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Domain constants associated with Schwarzian derivative

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Dedicated to Professor Albert Pfluger on his seventieth birthday

1. Definition of the constants

Let A be a simply connected domain in the extended plane with more than one boundary point. A non-euclidean metric $\rho(z)|dz|$ of A is defined by the condition $\rho(z)|dz| = (1-|w|^2)^{-1}|dw|$, where $z \to w$ is a conformal mapping of A onto the unit disc D. For a function φ holomorphic in A we introduce the norm

$$\|\varphi\|_A = \sup_{z \in A} |\varphi(z)|\, \rho(z)^{-2}.$$

Let f be a locally injective meromorphic function in A and S_f its Schwarzian derivative. At finite points of A which are not poles of f we have $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$, and the definition is extended to ∞ and to the poles of f by means of inversion. Every function which is holomorphic in A is the Schwarzian of some meromorphic f. The Schwarzian vanishes identically if and only if f is a Möbius transformation. A function with a prescribed Schwarzian is determined up to a Möbius transformation.

If $g : A \to B$ is a conformal mapping, then

$$|S_f(z) - S_g(z)|\, \rho_A(z)^{-2} = |S_{f \circ g^{-1}}(\zeta)|\, \rho_B(\zeta)^{-2}, \qquad \zeta = g(z). \tag{1}$$

In particular,

$$\|S_g\|_A = \|S_{g^{-1}}\|_B. \tag{2}$$

We associate with the domain A the following three constants:

$\sigma_1 = \|S_f\|_A$, where f is a conformal map of A onto a disc,

$\sigma_2 = \sup \{\|S_f\|_A \mid f \text{ univalent in } A\}$,

$\sigma_3 = \sup \{a \mid \|S_f\|_A \le a \text{ implies } f \text{ univalent in } A\}$.

2. Constant σ_1

In the definition of σ_1, a disc means an ordinary disc or a half-plane. The number σ_1 is well defined and equal to 0 if and only if A itself is a disc. It is well known that $\sigma_1 \le 6$, and the example $A = \{z \mid 0 < \arg z < k\pi\}$, $1 \le k \le 2$, shows that σ_1 can take any value of the closed interval $[0, 6]$.

In view of (2), we could define σ_1 also with the aid of conformal mappings of a disc onto A. A further characterization is obtained as follows: Let f be a conformal mapping of the unit disc D onto A and h a conformal self-mapping of D, such that $h(0) = z_0$. Since $\rho_D(0) = 1$, it follows from (1) that

$$|S_f(z_0)|\,\rho_D(z_0)^{-2} = |S_{f\circ h}(0)|. \tag{3}$$

Hence

$$\sigma_1 = \sup\{|S_f(0)| \mid f : D \to A \text{ conformal}\}. \tag{4}$$

In some cases, information about the boundary of A makes it possible to improve the estimate $\sigma_1 \le 6$. Suppose that the boundary of A is a K-quasicircle, i.e. the image of a circle under a K-quasiconformal mapping of the plane. (Quasicircles were first investigated by Pfluger [5].) Then $\sigma_1 \le 6(K^2-1)/(K^2+1)$.

Another result of this type is that for convex domains $\sigma_1 \le 2$. This follows from known results on the coefficients of univalent functions, see e.g. [6]. We include here a simple proof which also gives the extremals. (Quite recently, Nehari (J. Analyse Math. 30 (1976)) also established this result by using variational techniques.)

THEOREM 1. *Let f be a conformal mapping of a disc onto a convex domain. Then*

$$|S_f(z)|\,\rho(z)^{-2} \le 2. \tag{5}$$

Equality holds if and only if the image domain is bounded by two parallel lines.

Proof. We may assume that f is a conformal map of the unit disc. In view of (3), inequality (5) follows if we prove that $|S_f(0)| \le 2$. Since we may replace f by the function $z \to cf(ze^{i\varphi})$, c complex, φ real, there is no loss of generality in assuming that $S_f(0) \ge 0$ and that $f'(0) = 1$.

