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Univalent functions and the Schwarzian derivative

F. W. GEHRING[(1)]

Dedicated to Professor A. Pfluger on his seventieth birthday

1. Introduction

This paper is concerned with the problem of extending to an arbitrary simply connected plane domain D the following two well known results relating the univalence of a function f analytic in the unit disk B with the magnitude of its Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

THEOREM 1. *If f is analytic and univalent in B, then*

$$|S_f(z)| \le 6(1-|z|^2)^{-2}$$

in B. The constant 6 is sharp.

THEOREM 2. *If f is analytic with*

$$|S_f(z)| \le 2(1-|z|^2)^{-2}$$

in B, then f is univalent in B. The constant 2 is best possible.

Theorem 1 is due to Kraus [7] and Theorem 2 to Nehari [10].

Suppose next that D is a simply connected proper subdomain of the finite

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complex plane $\mathbf{C}$. Then the hyperbolic metric in D is given by

$$\rho_D(z)=\frac{|g'(z)|}{1-|g(z)|^2},$$

where g is any conformal mapping of D onto B. The inequality

$$\tfrac{1}{4}\operatorname{dist}(z,\partial D)^{-1}\le\rho_D(z)\le\operatorname{dist}(z,\partial D)^{-1} \tag{1}$$

follows immediately from well known results due to Koebe and Schwarz. (See, for example, page 22 in [12].)

A Jordan curve γ in the extended complex plane $\bar{\mathbf{C}}$ is said to be a *K-quasiconformal circle*, $1\le K<\infty$, if there exists a K-quasiconformal mapping f of $\bar{\mathbf{C}}$ onto $\bar{\mathbf{C}}$ which maps the unit circle onto γ. The curve γ is said to be a quasiconformal circle if it is a K-quasiconformal circle for some K.

The following analogues of Theorems 1 and 2 for simply connected subdomains D of $\mathbf{C}$ are due to Lehto [8] and Ahlfors [1], respectively. See also [3].

THEOREM 3. *If f is analytic and univalent in D, then*

$$|S_f(z)|\le 12\rho_D(z)^2$$

in D. The constant 12 *is sharp.*

THEOREM 4. *Suppose that ∂D is a K-quasiconformal circle. Then there exists a positive constant a which depends only on K such that f is univalent in D whenever f is analytic with*

$$|S_f(z)|\le a\rho_D(z)^2 \tag{2}$$

in D.

Remark. Ahlfors actually proved more than the conclusion given above, namely that one can choose $a=a(K)$ so that f has a quasiconformal extension to $\bar{\mathbf{C}}$ whenever f is analytic and satisfies (2) in D.

In view of the above remark, it is natural to ask if the hypothesis that ∂D be a quasiconformal circle is necessary in Theorem 4. We shall show that this is indeed the case by establishing the following result.

THEOREM 5. *Suppose there exists a positive constant a such that f is* *univalent in D whenever f is analytic with*

$$|S_f(z)| \le a\rho_D(z)^2$$

in D. Then ∂D is a K-quasiconformal circle where K depends only on a.

2. Schwarzian univalence criterion

We obtain Theorem 5 as a corollary of an analogous result for proper subdomains D of $\mathbf{C}$ with arbitrary connectivity. For such domains D we have the following consequence of Theorem 1.

COROLLARY 1. *If f is analytic and univalent in D, then*

$$|S_f(z)| \le 6 \operatorname{dist}(z, \partial D)^{-2} \tag{3}$$

in D. The constant 6 *is best possible.*

Proof. Fix $z_0 \in D$, choose r so that $0 < r < \operatorname{dist}(z_0, \partial D)$ and let $g(z) = f(rz + z_0)$. Then g is analytic and univalent in B,

$$|S_f(z_0)| = |S_g(0)| r^{-2} \le 6r^{-2}$$

by Theorem 1, and we obtain (3) for $z = z_0$ by letting $r \to \operatorname{dist}(z_0, \partial D)$. There is equality in (3) when f is the Koebe function $z(1-z)^{-2}$, $D = B$ and $z = 0$.

Corollary 1 and inequality (1) suggest that $\operatorname{dist}(z, \partial D)^{-1}$ is a reasonable substitute for the hyperbolic metric $\rho_D(z)$ in the case where D is multiply connected.

