The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces.

Comment. Math. Helvetici **52** (1977) 573–590 Birkhäuser Verlag, Basel

The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces

WOLFGANG ZILLER*

It is known that the Jacobi equation on a globally symmetric space has particularly simple solutions. Some results have been obtained for the Jacobi equation on a normal homogeneous or naturally reductive space. Compare [3]–[5], [7], [9]. One knows that there exists a connection D on a naturally reductive homogeneous space so that the curvature B and torsion T of D is D parallel and D has the same geodesics, and thus also the same Jacobi fields, as the given metric. But the Jacobi equation written down in terms of D is nicer since it is a differential equation with constant coefficients:

$$D_{\dot c}^2 Y - T(\dot c, D_{\dot c} Y) + B(Y, \dot c)\dot c = 0.$$

Rauch mentioned in [9] that there are no exponential Jacobi fields appearing if the curvature of D is positive, but this condition is seldom satisfied. One can express a basis of (complexified) solutions of the Jacobi equation as:

$$Y(t) = A(t) \cdot e^{mt}$$

where $A(t)$ is a vector valued polynomial with complex, D parallel vector fields as coefficients and m is a complex number. In this paper we show that for a compact naturally reductive riemannian homogeneous space:

(i) m is imaginary or 0;
(ii) if m is imaginary and $\neq 0$, then $A(t)$ is a constant polynomial;
(iii) if $m = 0$, then $A(t) = A_1 \cdot t + A_0$.

The statement of (i) is a generalization of Rauch's result since there are thus no exponential Jacobi fields, but here we need no condition on the curvature of D.

---

* This work was supported by the "Sonderforschungsbereich Theoretische Mathematik (SFB 40)" at the University of Bonn and completed at the Institute for Advanced Study with partial support from a National Science Foundation grant.

(ii) and (iii) say that the Jacobi fields are either oscillatory or of linear polynomial growth compared with a D parallel basis.

The proof is given by combining local information one gets from the differential equation with constant coefficients with global information one gets from the structure of the set of Killing vector fields.

We finally give a large class of examples of metrics (the ones studied in [6]) which are naturally reductive but not normal homogeneous. In some examples some of the eigenvalues of B, and of the sectional curvature of the metric, become negative. But the influence of the torsion in the Jacobi equation still guarantees that no exponential Jacobi fields exist.

This is not true anymore for a general riemannian homogeneous metric. In fact, H. Karcher gave an example of a riemannian homogeneous metric which is not naturally reductive and which has exponential Jacobi fields. This example is written down in [10].

The methods used in this paper are similar to the ones in [10], but we restate the ideas for completeness.

In 1. we give the necessary preliminaries about riemannian homogeneous spaces. In 2. we prove the main theorem and in 3. we give examples.

1. Preliminaries

Let $M = G/H$ be a homogeneous space. The residue class $g \cdot H$ is denoted by $\bar{g}$. If a metric on G/H is invariant under the operation of G on G/H it is called riemannian homogeneous. Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G and $\mathfrak{h}$ the Lie algebra of H. We can assume that G/H is reductive, i.e., there exists a complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}: \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ so that $Ad(H)$ leaves $\mathfrak{p}$ invariant. We can associate to each $X \in \mathfrak{g}$ a Killing vector field X^* on G/H defined by the one parameter group $\exp_G tX$ acting on G/H. Then $[X^*, Y^*] = -[X, Y]^*$. $\mathfrak{p}$ can be identified with $T_{\bar{e}}(G/H)$ by sending $X \in \mathfrak{p}$ to $X^*(\bar{e})$. We will always make this identification and compute Lie brackets in $\mathfrak{g}$. The metric on $T_{\bar{e}}(G/H)$ thus induces a metric on $\mathfrak{p}$ denoted by $\langle\, ,\rangle$. $Ad(H)$ acting on $\mathfrak{p}$ is identified with taking the derivative at $\bar{e}$ of the corresponding left translation on G/H. Left translation by g on G/H is denoted by L_g. We will assume that G acts (almost) effectively on G/H which is equivalent to saying that $Ad(H)$ operates (almost) faithfully on $\mathfrak{p}$. M being reductive implies $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. If $X, Y \in \mathfrak{p}$ then we denote by $[X, Y]_{\mathfrak{h}}$, $[X, Y]_{\mathfrak{p}}$ the $\mathfrak{h}$ and $\mathfrak{p}$ component of $[X, Y]$. M is called naturally reductive (with respect to the complement $\mathfrak{p}$) if:

$$[X, \cdot]_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$$

is skew symmetric for all $X \in \mathfrak{p}$ and is called normal homogeneous if there exists a biinvariant metric on $\mathfrak{g}$ whose restriction to $\mathfrak{p}=\mathfrak{h}^{\perp}$ is the given metric. In particular all $[X,\cdot]:\mathfrak{g}\to\mathfrak{g}$ are skew symmetric, and thus a normal homogeneous metric is also naturally reductive. We will denote the Levi-Cevita connection and the curvature tensor of $\langle,\rangle$ by ∇ and R.

If M is naturally reductive, the connection ∇ can be described as follows: Let X^* be a Killing vector field and $v\in T_{\bar e}M$. Then

$$(\nabla_v X^*)(\bar e)=\begin{cases}[X,v] & \text{if } X\in\mathfrak{h}\\ \frac{1}{2}[X,v]_{\mathfrak{p}} & \text{if } X\in\mathfrak{p}\end{cases}.$$

The curvature tensor at $\bar e$ is given by:

$$R(X,Y)Y=[Y,[Y,X]_{\mathfrak{h}}]-\tfrac{1}{4}[Y,[Y,X]_{\mathfrak{p}}]_{\mathfrak{p}},\qquad Y,X\in\mathfrak{p}.$$

If M is normal homogeneous it follows that the sectional curvature is:

$$K(X,Y)=\|[X,Y]_{\mathfrak{h}}\|^2+\tfrac{1}{4}\|[X,Y]_{\mathfrak{p}}\|^2$$

so that $K\geq 0$. But for a naturally reductive space one can have negative sectional curvature too.

One knows that on a naturally reductive space there exists a metric connection D with torsion T and curvature B so that T and B are D parallel.