It is well known that f' admits a representation

$$f'(z) = \exp\left(-\int_0^{2\pi} (\log(1 - ze^{-i\theta})\, d\psi(\theta)\right), \tag{6}$$

where ψ is increasing and

$$\int_0^{2\pi} d\psi(\theta) = 2.$$

Direct computation yields

$$S_f(0) = \int_0^{2\pi} e^{-2i\theta}\, d\psi(\theta) - \tfrac{1}{2}\left(\int_0^{2\pi} e^{-i\theta}\, d\psi(\theta)\right)^2.$$

Since $S_f(0)$ is real and $d\psi(\theta) \geq 0$, we obtain

$$S_f(0) = \int_0^{2\pi} \cos 2\theta\, d\psi(\theta) - \tfrac{1}{2}\left(\int_0^{2\pi} \cos\theta\, d\psi(\theta)\right)^2 + \tfrac{1}{2}\left(\int_0^{2\pi} \sin\theta\, d\psi(\theta)\right)^2$$

$$\leq \int_0^{2\pi} \cos 2\theta\, d\psi(\theta) - \tfrac{1}{2}\left(\int_0^{2\pi} \cos\theta\, d\psi(\theta)\right)^2 + \int_0^{2\pi} \sin^2\theta\, d\psi(\theta)$$

$$= \int_0^{2\pi} \cos^2\theta\, d\psi(\theta) - \tfrac{1}{2}\left(\int_0^{2\pi} \cos\theta\, d\psi(\theta)\right)^2 \leq \int_0^{2\pi} \cos^2\theta\, d\psi(\theta) \leq 2.$$

Because $S_f(0) \geq 0$, we have proved (5).

Equality holds only if

$$\int_0^{2\pi} \cos^2\theta\, d\psi(\theta) = 2, \quad \int_0^{2\pi} \cos\theta\, d\psi(\theta) = 0.$$

These conditions are fulfilled if and only if ψ has a jump $+1$ at the points 0 and π and is constant on the intervals $(0, \pi)$ and $(\pi, 2\pi)$. Then $S_f(0) = 2$, and it follows from (6) that $f'(z) = (1 - z^2)^{-1}$. We conclude that the image of D is a parallel strip.

3. Constant σ_2

The number σ_2 is 6 if A is a disc and $\sigma_2 \leq 12$ for all domains A. In fact, there is a simple relation between σ_1 and σ_2:

THEOREM 2. *In every domain A,*

$$\sigma_2 = \sigma_1 + 6. \tag{7}$$

Proof. Let f be univalent in A and $h : D \to A$ conformal. By (1).

$$\|S_f\|_A = \|S_{f\circ h} - S_h\|_D \leq 6 + \|S_h\|_D = 6 + \sigma_1. \tag{8}$$

In order to show that the estimate $\|S_f\|_A \leq 6 + \sigma_1$ cannot be improved, let an $\varepsilon > 0$ be given. Considering (4), we can choose h such that $|S_h(0)| > \sigma_1 - \varepsilon$. The mapping w, defined by $w(z) = z + e^{i\theta}/z$, is univalent in D, and

$$S_w(z) = -6e^{i\theta}(e^{i\theta} - z)^{-2}.$$

Set $f = w \circ h^{-1}$. Then f is univalent in A, and

$$\|S_f\|_A = \|S_w - S_h\|_D \geq |S_w(0) - S_h(0)| = |-6e^{i\theta} - S_h(0)|.$$

By choosing θ suitably we obtain

$$\|S_f\|_A \geq 6 + |S_h(0)| > 6 + \sigma_1 - \varepsilon.$$

Combined with (8), this yields (7).

4. Constant σ_3

In the definition of σ_3, sup can be replaced by max. To prove this, let us suppose that f is meromorphic in A with $\|S_f\|_A = \sigma_3$. Let f_n, $n = 1, 2, \ldots,$ be determined by the condition

$$S_{f_n} = r_n S_f,$$

where $r_n < 1$ and $r_n \to 1$ as $n \to \infty$. All functions f_n are univalent, and we can normalize them so that they agree with f at three fixed points of A. Then the functions f_n form a normal family, and there is a sub-sequence which converges locally uniformly in A towards a conformal mapping of A. This limit function has the same Schwarzian derivative as f, and it follows that f is univalent.

If A is a disc, then $\sigma_3 = 2$. This has been known for almost thirty years: the estimate $\sigma_3 \geq 2$ follows from a theorem of Nehari [4], and examples given by Hille [3] show that $\sigma_3 \leq 2$.

There is an intimate connection between the constant σ_3 and quasiconformal mappings: $\sigma_3 > 0$ if and only if the boundary of A is a quasicircle. The sufficiency of the condition was proved by Ahlfors [1], the necessity by Gehring [2].