DEFINITION. *Suppose that D is an arbitrary proper subdomain of* $\mathbf{C}$. *We say that D satisfies the Schwarzian univalence criterion if there exists a positive constant a such that f is univalent in D whenever f is analytic with*

$$|S_f(z)| \le a \operatorname{dist}(z, \partial D)^{-2}$$

in D.

The purpose of this paper is to establish the following result.

THEOREM 6. *If D satisfies the Schwarzian univalence criterion with constant a, then each component of ∂D is either a point or a K-quasiconformal circle where K depends only on a.*

Proof of Theorem 5. Suppose that D is a simply connected proper subdomain of $\mathbf{C}$ which satisfies the hypotheses of Theorem 5. Then by inequality (1), D satisfies the Schwarzian univalence criterion with constant $a/16$. Since ∂D is connected and contains at least two points, Theorem 6 implies that ∂D is a K-quasiconformal circle where K depends only on a.

COROLLARY 2. *Suppose that D is a simply connected proper subdomain of $\mathbf{C}$. Then D satisfies the Schwarzian univalence criterion if and only if ∂D is a quasiconformal circle.*

Proof. Theorem 4 and inequality (1) imply that D satisfies the Schwarzian univalence criterion whenever ∂D is a quasiconformal circle. The converse follows from Theorem 6.

3. Proof of Theorem 6

The proof of Theorem 6 depends on five lemmas given below. In what follows we let D denote an arbitrary domain in $\bar{\mathbf{C}}$, $B(z_0, r)$ the open disk with center $z_0 \in \mathbf{C}$ and radius $r \in (0, \infty)$, and b a constant in $(1, \infty)$. Next we say that two points z_1, z_2 can be *joined* in a set $E \subset \bar{\mathbf{C}}$ if there exists an arc $\alpha \subset E$ with z_1, z_2 as its endpoints. Finally for each set $E \subset \bar{\mathbf{C}}$ we let ∂E, $\bar{E}$ and $C(E)$ denote respectively the boundary, closure and complement of E in $\bar{\mathbf{C}}$.

LEMMA 1. *Suppose that for some z_0 and r there exist two points in $D \cap \bar{B}(z_0, r)$ which cannot be joined in $D \cap \bar{B}(z_0, br)$. Then there exist finite points z_1, z_2 in D and w_1, w_2 in $C(D)$ such that*

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D with

$$|h(z_1) - h(z_2) - 2\pi i| \le \frac{4}{b-1}. \tag{4}$$

Proof. By hypothesis there exist two points z_1', z_2' in $D \cap \bar{B}(z_0, r)$ which cannot be joined in $D \cap \bar{B}(z_0, br)$. Let α' denote the closed segment from z_1' to z_2'

and let $B_0 = B(z_0, br)$. Since $z_1', z_2' \in D$, there exists an open polygonal arc β' from z_2' to z_1' in D which meets α' in at most a finite set of points; when $z_1', z_2' \neq z_0$, we choose β' so that it lies in $D - \{z_0\}$. Then $\beta' - (\alpha' \cap \beta')$ is the union of a finite number of open subarcs β with endpoints in α'. Since z_1', z_2' cannot be joined in $D \cap \bar{B}_0$, we can choose a β whose endpoints cannot be joined in $D \cap \bar{B}_0$. Let z_1 and z_2 denote respectively the terminal and initial points of β, and let α denote the closed segment from z_1 to z_2. Note that $z_1, z_2 \neq z_0$ whenever $z_1', z_2' \neq z_0$.

We want next to find finite points $w_1, w_2 \in C(D)$ so that the function h is analytic in D and satisfies (4). Now z_1 and z_2 are separated in $\bar{B}_0$ by the closed set $C(D)$. Using Theorem VI.7.1 in [11] it is easy to show that z_1 and z_2 are separated in $\bar{B}_0$ by a component C_0 of $C(D)$. Let $D_0 = C(C_0)$. Then D_0 is a simply connected domain by Theorem IV.3.3 in [11], $D \subset D_0$, and the points z_1, z_2 cannot be joined in $D_0 \cap \bar{B}_0$. Hence by replacing D by D_0, we may assume without loss of generality that D is simply connected.