D has the same geodesics as ∇ so that

$$\nabla_X Y=D_X Y-\tfrac{1}{2}T(X,Y)$$

and D, T and B at $\bar e$ can be expressed in terms of the Lie brackets:

$$(D_v X^*)(\bar e)=\begin{cases}[X,v] & \text{if } X\in\mathfrak{h}\\ [X,v]_{\mathfrak{p}} & \text{if } X\in\mathfrak{p}\end{cases}\qquad v\in\mathfrak{p}$$

$$\left.\begin{aligned}T(X,Y)&=-[X,Y]_{\mathfrak{p}}\\ B(X,Y)Z&=-[[X,Y]_{\mathfrak{h}},Z]\end{aligned}\right.\qquad X,Y,Z\in\mathfrak{p}.$$

Notice that $R(X,Y)Y=B(X,Y)Y-\frac{1}{4}T(T(X,Y),Y)$. Since T is skew symmetric it follows from the symmetry of R that $B(\cdot,Y)Y:\mathfrak{p}\to\mathfrak{p}$ is symmetric.

If M is normal homogeneous, one has in addition that $B(\cdot,Y)Y$ is positive semidefinite. The geodesics in a naturally reductive space are images of one parameter groups in G: For $v\in\mathfrak{p}$ $L_{\exp_G t\cdot v}(\bar e)$ is the geodesic through $\bar e$ with initial condition $v^*(\bar e)$.

The derivative of $L_{\exp_G v}$ at $\bar{e}$ is parallel translation along $L_{\exp_G tv}(\bar{e})$ with respect to the connection D.

Since $DB = 0$, the curvature tensor B of D is invariant under $d(L_{\exp_G v})_{\bar{e}}$ and $B(\cdot, v)v$ commutes with $d(L_{\exp_G v})_{\bar{e}}$.

Since ∇ and D have the same geodesics, they also have the same Jacobi fields. But the Jacobi equation with respect to D along $c(t) = L_{\exp_G tv}(\bar{e})$, $\dot{c}(0) = v$:

$$D_{\dot{c}}^2 X - T(\dot{c}, D_{\dot{c}} X) + B(X, \dot{c})\dot{c} = 0$$

is much simpler since T and B are D parallel.

If we write X as $X(t) = d(L_{\exp_G tv})_{\bar{e}}(Y(t))$, then the Jacobi equation reads:

$$Y'' - T(Y') + B(Y) = 0$$

where $T(Y) = T(v, Y) = -[v, Y]_{\mathfrak{p}}$, $B(Y) = B(Y, v)v = -[v, [v, Y]_{\mathfrak{h}}]$.
This is a differential equation in the vector space $\mathfrak{p}$ with constant coefficients, T is skew symmetric and B is symmetric. The solutions of this equation are obtained by substituting $Y(t) = A(t) \cdot e^{mt}$, where m is a complex number and $A(t)$ a complex vector valued polynomial. The real and complex parts of these solutions then give a basis of the Jacobi fields along c.

Since the differential equation is linear with Y also Y' is a solution. Therefore, with $Y(t) = (A_n t^n + \cdots + A_0)e^{mt}$, also $A_n \cdot e^{mt}$ is a solution.

But substituting we get that $A_n e^{mt}$ is a solution iff:

$$(m^2 Id - mT + B)A_n = 0.$$

Therefore only the solutions of

$$\det(m^2 Id - mT + B) = 0$$

are possible exponents for Jacobi fields.

For later purposes we remark that $(A_1 t + A_0)e^{mt}$ is a solution iff

$$(m^2 Id - mT + B)A_1 = 0$$

$$(m^2 Id - mT + B)A_0 = -(2m - T)A_1.$$

We will be particularly interested in the case $m = 0$ later on:

$m = 0$ is only possible if $\det B = 0$ and $X(t) = d(L_{\exp_G tv})_{\bar{e}}(A_0)$ is a Jacobi field ($X(t)$ is D parallel) iff $B(A_0) = 0$.

Furthermore, $X(t)=d(L_{\exp_G tv})_{\bar{e}}(A_1t+A_0)$ is a Jacobi field iff

$$B(A_1)=0$$

and

$$B(A_0)=T(A_1).$$

Notice that by complexifying, B becomes hermitian and T skew hermitian, so that $m^2Id-mT+B$ is hermitian if m is imaginary.

2. The Jacobi equation

A vector field X is called a Killing vector field if the operator $A_X=\nabla X$ is skew symmetric. This is equivalent to saying that the one parameter group φ_s generated by X consists of isometries.

The vector fields X^* in 1. are Killing vector fields. A Killing vector field X restricted to a geodesic c is a Jacobi field since $X\circ c(t)=d/ds_{/s=0}\varphi_s\circ c(t)$ and for each s, $\varphi_s\circ c(t)$ is a geodesic. Jacobi fields which are restrictions of Killing vector fields are called isotropic Jacobi fields.

Since in our case we have a transitive group of isometries we expect a lot of isotropic Jacobi fields.

In fact, it is known that on a globally symmetric space all Jacobi fields which vanish at two points are isotropic Jacobi fields [2] and all periodic Jacobi fields along closed geodesics are isotropic [11].

But not all Jacobi fields need to be isotropic; in fact, the Jacobi fields $t\cdot X$ with X parallel and $R(X,\dot{c})\dot{c}=0$ are not isotropic on a compact globally symmetric space.

It is also known that on a normal homogeneous space the Jacobi fields which vanish at two points need not be isotropic [4] and also the periodic Jacobi fields along closed geodesics need not be isotropic [10]. From 1. it follows that an isotropic Jacobi field Y along $c(t)=\exp tv$, $v\in\mathfrak{p}$ coming from the Killing vector field X^*, $X\in\mathfrak{g}$, i.e., $Y(t)=X^*\circ c(t)$, satisfies:

$$Y(0)=\begin{cases}0 & \text{if}\quad X\in\mathfrak{h}\\ X & \text{if}\quad X\in\mathfrak{p}\end{cases}$$

$$\nabla Y(0)=\begin{cases}[X,v] & \text{if}\quad X\in\mathfrak{h}\\ \frac{1}{2}[X,v]_{\mathfrak{p}} & \text{if}\quad X\in\mathfrak{p}\end{cases}.$$

Let $E=\{w\in T_{\bar{e}}M\mid\langle v,w\rangle=0\}$. We are only interested in the Jacobi fields orthogonal to $\dot{c}$, i.e., in Jacobi fields Y having initial condition $(Y(0),\nabla Y(0))$ in

$E \oplus E$. If $X \in E \subset \mathfrak{p}$ then also $[X, v]_{\mathfrak{p}} \in E$:

$$\langle [X, v]_{\mathfrak{p}}, v\rangle = -\langle X, [v, v]_{\mathfrak{p}}\rangle = 0.$$

Thus the Jacobi fields coming from $X \in \mathfrak{p}$ can be restricted to $X \in E$ and all Jacobi fields with initial condition:

$$(Y(0), \nabla Y(0)) = (X, \tfrac{1}{2}[X, v]_{\mathfrak{p}}), \qquad X \in E,$$

are isotropic. These are already half of all Jacobi fields.

