admissible one checks by induction on r the inequality

$$j_1 + \cdots + j_r \geq \frac{p^r - 1}{p-1}(p-2) \quad \text{(a)}$$

Let $a_1 \cdots a_k$ be a monomial in $H_i(S_n; \mathbf{F}_p^-)$ where: $a_t \in H_{\alpha(t)}(S_{\beta(t)}; \mathbf{F}_p^-)$

$$i = \sum_{t=1}^{k} \alpha(t), \qquad n = \sum_{t=1}^{k} \beta(t), \qquad \beta(t) = p^{\mu(t)}$$

Since there is at most one a_t of dimension zero (i.e. $a(\varnothing) \in H_0(S_1; \mathbf{F}_p^-) = \mathbf{F}_p$), one must have $n \equiv 0$ or $1 \pmod p$. By (a) one has

$$\alpha(t) \geq \frac{p^{\mu(t)} - 1}{p - 1}\ (p-2)$$

whence

$$i \geq (n-k)\frac{p-2}{p-1} \qquad \text{(b)}$$

If $n = \lambda p$, one must have $\mu(t) \geq 1$ for all t; then $k \leq \lambda$ and (b) gives $i \geq \lambda(p-2)$. If $n = \lambda p + 1$, one has a unique t for which $\mu(t) = 0$; thus $k \leq \lambda + 1$ and (b) gives also $i \geq \lambda(p-2)$. When $i = \lambda(p-2)$ and $n = \lambda p$ or $\lambda p + 1$, one checks similarly that $a_t = a(p-2)$ for all t, whence $H_{\lambda(p-2)}(S_{\lambda p}; \mathbf{F}_p^-)$ and $H_{\lambda(p-2)}(S_{\lambda p+1}; \mathbf{F}_p^-)$ are both isomorphic to F_p. ■

Proof of Proposition A. The map $\pi_i(BA_\infty^+) \to \pi_i(BS_\infty^+)$ is an isomorphism for $i \geq 2$, since BA_∞^+ is the universal cover of BS_∞^+. Thus it suffices to prove the isomorphism for $\pi_i(BA_n^+) \to \pi_i(BA_\infty^+)$ or equivalently for $H_i(A_n; \mathbf{F}_p) \to H_1(A_\infty; \mathbf{F}_p)$ for all prime p. One has the exact sequence of $\mathbf{F}_pC$-modules:

$$0 \longrightarrow \mathbf{F}_p^- \xrightarrow{\ \alpha\ } \mathbf{F}_pC \xrightarrow{\ r\ } \mathbf{F}_p^+ \longrightarrow 0$$

where $\alpha(1) = 1 - t$ and $r(1) = 1$

This gives a long exact sequence:

$$\to H_{i+1}(S_n; \mathbf{F}_p^+) \to H_i(S_n; \mathbf{F}_p^-) \xrightarrow{\ \alpha_*\ } H_i(S_n; \mathbf{F}_pC) \to H_i(S_n; \mathbf{F}_p^+) \to$$

One has $H_*(S_n; \mathbf{F}_pC) \cong H_*(A_n; \mathbf{F}_p)$ under which identification α_* is the homomorphism induced by the inclusion. Using the five lemma, it suffices to prove the corresponding isomorphisms for $H_*(S_n; \mathbf{F}_p^\pm)$.

The isomorphism $H_*(S_n; \mathbf{F}_p^+) \to H_*(S_\infty; \mathbf{F}_p^+)$ for $i<(n+1)/2$ was proven by Nakaoka [N Corollary 6.7.]. In the case $p=2$, one has $\mathbf{F}_2^- = \mathbf{F}_2$. So, using Proposition B, one deduces (1) and (2) and the fact that β_i^n is an epimorphism for $n=3i+\varepsilon$, $\varepsilon=0$ or 1. To compute Ker β_i^n, one considers the diagram

$$\begin{array}{ccc} H_{i+1}(BS_\infty^+; BS_n^+; \mathbf{Z}^-) & \xrightarrow{\simeq} & H_{i+1}(BA_\infty^+; BA_n^+; \mathbf{Z}) \\ \downarrow{\scriptstyle\partial} & & \downarrow \\ H_i(S_n; \mathbf{Z}^-) & \xrightarrow{\alpha_*} & H_i(A_n; \mathbf{Z}) \end{array}$$

From above, we deduce that ∂ is an isomorphism modulo 2-torsion and α_* has a 2-torsion kernel by transfer. As $H_{i+1}(BA_\infty^+; BA_n^+; \mathbf{Z}^-)$ is a 3-torsion group, one has:

$$\ker \beta_i^n \twoheadleftarrow \pi_{i+1}(BA_\infty^+, BA_n^+) \simeq H_{i+1}(BA_\infty^+ BA_n^+; \mathbf{Z}) \simeq H_i(S_n; \mathbf{Z}^-) \otimes \mathbf{Z}[\tfrac{1}{2}]$$

Thus it suffices to prove that $H_i(S_n; \mathbf{Z}^-) \otimes \mathbf{Z}[\frac{1}{2}] \simeq \mathbf{Z}/3\mathbf{Z}$.

Let β and $\bar{\beta}$ be the Bockstein homomorphisms for the sequences

$$0 \to \mathbf{Z}/3\mathbf{Z}^- \to \mathbf{Z}/9\mathbf{Z} \to \mathbf{Z}/3\mathbf{Z}^- \to 0$$

and

$$0 \to \mathbf{Z}^- \to \mathbf{Z}^- \to \mathbf{Z}/3\mathbf{Z}^- \to 0$$

respectively. The long homology exact sequence shows that

$$\beta : H_2(S_3; \mathbf{F}_3^-) \to H_1(S_3; \mathbf{F}_3^-)$$

is surjective. Proposition B shows that

- $-H_0(S_3; \mathbf{F}_3^-) = 0$
- $-H_1(S_3; \mathbf{F}_3^-) \cong \mathbf{F}_3$, generator $a^{(1)}(1)$
- $-H_2(S_3; \mathbf{F}_3^-) \simeq \mathbf{F}_3$, generator $a^{(1)}(2)$.