5. Universal Teichmüller space

Suppose the domain A is bounded by a quasicircle. Let $Q(A)$ be the Banach space consisting of all holomorphic functions of A with finite norm. We introduce the subsets

$$\Delta(A) = \{\varphi = S_f \mid f \text{ univalent in } A\},$$

$\Delta_0(A)=\{\varphi=S_f\in\Delta(A)\mid f$ can be extended to a quasiconformal mapping of the plane$\}$.

Both sets are well defined. The set $\Delta_0(A)$ is called the universal Teichmüller space of A.

The sets $\Delta(A)$ and $\Delta_0(A)$ are connected as follows:

$$\Delta_0(A)=\text{interior of }\Delta(A). \tag{9}$$

This was proved by Gehring [2] in the case where A is a disc. The same reasoning yields the result for an arbitrary domain A also.

THEOREM 3. *If f is univalent in A and $\|S_f\|_A<\sigma_3$, then f can be extended to a quasiconformal mapping of the plane.*

Proof. By the remark in Section 4, the closed ball $\{\varphi\in Q(A)\mid\|\varphi\|_A\le\sigma_3\}$ is contained in $\Delta(A)$. Hence, if $\|S_f\|<\sigma_3$, then S_f is an inner point of $\Delta(A)$, and the theorem follows from (9).

In the special case where A is the upper half-plane H, we write briefly Q, Δ and Δ_0, without indicating the domain H.

It is not known whether every point of Δ is in the closure of Δ_0. We need the following much weaker result:

LEMMA 1. *On every sphere $\|\varphi\|=r$ of Q, $2\le r\le 6$, there are points of $\Delta-\Delta_0$ belonging to the closure of Δ_0.*

Proof. Let f be a conformal mapping of H onto a rectilinear quadrilateral B with symmetry $f(-\bar z)=\overline{f}(z)$, with vertices at the points $f(0)=0$, $f(\pm1)$, $f(\infty)<0$, and with the angles $\alpha\pi$ at 0, $1\le\alpha<2$, and $(1-\alpha/2-\eta)\pi$ at $f(\pm1)$, where $\eta\ge0$ is small. If $\eta=0$, then $f(\infty)=\infty$, and two sides of B are half-lines parallel to the real axis.

Direct computation yields

$$\frac{f''(z)}{f'(z)}=\frac{\alpha-1}{z}-\left(\frac{\alpha}{2}+\eta\right)\left(\frac{1}{z-1}+\frac{1}{z-1}\right).$$

Hence,

$$S_f(z)=\frac{1-\alpha^2}{2z^2}+\frac{a}{(z-1)^2}+\frac{a}{(z+1)^2}+\frac{b}{z^2-1}, \tag{10}$$

where

$$a=\tfrac{1}{8}(\alpha+2\eta)(4-\alpha-2\eta),\qquad b=\tfrac{1}{4}(\alpha+2\eta)(4-3\alpha+2\eta). \tag{11}$$

From (10) we see that

$$4y^2 S_f(iy) = 2(\alpha^2 - 1) + o(1), \qquad z = x + iy,$$

as $y \to 0$. Because $\rho_H(z) = (2y)^{-1}$, it follows that $\|S_f\|_H \geq 2(\alpha^2 - 1)$. In particular, $\|S_f\|_H \to 6$ as $\alpha \to 2$. On the other hand, for $\alpha = 1$ the domain B is convex, and by Theorem 1, $\|S_f\|_H \leq 2$.

Suppose, for a moment, that $\eta = 0$. From (10) and (11) we deduce, since $y^2 |z \pm 1|^{-2} \leq 1$, $y^2 |z^2 - 1|^{-1} \leq 1$, that $\|S_f\|_H$ depends continuously on α. Therefore, given any r, $2 \leq r \leq 6$, there is a quadrilateral B such that $\|S_f\|_H = r$. The boundary of B, having a cusp at ∞, is not a quasicircle and so $S_f \in \Delta - \Delta_0$.

On the other hand, for $\eta > 0$ the domain B is bounded by a quasicircle, and hence $S_f \in \Delta_0$. If we write $f = f_{\alpha,\eta}$, then it is again immediate from (10) and (11) that for every α,

$$\lim_{\eta \to 0} \|S_{f_{\alpha,\eta}} - S_{f_{\alpha,0}}\|_H = 0.$$

Consequently, the Schwarzian of $f_{\alpha,0}$ is in the closure of Δ_0.