To study the Jacobi fields coming from $X \in \mathfrak{h}$ we examine the symmetric endomorphism $B(X) = -[v, [v, X]\mathfrak{h}]$. Since $B(v) = 0$, B maps E into itself and we let X_i, λ_i be the eigenvectors resp. eigenvalues of $B/E : B(X_i) = \lambda_i X_i$ and we set $Z_i = [v, X_i]_{\mathfrak{h}} \in \mathfrak{h}$. Then $[Z_i, v] = [[v, X_i]_{\mathfrak{h}}, v] = -B(X_i) = -\lambda_i X_i$. Therefore, if $\lambda_i \neq 0$, the Jacobi field Y_i corresponding to $Z_i \in \mathfrak{h}$ does not vanish identically since $\nabla Y_i(0) = [Z_i, v] \neq 0$.

Let $E = E_0 \oplus E_1$ with E_0 the 0-eigenspace of B and E_1 the sum of the eigenspaces with $\lambda_i \neq 0$. Then the Jacobi fields with initial condition

$$(Y(0), \nabla Y(0)) = (0, X), \qquad X \in E_1,$$

are isotropic Jacobi fields.

If $X \in E_0$, i.e., $B(X) = 0$, we showed in 1. that the D parallel vector field $Y(t) = d(L_{\exp_G tv})_{\bar{e}}(X)$ is a Jacobi field (which is not necessarily isotropic). The initial conditions are:

$$Y(0) = X \in E_0$$

$$\nabla_v Y(0) = D_v Y(0) - \tfrac{1}{2}T(v, Y(0)) = \tfrac{1}{2}[v, X]_{\mathfrak{p}}.$$

These Jacobi fields together with the two sets of isotropic Jacobi fields previously mentioned would generate all Jacobi fields if they were linearly independent, but $(X, \frac{1}{2}[v, X]_{\mathfrak{p}})$, $X \in E_0$, could be a linear combination of $(X, \frac{1}{2}[X, v]_{\mathfrak{p}})$, $X \in E_0$, and $(0, Z)$, $Z \in E_1$, if $[X, v]_{\mathfrak{p}} \in E_1$. But in 1. we also pointed out that

$$Y(t) = d(L_{\exp_G tv})_{\bar{e}}(tX + Z)$$

is a Jacobi field iff

$$B(X) = 0 \quad \text{and} \quad B(Z) = T(X) = [X, v]_{\mathfrak{p}}.$$

Since $B/E_0 = 0$ and B/E_1 is an isomorphism, we have in the above situation

($[X, v]_p \in E_1$) a vector Z with $B(Z) = T(X) = [X, v]_p$ and thus we have a new Jacobi field

$$Y(t) = d(L_{\exp_G tv})_{\bar{e}}(tX + Z)$$

with initial conditions

$$Y(0) = Z$$

$$\nabla_v Y(0) = D_v Y(0) - \tfrac{1}{2}T(v, Y(0)) = X + \tfrac{1}{2}[v, Z]_p.$$

We will now show that these Jacobi fields together with the previous ones generate all Jacobi fields. Here the compactness of $M = G/H$, which has not been used up to now, comes in in an essential way. Set $E_0 = E_2 \oplus E_3$ with

$$E_2 = \{X \in E_0 \mid [X, v]_p \in E_1\}$$

and $E_3 = E_2^{\perp}$. Thus $E = E_1 \oplus E_2 \oplus E_3$. Define the subspaces $V_i \subset E \oplus E$ by:

$$V_1 = \{(X, \tfrac{1}{2}[X, v]_p) \mid X \in E_1 \oplus E_3\}$$

$$V_2 = \{(0, X) \mid X \in E_1\}$$

$$V_3 = \{(X, \tfrac{1}{2}[v, X]_p) \mid X \in E_2\}$$

$$V_4 = \{(X, \tfrac{1}{2}[v, X]_p) \mid X \in E_3\}$$

$$V_5 = \{(Z, X + \tfrac{1}{2}[v, Z]_p) \mid X \in E_2, \quad B(Z) = T(X) = [X, v]_p\}.$$

We will now show that $E \oplus E = \oplus_{i=1}^{5} V_i$. The elements of $V_1 + V_2 + V_3 + V_4$ are linearly independent as one sees by looking at the first and second components. But also the elements of V_5 cannot be a linear combination of the others for the following reason: The Jacobi fields with initial condition in V_1 and V_2 are isotropic and so are the Jacobi fields with initial conditions in V_3 since for $X \in E_2$:

$$(X, \tfrac{1}{2}[v, X]_p) = (X, \tfrac{1}{2}[X, v]_p) + (0, [v, X]_p)$$

and the two Jacobi fields with initial condition given by the right-hand side are both isotropic. But since isotropic Jacobi fields are restrictions of Killing vector fields and since M is compact they are bounded in length.

The Jacobi fields with initial condition in V_4 are also bounded in length; in fact, they have constant length since $L_{\exp_G tv}$ are isometries.

But the Jacobi fields with initial condition in V_5 are of the form

$$Y(t) = d(L_{\exp_G tv})_{\bar{e}}(tX+Z)$$

and are thus unbounded in length since M is complete. They can therefore not be linear combinations of the others. Thus we have proved:

THEOREM 1. $E \oplus E = \oplus_{i=1}^{5} V_i$. *On a compact naturally reductive homogeneous space, the Jacobi fields along c can be written as linear combinations of Jacobi fields with initial conditions in* V_i.