Thus $\beta(a^{(1)}(2)) = \pm a^{(1)}(1)$ and $\beta(a^{(1)}(1)) = 0$.

Since the Bockstein homomorphism behaves like a derivation for the product of $H_*(S_*; k^\pm)$, the generator $a^{(1)}(1)^i a^{(1)}(\varnothing)^\varepsilon$ of $H_i(S_{3i+\varepsilon}; \mathbf{F}_3^-) \simeq \mathbf{F}_3$ is equal to

$\beta(a^{(1)}(2)a^{(1)}(1)^{i-1}a^{(1)}(\varnothing)^{\varepsilon})$. Then the exact sequence

$$\begin{array}{ccccccc} H_{i+1}(S_n;\mathbf{F}_3^-) & \xrightarrow{\beta} & H_i(S_n;\mathbf{Z}^-) & \xrightarrow{.3} & H_i(S_n;\mathbf{Z}^-) & \longrightarrow & H_i(S_n;\mathbf{F}_3^-)\simeq\mathbf{F}_3 \\ & \searrow^{\beta} & \downarrow & & & & \\ & & H_i(S_n;\mathbf{F}_3^-) & & & & \end{array}$$

proves that $H_i(S_{3i+\varepsilon};\mathbf{Z}^-)\otimes\mathbf{Z}\left[\frac{1}{2}\right]\simeq\mathbf{Z}/3\mathbf{Z}$. The proof of Proposition B is thus complete.

Remark. As in the proof of Proposition B one can actually show that $H_{i+1}(S_{3i+\varepsilon};\mathbf{F}_3^-)\simeq\mathbf{F}_3$, generated by $a^{(1)}(2)a^{(1)}(1)^{i-1}a^{(1)}(\varnothing)^{\varepsilon}$.

REFERENCES

[B-E] BIERI R., ECKMANN B.-*Finiteness properties of duality groups.* Comment. Math. Helv. *49* (1974) 74–83.

[B] BROWDER W. *Poincaré Spaces, Their normal fibrations and Surgery.* Inv. Math. *17* (1972) 191–202.

[C-E] CARTAN H., EILENBERG S.-*Homological Algebra.* Princeton University Press 1956.

[De] DEHN M. *Über die Topologie des dreidimensional Raumes.* Math. Ann. *69* (1910) 127–168.

[D1] DROR E. *Acyclic Spaces.* Topology *11* (1972). 339–348.

[D2] —— *Homology Spheres.* Israeli J. of Math. *15* (1973).

[D3] —— *A generalization of the Whitehead Theorem.* Springer Lect. Notes 249, 13–22.

[D-V] DYER E., VASQUEA A. T.-*Some small aspherical spaces.* J. of the Australian Math. Soc. XVI (1973) 332–352.

[HO] HAUSMANN J.-Cl.-*Groupes de sphères d'homologie entière.* Thesis, Univ. of Geneva, 1974.

[H1] —— *Classification of homology spheres.* Note Univ. of Geneva 1975.

[H2] —— *Homological surgery.* Ann. of Math. *104* (1976), 573–584.

[H3] —— *Homology sphere bordism and Quillen plus construction. Algebraic K-theory,* Evanston 1976, Springer Lect. Notes 551, 170–181.

[H4] ——, *Variétés avec une homologie et un groupe fondamental donné.* C. R. Acad. Sc. Paris *283* (1976), 241–244.

[H-V] HAUSMANN J.-Cl., VOGEL P.-*The plus construction and lifting maps from manifolds. To appear in Proc.* AMS Summer Institute Stanford, 1976.

[Hi] HIGHMAN G. *A finitely generated infinite simple group.* J. London Math. Soc. *26* (1951) 61–64.

[Hu] HU S. T. *Homotopy theory.* Academic Press 1959.

[Ki] KERVAIRE M. *Smooth homology spheres and their fundamental groups.* Trans. AMS *144* (1969) 67–72.

[K2] —— *Multiplicateurs de Shur et K-theorie* Essays on topology and related topics Springer 1970, 212–225.

[K-M] KERVAIRE M., MILNOR J.-*Groups of homotopy spheres* Ann. of Math. *77* (1963) 504–537.

[M] MILNOR J. *A unique decomposition theorem for 3-manifold.* Amer. J. of Math. *84* (1962) 1–7.

[N] NAKAOKA M. *Decomposition Theorem for homology groups of symmetric groups.* Ann. of Math. *71* (1960) 16–42.

[P] PRIDDY S. *Transfer, symmetric groups and stable homotopy theory. Algebraic K-theory I.* Springer Lect. Notes 341, 244–259.

[T] TODA H. *Composition methods in homotopy groups of spheres.* Princeton Univ. Press 1962.
[V1] VOGEL P. *Cobordisme d'immersions.* Ann. Ec. norm. Sup. 7 (1974), 316–357.
[V2] —— *Un théorème d'Hurewicz homologique to appear in Comment. Math. Helv.*
[Wg] WAGONER J. *Delooping classifying spaces in Algebraic K-theory.* Topology II (1972). 349–370.
[W] WALL C. T. C. *Surgery on compact manifolds.* Academic Press 1970.

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