6. New characterization of σ_3

Given a domain A bounded by a quasicircle, let $f: H \to A$ be conformal. We let the point $\varphi_A = S_f$ represent A in Δ_0. Then $\|\varphi_A\| = \sigma_1$.

The point $\varphi_A \in \Delta_0$ is not uniquely determined by A, nor does a point of Δ_0 determine a unique domain. We obtain a well defined bijection by identifying two domains if they are equivalent under Möbius transformations, and two points S_f and S_g of Δ_0 if $g^{-1} o f$ is a conformal self-mapping of H. For our purposes, the choice of the representative of A in Δ_0 is immaterial.

THEOREM 4. *The constant σ_3 of A is equal to the distance of φ_A to the set $\Delta - \Delta_0$.*

Proof. Let d denote the distance between $\Delta - \Delta_0$ and the point $\varphi_A = S_h$, where h is a conformal map of H onto A. Let f be meromorphic in A. From

$$\|S_f\|_A = \|S_{f \circ h} - S_h\|_H$$

we see that if $\|S_f\|_A < d$, then $S_{f \circ h} \in \Delta_0$. But then $f = (f \circ h) \circ h^{-1}$ is univalent, and consequently $\sigma_3 \geq d$.

On the other hand, it follows from Theorem 3 that $\sigma_3 \leq d$.

7. Estimates for σ_3

Theorem 4 and Lemma 1 give sharp lower estimates for σ_3 if σ_1 is given.

THEOREM 5. *For domains with given σ_1 and bounded by quasicircles,*

$$\min \sigma_3 = 2 - \sigma_1 \quad \text{if} \quad 0 \leq \sigma_1 < 2, \tag{12}$$

$$\inf \sigma_3 = 0 \quad \text{if} \quad 2 \leq \sigma_1 < 6. \tag{13}$$

Proof. Suppose first that $\sigma_1 < 2$. Since the ball $\{\varphi \in Q \mid \|\varphi\|_H < 2\}$ lies in Δ_0, Theorem 4 yields the lower estimate $\sigma_3 \geq 2 - \sigma_1$.

In order to prove that this inequality is sharp, we consider the point S_w, where w is the restriction to H of a branch of the logarithm. Then $S_w \in \Delta - \Delta_0$ and $\|S_w\|_H = 2$. Let h be determined by the condition $S_h = rS_w$, $r < 1$, and set $A = h(H)$, $f = w \circ h^{-1}$. From $\|S_h\|_H < 2$ it follows that $S_h \in \Delta_0$, and so A is bounded by a quasicircle. Furthermore, $\sigma_1 = \|S_h\|_H = 2r$, and

$$\|S_f\|_A = \|S_f - S_w\|_H = 2(1 - r) = 2 - \sigma_1.$$

From $S_w \in \Delta - \Delta_0$ we conclude that $S_f \in \Delta(A) - \Delta_0(A)$. Consequently, by Theorem 3, $\sigma_3 \leq 2 - \sigma_1$, and (12) follows.

Since $\sigma_3 = 0$ for a domain not bounded by a quasicircle, equation (13) follows immediately from Theorem 4 and Lemma 1.

The following upper estimates complement Theorem 5.

THEOREM 6. *The constant σ_3 satisfies the inequality*

$$\sigma_3 \leq \min(2, 6 - \sigma_1).$$

Proof. Since Δ is contained in the ball of radius 6, the estimate $\sigma_3 \leq 6 - \sigma_1$ follows immediately from Theorem 4.

In order to prove that

$$\sigma_3 \leq 2 \tag{14}$$

we note that every Jordan domain is Möbius equivalent to a subdomain of H having 0 and ∞ as boundary points. Therefore, we may assume that A is such a domain.

Set $f(z) = \log z$. From $S_f(z) = z^{-2}/2$ and $\rho_A(z) \geq \rho_H(z)$ it follows that

$$|S_f(z)|\, \rho_A(z)^{-2} \leq 2\left(\frac{y}{|z|}\right)^2 \leq 2. \qquad \bullet$$

Since the boundary of $f(A)$ is not a Jordan curve, $S_f \in \Delta(A) - \Delta_0(A)$. Thus (14) follows from Theorem 3.

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