We can draw several conclusions from this.

THEOREM 2. *If one solves the Jacobi equation on a compact naturally reductive riemannian homogeneous space in the form* $Y(t) = A(t) \cdot e^{mt}$ *with* $A(t)$ *a polynomial with D parallel complex vector fields as coefficients and m a complex number, one has*:

(i) *m is imaginary or* 0;
(ii) *if m is imaginary and* $\neq 0$, *then* $A(t)$ *is a constant polynomial so that the corresponding Jacobi fields are of the form:*

$$Y(t) = \operatorname{Re} A \cos at - \operatorname{Im} A \sin at$$

$$Y(t) = \operatorname{Re} A \sin at + \operatorname{Im} A \cos at$$

with $m = i \cdot a$ *and* $A(t) = A$ *a D parallel vector field with* $(m^2 Id - mT + B)A = 0$;
(iii) *if* $m = 0$, *then* $A(t) = A_1 t + A_0$ *with* A_1 *and* A_0 *D parallel* (*real*) *vector fields are the only possible Jacobi fields where* $B(A_1) = 0$ *and* $B(A_0) = T(A_1)$.

Proof. (i) If there exists a solution $Y(t) = A(t)e^{mt}$ where m has a nonzero real part, then either $\|Y(t)\| \to \infty$ as $t \to \infty$ and $\|Y(t)\| \to 0$ as $t \to -\infty$ or $\|Y(t)\| \to 0$ as $t \to \infty$ and $\|Y(t)\| \to \infty$ as $t \to -\infty$. But from our description of the Jacobi fields we see that all Jacobi fields have either bounded length or their length goes to ∞ as $t \to \infty$ and as $t \to -\infty$. Thus m cannot have a nonzero real part.
(ii) If $X(t) = A(t)e^{mt}$, $m \neq 0$ and imaginary and degree $A(t) \geq 1$ is a solution with $A(t) = A_n t^n + \cdots + A_0$ $(A_n \neq 0)$, then $(nA_n t + A_{n-1})e^{mt}$ is a solution too.

We will now show that there are no solutions of the form $(A_1 t + A_0)e^{mt}$ with $m \neq 0$ and $A_1 \neq 0$. From 1. we know that $(A_1 t + A_0)e^{mt}$ is a solution iff

$$(m^2 Id - mT + B)A_1 = 0$$

$$(m^2 Id - mT + B)A_0 = -(2m - T)A_1.$$

From this we can conclude that $B(A_1) \neq 0$ since if $B(A_1)=0$ we get $T(A_1)=mA_1$ from the first equation and

$$(m^2 Id - mT + B)A_0 = -mA_1$$

from the second equation. Then

$$-m\langle A_1, A_1\rangle = \langle (m^2 Id - mT + B)A_0, A_1\rangle = \langle A_0, (m^2 Id - mT + B)A_1\rangle = 0$$

so that $A_1 = 0$ which we assumed not to be the case. Since $B(A_1) \neq 0$ it follows that

$$B((A_1 t + A_0)e^{mt}) = (B(A_1)t + B(A_0))e^{mt}$$

is a vector field of unbounded length.

But from our description of the Jacobi fields we know that for a Jacobi field Y the vector field $B(Y)$ is of bounded length: This is clear for Jacobi fields Y which are of bounded length themselves since $B = B(\cdot, \dot{c})\dot{c}$ is bounded. (The curvature tensor B is bounded since M is compact and $\dot{c}$ has constant length.) The only Jacobi fields of unbounded length are of the form

$$Y(t) = d(L_{\exp_G tv})_{\bar{e}}(tX + Z)$$

and since in this case $B(X)=0$ and since $d(L_{\exp_G tv})_{\bar{e}}$ commutes with B we have that

$$B(Y) = d(L_{\exp_G tv})_{\bar{e}}(B(Z))$$

is of constant length.

Thus $(A_1 t + A_0)e^{mt}$ cannot be a Jacobi field.

(iii) If $m = 0$ then $A(t)$ cannot have degree ≥ 2 since no Jacobi field has this kind of growth.

Remarks

(1) In [10] we proved Theorem 1 for Jacobi fields along a closed geodesic without using the compactness of M.

(2) If $X(t)$ is a D parallel vector field, then

$$\bar{X}(t) = e^{(t/2)T(\dot{c},\,\cdot)} \cdot X(t)$$

is the ∇ parallel vector field with $\bar{X}(0) = X(0)$ [5]. Thus the description of Jacobi

fields in Theorem 2 can be easily interpreted in terms of a ∇ parallel basis too.
(3) In [9] Rauch mentioned that m is 0 or imaginary if B is positive definite. But notice that this condition is satisfied globally only if M is a symmetric space of rank 1 since $B>0$ implies that the sectional curvature is positive, in which case M is symmetric of rank 1 or one of the two Berger examples [1]. But for the two Berger examples B has 0 eigenvalues [4] and [5]. Of course $B>0$ is possible along a particular geodesic. Notice also that positive sectional curvature does not imply $B>0$ as seems to be assumed in [9] and that C in Theorem 4 in [9] has to be a complex vector valued polynomial and not just a vector.
(4) As J. Rawnsley pointed out to me, the proof of Theorem 2 is easier if M is normal homogeneous. In this case one does not have to apply Theorem 1, which uses global properties of Jacobi fields, but can derive the claims from the local properties of the differential equation. We give a sketch of the proof here.

For M normal homogeneous one has the additional information that B is positive semidefinite:

$$\langle B(X), X\rangle = -\langle [v, [v, X]_{\mathfrak{h}}], X\rangle = \langle [v, X]_{\mathfrak{h}}, [v, X]_{\mathfrak{h}}\rangle \geq 0.$$

To prove (i), if $(A_n t^n + \cdots + A_0)e^{mt}$ is a solution, then also $A_n e^{mt}$ is a solution and thus

$$(m^2 Id - mT + B)A_n = 0$$

and $m^2\langle A_n, A_n\rangle - m\langle TA_n, A_n\rangle + \langle BA_n, A_n\rangle = 0$. But $\langle A_n, A_n\rangle$ and $\langle BA_n, A_n\rangle$ are real and ≥ 0 and $\langle TA_n, A_n\rangle$ is purely imaginary or 0. Thus m is imaginary or 0.

To prove (ii) we show that if $(A_1 t + A_0)e^{mt}$ is a solution and $m \neq 0$, then $A_1 = 0$. We have from 1.:

$$(m^2 Id - mT + B)A_1 = 0$$

$$(m^2 Id - mT + B)A_0 = -(2m - T)A_1.$$

Since $B - m^2$ is a positive definite symmetric operator and since

$$\begin{aligned}\langle A_1, (B - m^2)A_1\rangle &= \langle A_1, m(-2m + T)A_1\rangle = \langle A_1, m(m^2 Id - mT + B)A_0\rangle \\ &= \langle (m^2 Id - mT + B)A_1, mA_0\rangle = 0\end{aligned}$$

we have $A_1 = 0$.

To prove (iii) we show that $A_2t^2+A_1t+A_0$ cannot be a solution. If it were, it would satisfy

$$BA_2=0$$
$$2TA_2=BA_1$$
$$2A_2-TA_1+BA_0=0.$$

Multiplying the third equation with A_2 we get:

$$0=2\langle A_2, A_2\rangle-\langle TA_1, A_2\rangle+\langle BA_0, A_2\rangle=2\langle A_2, A_2\rangle-\langle TA_1, A_2\rangle.$$

Thus $\langle TA_1, A_2\rangle$ is real and $\langle A_1, TA_2\rangle=-2\langle A_2, A_2\rangle$. Multiplying the second equation with A_1 we get

$$0=2\langle TA_2, A_1\rangle-\langle BA_1, A_1\rangle=-4\langle A_2, A_2\rangle-\langle BA_1, A_1\rangle$$

and thus $\langle A_2, A_2\rangle=0$.
(5) On a naturally reductive space $B\geq 0$ is not necessarily satisfied. In fact, in 3. we give an example where $B\leq 0$ and also some of the sectional curvatures become negative. But the influence of the torsion T in the Jacobi equation guarantees that no exponential Jacobi fields exist.

There are lots of naturally reductive spaces which are not normal homogeneous as will be shown in 3.

COROLLARY. *If $G=I_0(M)$ and $M=G/H$ is a compact normal homogeneous space, then a Jacobi field is isotropic iff it has initial conditions in $V_1\oplus V_2\oplus V_3$. Thus if $V_4\oplus V_5\neq 0$, there exist nonisotropic Jacobi fields.*

Proof. As we mentioned before, the Jacobi fields with initial conditions in $V_1\oplus V_2\oplus V_3$ are isotropic. If $G=I_0(M)$ the isotropic Jacobi fields have initial conditions

$$(X, \tfrac{1}{2}[X, v]_{\mathfrak{p}}) \qquad X\in\mathfrak{p}$$
$$(0, Z) \qquad Z\in[\mathfrak{h}, v].$$

The first ones are contained in $V_1\oplus V_2\oplus V_3$, and we claim that the second ones are contained in V_2 which is equivalent to saying:

$$[\mathfrak{h}, v]=E_1.$$

Since M is normal homogeneous:

$$B(X)=0 \Leftrightarrow [v, X]_{\mathfrak{h}}=0 \Leftrightarrow 0=\langle [v, X]_{\mathfrak{h}}, \mathfrak{h}\rangle = -\langle X, [v, \mathfrak{h}]\rangle \Leftrightarrow X \in [v, \mathfrak{h}]^{\perp}$$

and thus $E_0=[v, \mathfrak{h}]^{\perp}$ or $E_1=[v, \mathfrak{h}]$.

Remark. The relationship between the description of the Jacobi fields in Theorem 2 and in Theorem 1 seems complicated.

Clearly the Jacobi fields in (iii) are contained in $V_3 \oplus V_4 \oplus V_5$, but it is not clear which Jacobi fields in (ii) are contained in $V_1 \oplus V_2$, and are thus isotropic, and which ones are linear combinations of isotropic Jacobi fields and Jacobi fields with initial condition in $V_3 \oplus V_4 \oplus V_5$.

Such a relationship would also give a description of the Jacobi fields vanishing at two points where one could see which ones are isotropic and which ones are not. One could then examine conjectures like: All naturally reductive spaces with the property that all Jacobi fields vanishing at two points are isotropic are locally symmetric.

3. Examples

We will now study a general class of homogeneous metrics which is naturally reductive but not normal homogeneous.

If g is a G invariant metric on $M=G/H$ we will say that g is G-naturally reductive if there exists some $Ad(H)$ invariant splitting $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ with respect to which g is naturally reductive. If g is G-naturally reductive with respect to one splitting, it is in general not G-naturally reductive with respect to another splitting, but the spitting does not have to be unique either.

Similarly we say that g is G-normal homogeneous if there exists a biinvariant metric on $\mathfrak{g}$ whose restriction to $\mathfrak{p}=\mathfrak{h}^{\perp}$ is g.

Let $G_1 \subset G_2 \subset I_0(M)$ be two subgroups which act transitively on M. Notice that G_1-naturally reductive (resp. normal homogeneous) does not necessarily imply G_2-naturally reductive (resp. normal homogeneous) nor vice versa. For normal homogeneous metrics we will demonstrate this in a simple example. Therefore we will always mention the group G with respect to which the metric is or is not naturally reductive resp. normal homogeneous.

Let $M=G/H$ be a compact homogeneous space with a normal homogeneous metric g and an $Ad(H)$ invariant splitting $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$. Assume that $\mathfrak{p}=\mathfrak{p}_1\oplus\mathfrak{p}_2$ (orthogonal splitting) and $[\mathfrak{h}, \mathfrak{p}_2]=0$, $[\mathfrak{p}_2, \mathfrak{p}_2]\subset\mathfrak{p}_2$.

Then we define a variation g_s of the normal homogeneous metric g on G/H by:

$$g_s = g/\mathfrak{p}_1 \times \mathfrak{p}_1 + s^2 \cdot g/\mathfrak{p}_2 \times \mathfrak{p}_2, \qquad s > 0.$$

Let K be the connected subgroup of G with Lie algebra $\mathfrak{p}_2$. Then if H is connected, the right translation on G/H with elements of K are well defined since $[\mathfrak{h}, \mathfrak{p}_2] = 0$ and are thus isometries of M, which differ from the left translations if the center of G is empty.

We will assume from now on that the center of $\mathfrak{g}$ is empty (which is equivalent to saying that G is semisimple, since G is compact) and that H is connected. We therefore have $G \times K$ as a subgroup of the isometry group (at least locally). We will show that the metrics g_s on G/H are all $G \times K$-naturally reductive.

Note that we could also define our spaces as follows: In the above situation $H \times K$ is locally a subgroup of G and conversely if $H \times K$ is locally a subgroup of G, then G/H satisfies the above properties with $\mathfrak{p}_2 = \mathfrak{k}$. Thus our class of metrics coincides with the ones studied in [6]. But notice that in [6] the author studied the question whether g_s is naturally reductive or not only with respect to a fixed splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and thus obtains that only $g_1 = g$ is naturally reductive.

Let $\bar{G} = G \times K$ where (g, k) operates by left translation with g and right translation with k^{-1} on G/H. The isotropy group is then $\bar{H} = H \times K$ with imbedding $(h, k) \to (hk, k)$.
Thus

$$\bar{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{k}$$

and

$$\bar{\mathfrak{h}} = \mathfrak{h} \oplus \{(0, X, X) \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{k} \mid X \in \mathfrak{p}_2 \cong \mathfrak{k}\}.$$

As an $Ad(\bar{H})$ invariant complement $\bar{\mathfrak{p}}$ we can choose $\bar{\mathfrak{p}} = \mathfrak{p}_1 \oplus \bar{\mathfrak{p}}_2$ where $\bar{\mathfrak{p}}_2 = \{(0, aX, bX) \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{k} \mid X \in \mathfrak{p}_2 \cong \mathfrak{k}\}$, and we normalize $a - b = 1$.

The isomorphism between $\bar{G}/\bar{H}$ and G/H on Lie algebra level sends $\mathfrak{p}_1$ to $\mathfrak{p}_1$ as *id* and $(0, aX, bX)$ to $aX - bX = X \in \mathfrak{p}_2$, so that the above metric g_s looks as follows on $\bar{\mathfrak{p}}$:

$$g_s/\mathfrak{p}_1 \times \mathfrak{p}_1 \text{ as before}, \qquad g_s(\mathfrak{p}_1, \bar{\mathfrak{p}}_2) = 0,$$

and on $\bar{\mathfrak{p}}_2$, $g_s((0, aX, bX), (0, aY, bY)) = s^2 \cdot g(X, Y)$. For g_s to be naturally reductive we need. e.g., for $X, Y \in \mathfrak{p}_1, Z \in \mathfrak{p}_2$:

$$g_s([(X,0,0),(Y,0,0)],(0,aZ,bZ)) = -g_s([(X,0,0),(0,aZ,bZ)],(Y,0,0)).$$

The left-hand side is equal to

$$\begin{aligned} &g_s(([X,Y]_{\mathfrak{p}_1},[X,Y]_{\mathfrak{p}_2},0),(0,aZ,bZ)) \\ &\quad = g_s((0,-b[X,Y]_{\mathfrak{p}_2},-b[X,Y]_{\mathfrak{p}_2}) \\ &\qquad\qquad + ([X,Y]_{\mathfrak{p}_1},a[X,Y]_{\mathfrak{p}_2},b[X,Y]_{\mathfrak{p}_2}),(0,aZ,bZ)) \\ &\quad = s^2 \cdot g([X,Y]_{\mathfrak{p}_2},Z) \end{aligned}$$

and the right-hand side is equal to

$$-g_s((a[X,Z]_{\mathfrak{p}_1},a[X,Z]_{\mathfrak{p}_2},0),(Y,0,0)) = -ag([X,Z]_{\mathfrak{p}_1},Y).$$

Using the fact that g is naturally reductive we get the condition $a = s^2$ and one can easily check that $a = s^2$ is also sufficient for g_s to be naturally reductive.

Thus each g_s has exactly one complement $\bar{\mathfrak{p}}_s$ with respect to which it is naturally reductive. We will now examine which metrics are $G \times K$ normal homogeneous. For that purpose let us assume that G is simple. (Thus only $g_1 = g$ is G-normal homogeneous.) Let us first restrict ourselves to K being simple too. Then the biinvariant metrics on $G \times K$ are of the form

$$\langle , \rangle / \mathfrak{g} \times \mathfrak{g} + d^2 \langle , \rangle / \mathfrak{k} \times \mathfrak{k}$$

where $\langle , \rangle$ is a biinvariant metric on $\mathfrak{g}$. $\bar{\mathfrak{p}} = \bar{\mathfrak{h}}^\perp$ is then equal to

$$\bar{\mathfrak{p}} = \mathfrak{p}_1 \oplus \bar{\mathfrak{p}}_2$$

with $\mathfrak{p}_2 = \{(0, d^2X, X) \in \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{k} \mid X \in \mathfrak{p}_2 \cong \mathfrak{k}\}$. Thus $s^2 = a = d^2/(d^2+1)$, and we see that g_s is $G \times K$-normal homogeneous iff $s < 1$ (notice that $g_1 = g$ is not $G \times K$-normal homogeneous). If K were not simple, then the metric on $\mathfrak{k} \times \mathfrak{k}$ would be a multiple of $\langle , \rangle$ on each simple factor of $\mathfrak{k}$. Then $\mathfrak{p}_1$ is still contained in $\bar{\mathfrak{h}}^\perp$ but $\bar{\mathfrak{p}}_2$ would not consist of $(0, d^2X, X)$ anymore, unless the metric is a multiple of $\langle , \rangle / \mathfrak{k} \times \mathfrak{k}$ and thus $\mathfrak{p}_1$ and $\mathfrak{p}_2$ would not be orthogonal anymore, and the metric is not of the form g_s. But we do need G simple since if $G = K_1 \times K_2$, $\mathfrak{k}_1 = \mathfrak{p}_1$, $\mathfrak{k}_2 = \mathfrak{p}_2$, and $H \subset K_1$ we have $G/H = (K_1/H) \times K_2$ and all metrics g_s are G and $G \times K_2$-normal homogeneous. We summarize:

THEOREM 3. *Let $M = G/H$ be a compact normal homogeneous space with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and assume that G is semisimple, H connected and that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ with*

$[\mathfrak{h}, \mathfrak{p}_2]=0$, $[\mathfrak{p}_2, \mathfrak{p}_2]\subset \mathfrak{p}_2$. *Then the metric* $g_s = g/\mathfrak{p}_1\times \mathfrak{p}_1 + s^2\cdot g/\mathfrak{p}_2\times\mathfrak{p}_2$ *is* $G\times K$-*naturally reductive, where K is the connected subgroup of G with* $\mathfrak{k}\cong\mathfrak{p}_2$.

If G is simple g_s, $s\neq 1$, is not G-normal homogeneous and g_s is $G\times K$-normal homogeneous iff $s<1$.

Remark. One can compute the sectional curvature of g_s in terms of the sectional curvature of g and show that it is not nonnegative anymore if G is simple and s is large enough >1. These metrics are then not normal homogeneous with respect to any transitive subgroup $G\subset I_0(M)$.

We will now give some specific examples.
(1) *The Berger spheres:* $S^{2n+1}=SU(n+1)/SU(n)$. Since $SU(n+1)$ contains the subgroup $S(U(n)\times U(1))$ which is a product $SU(n)\times S^1$, the conditions of the above theorem are satisfied. For $s\leq 1$, g_s is known as the Berger sphere [1], [3] and for $s>1$, g_s is a naturally reductive metric which is not $SU(n+1)\times S^1$-normal homogeneous. We will now compute T, B, and R for this example. Let $\bar{G}=SU(n+1)\times S^1$ and $\bar{H}=S(U(n)\times U(1))\cong SU(n)\times S^1$. We will use the notation in [3].

Let $A_{jk}=i(E_{jj}-E_{kk})$, $B_{jk}=E_{jk}-E_{kj}$, $C_{jk}=i(E_{jk}+E_{kj})$ and $S_j=(1/a_j)\sum_{l=1}^{j} lA_{l,l+1}$, $a_j=(j(j+1)/2)^{1/2}$. For the biinvariant metric $\langle X, Y\rangle = -\frac{1}{2}$ trace XY, $\mathfrak{p}=\mathfrak{h}^{\perp}$ has as an orthonormal basis:

$$A=S_n,\qquad e_r=B_{r,n+1},\qquad f_r=C_{r,n+1},\qquad r=1,2,\ldots,n.$$

One easily shows that $[\mathfrak{h}, A]=0$ so that

$$\mathfrak{p}_1=\langle\langle e_r, f_r\rangle\rangle,\qquad \mathfrak{p}_2=\mathbf{R}\cdot A.$$

where the brackets $\langle\langle\ ,\ \rangle\rangle$ mean that these vectors are a basis of $\mathfrak{p}_1$. Let D be a basis of the Lie algebra of S^1. Then, according to the above, the metric g_s is naturally reductive with respect to the decomposition $\bar{\mathfrak{g}}=\mathfrak{h}\oplus\bar{\mathfrak{p}}_s$, $\bar{\mathfrak{p}}_s=\langle\langle s^2A+(s^2-1)D, e_r, f_r\rangle\rangle$ where we will abbreviate

$$d_s=(s^2A+(s^2-1)D)/\|s^2A+(s^2-1)D\|_s=\frac{1}{s}(s^2A+(s^2-1)D).$$

Thus d_s, e_r, f_r, $r=1,\ldots,n$, is a g_s orthonormal basis of $\bar{\mathfrak{p}}_s$. We will now determine $B_v=B(\cdot, v)v$ and $T_v=T(v,\cdot)$. Since $Ad(\bar{H})$ maps any v, $\|v\|_s=1$ into a vector

$$v_\alpha=\cos\alpha d_s+\sin\alpha e_1,$$

we can restrict ourselves to $v=v_\alpha$.

Then $E = \langle\langle \bar{e}_1 = \sin \alpha d_s - \cos \alpha e_1, e_2, \ldots, e_n, f_1, \ldots, f_n \rangle\rangle$. The Lie brackets $[\mathfrak{p}, \mathfrak{p}]$ are easily determined to be:

$$[d_s, e_r] = s \cdot \frac{n+1}{\alpha_n} \cdot f_r, \quad [d_s, f_r] = -s \frac{n+1}{\alpha_n} \cdot e_r$$

$$[e_r, f_t] = C_{r,t} \quad \text{if} \quad r \neq t$$

$$[e_r, f_r] = C_{r,r} - C_{n+1,n+1} = s \frac{n+1}{\alpha_n} d_s + (1-s^2) \frac{n+1}{\alpha_n} \cdot (A+D)$$
$$+ \frac{1-r}{\alpha_{r-1}} S_{r-1} + \frac{1}{\alpha_r} S_r + \cdots + \frac{1}{\alpha_{n-1}} S_{n-1}$$

and thus

$$T_v(\bar{e}_1) = -s \frac{n+1}{\alpha_n} f_1, \qquad T_v(f_1) = s \frac{n+1}{\alpha_n} \cdot e_1$$

$$T_v(e_i) = s \frac{n+1}{\alpha_n} \cos \alpha \cdot f_i, \qquad T_v(f_i) = -s \frac{n+1}{\alpha_n} \cos \alpha e_i, \qquad i \geq 2$$

$$B_v(\bar{e}_1) = 0, \qquad B_v(f_1) = \left\{ 4 - s^2 \frac{(n+1)^2}{\alpha_n^2} \right\} \sin^2 \alpha \cdot f_1$$

$$B_v(e_i) = \sin^2 \alpha e_i, \qquad B_v(f_i) = \sin^2 \alpha \cdot f_i, \qquad i \geq 2.$$

Thus for $s^2 > 2n/(n+1)$ one eigenvalue of B_v becomes negative. If $n = 1$ ($M \cong S^3$) one even has $B_v \leq 0$. Since $R_v = B_v - \frac{1}{4} T_v^2$ we have

$$R_v(\bar{e}_1) = s^2 \frac{(n+1)^2}{4\alpha_n^2} \cdot \bar{e}_1$$

$$R_v(f_1) = \left[s^2 \frac{(n+1)^2}{4\alpha_n^2} + \left\{ 4 - s^2 \frac{(n+1)^2}{\alpha_n^2} \right\} \sin^2 \alpha \right] \cdot f_1$$

$$R_v(e_i) = \left\{ \sin^2 \alpha + s^2 \frac{(n+1)^2}{4\alpha_n^2} \cos^2 \alpha \right\} e_i$$

$$R_v(f_i) = \left\{ \sin^2 \alpha + s^2 \frac{(n+1)^2}{4\alpha_n^2} \cos^2 \alpha \right\} f_i.$$

Thus for $s^2 > 8n/(3(n+1))$ some of the sectional curvature becomes negative, and we can conclude that such a g_s is not G-normal homogeneous for any transitive

$G \subset I_0(M)$. The minimum and maximum sectional curvatures are:

$$s^2\frac{n+1}{2n}, \qquad 4-3s^2\frac{n+1}{2n} \quad \text{for} \quad s^2 \leq \frac{2n}{n+1}$$

and

$$4-3s^2\frac{n+1}{2n}, \qquad s^2\frac{n+1}{2n} \quad \text{for} \quad s^2 \geq \frac{2n}{n+1}$$

so that the pinching is equal to

$$\frac{s^2(n+1)}{8n-3s^2(n+1)} \quad \text{if} \quad s^2 \leq \frac{2n}{n+1}$$

$$\frac{8n-3s^2(n+1)}{s^2(n+1)} \quad \text{if} \quad s^2 \geq \frac{2n}{n+1}.$$

Thus g_s with $s^2 = 2n/(n+1)$ is the standard metric on S^{2n+1}. It is the only metric g_s, $s > 1$, which is G-normal homogeneous with respect to some G. This is clear for $n = 1$ since the only transitive groups G are S^3, $S^3 \times S^1$ and $SO(4)$. Note that the standard metric on S^3 is S^3-normal homogeneous and $SO(4)$-normal homogeneous but not $S^3 \times S^1$-normal homogeneous. We can conclude that if g_t is the metric on S^{2n+1} which is obtained from the standard metric by multiplying with t^2 in the direction of A^* then g_t is isometric to g_s with $s^2 = t^2(2n/(n+1))$. Notice that A^* is the vector field on $S^{2n+1} \subset \mathbf{C}^{n+1}$ obtained by multiplying the base point with i. Thus g_t is naturally reductive and has some negative sectional curvature for $t^2 > \frac{4}{3}$ and g_t with $t^2 \leq (n+1)/2n$ is normal homogeneous, whereas g_t with $(n+1)/2n < t^2 < 1$ or $t^2 > 1$ is not normal homogeneous.

Looking at the computations in [3], one sees that the metrics g_s, $s \leq 1$, (resp. g_t, $t \leq (n+1)/2n$) are isometric to the Berger metrics where $\sin \alpha = s$. One can also compare the metrics g_s with the metrics on the distance spheres in complex projective space [12]. Comparing minimum and maximum sectional curvature one can see that the metric on the distance sphere of radius r is isometric to the metric g_s, multiplied with a factor $4\sin^2(r/2)$ and where $\cos^2(r/2) = s^2(n+1)/2n = t^2$. Only the distance spheres with $r \geq 2 \arccos((n+1)/2n)^{1/2}$ are thus isometric to the Berger spheres (up to a factor) and the other distance spheres are not normal homogeneous. (This fact was also known to J. E. D'Atri.) One can solve the Jacobi equation for g_s explicitly, just like in [3]. One gets the same result as obtained there after substituting $\sin \alpha = s$. This example shows that all the Jacobi-fields described in Theorem 2 actually do occur.

(2) In [8] some of the sectional curvatures of the metric g_s on $SO(n+2)/SO(n)$ $(K=SO(2))$ were computed and at least for $s^2>\frac{4}{3}$ some of them become negative. (3) In [6] one finds a list of the spaces G/H satisfying the conditions of Theorem 3 under the additional hypothesis that $G/H\times K$ is a simply connected irreducible globally symmetric space, but there are lots of other spaces.

Remark. The following generalization of the metric g_s is also naturally reductive: If K is not simple, let $\mathfrak{k}=\mathfrak{k}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_r$ where $\mathfrak{k}_0$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_1,\ldots,\mathfrak{k}_r$ are simple. Then $g^*=g_{/\mathfrak{p}_1}+h_{/\mathfrak{k}_0}+t_1^2g_{/\mathfrak{k}_1}+\cdots+t_r^2g_{/\mathfrak{k}_r}$ with h arbitrary on $\mathfrak{k}_0$, is easily shown to be $G\times K$-naturally reductive.

REFERENCES

[1] Berger, M., Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, *Ann. Scoula Norm. Sup. Pisa, 15* (1961), 179–246.

[2] Bott, R., Samelson, H., Applications of Morse theory to symmetric spaces, *Amer. J. of Math., 80* (1958), 964–1029.

[3] Chavel, I., A class of riemannian homogeneous spaces, *J. of Diff. Geom., 4* (1970), 13–20.

[4] Chavel, I., On normal riemannian homogeneous spaces of rank 1, *Bull. Amer. Math. Soc., 73* (1967), 477–481.

[5] Chavel, I., Isotropic Jacobi fields and Jacobi's equation on riemannian homogeneous spaces, *Comm. Math. Helv., 42* (1967), 237–248.

[6] Jensen, G. R., Einstein metrics on principal fibre bundles, *J. Diff. Geom., 8* (1973), 599–614.

[7] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry II*, New York, Interscience, 1969.

[8] Lukesh, G. W., Variations of metrics on homogeneous manifolds, preprint, University of Massachusetts, 1976.

[9] Rauch, H. E., Geodesics and Jacobi equations on homogeneous riemannian manifolds, *Proc. U.S. Japan Seminar in Differential Geometry*, (Kyoto Univ., 1965), 115–127.

[10] Ziller, W., Closed geodesics on homogeneous spaces, to appear in *Math Z.*

[11] Ziller, W., The free-loop space of globally symmetric spaces, preprint, Bonn, 1975.

[12] Weinstein, A., Distance spheres in complex projective spaces, *Proc. Am. Math. Soc., 39* (1973).

The Institute for Advanced Study
Princeton, New Jersey 08540

Received December 22, 1976