Now $\gamma = \alpha \cup \beta$ is a Jordan curve. Let D_1 and D_2 denote respectively the bounded and unbounded components of $C(\gamma)$. We shall show that there exist points w_1, w_2 such that

$$w_i \in C(D) \cap \partial B_0 \cap D_i \tag{5}$$

for $i = 1, 2$. Fix i. Since z_1, z_2 cannot be joined in $D \cap \bar{B}_0$, β and hence γ must meet ∂B_0 in at least two points. From Kerékjártó's theorem it follows that each component of

$$C(\gamma) \cap C(\partial B_0) = C(\gamma \cup \partial B_0)$$

is a Jordan domain, and hence that each component of $D_i \cap B_0$ is bounded by a Jordan curve. (See page 168 in [11].) Next since D_i is a Jordan domain and since $z_1 \in \partial D_i \cap B_0$, there exists a neighborhood U of z_1 such that points of $D_i \cap U$ can be joined in $D_i \cap B_0$. Hence $D_i \cap U$ is contained in a component D^* of $D_i \cap B_0$,

$$D^* \cap U = D_i \cap U, \tag{6}$$

and ∂D^* is a Jordan curve γ^*.

Choose $z \in \alpha - \{z_1\}$. Since α lies at a positive distance from ∂B_0, we can choose an open crosscut δ of D_i from z_1 to z which lies in B_0. Then (6) implies that $\delta \subset D^*$, that $z \in \gamma^*$, and hence that $\alpha \subset \gamma^*$. Thus $\beta^* = \gamma^* - \alpha$ is an open arc joining z_2 to z_1 in $\bar{B}_0$, and there exists a point

$$w_i \in \beta^* \cap C(D). \tag{7}$$

Since

$$\gamma^* \subset \partial(D_i \cap B_0) \subset \gamma \cup (\partial B_0 \cap D_i),$$

we have

$$\beta^* \subset \beta \cup (\partial B_0 \cap D_i) \subset D \cup (\partial B_0 \cap D_i), \tag{8}$$

and (5) follows from (7) and (8).

Since D is simply connected, we can define an analytic branch of

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

in D. Then

$$\begin{aligned} h(z_1) - h(z_2) &= \int_\beta \frac{dz}{z - w_1} - \int_\beta \frac{dz}{z - w_2} \\ &= 2\pi i(n(\gamma, w_1) - n(\gamma, w_2)) - \int_\alpha \frac{dz}{z - w_1} + \int_\alpha \frac{dz}{z - w_2}, \end{aligned}$$

where $n(\gamma, w_i)$ is the winding number of γ with respect to w_i. Since D_1 is the bounded component of $C(\gamma)$,

$$n(\gamma, w_1) = n = \pm 1, \qquad n(\gamma, w_2) = 0,$$

and we have

$$|h(z_1) - h(z_2) - 2n\pi i| \le \int_\alpha \frac{|dz|}{|z - w_1|} + \int_\alpha \frac{|dz|}{|z - w_2|}. \tag{9}$$

(See [2].) Then

$$\int_\alpha \frac{|dz|}{|z - w_i|} \le \frac{|z_1 - z_2|}{(b-1)r} \le \frac{2}{b-1} \tag{10}$$

for $i = 1, 2$, and (4) follows from (9) and (10) when $n = 1$. When $n = -1$, we obtain (4) by interchanging w_1 and w_2.

LEMMA 2. *Suppose that for some z_0 and r there exist two points in $D - B(z_0, r)$ which cannot be joined in $D - B(z_0, r/b)$. Then the conclusion of Lemma* 1 *again holds.*

Proof. By hypothesis there exist two points z_1', z_2' in $D-B(z_0, r)$ which cannot be joined in $D-B(z_0, r/b)$; we may assume without loss of generality that $z_1', z_2' \neq \infty$. Next let Δ and ζ_i' denote the images of D and z_i' under

$$f(z)=\frac{1}{z-z_0}+z_0.$$

Then ζ_1', ζ_2' are points in $\Delta \cap \bar{B}(z_0, 1/r)$ which cannot be joined in $\Delta \cap \bar{B}(z_0, b/r)$. By the argument for Lemma 1, there exist finite points

$$\zeta_1, \zeta_2 \in \Delta - \{z_0\}, \qquad \omega_1, \omega_2 \in C(\Delta) \cap \partial B(z_0, b/r)$$

such that

$$g(\zeta)=\log\frac{\zeta-\omega_1}{\zeta-\omega_2}$$

is analytic in Δ with

$$|g(\zeta_1)-g(\zeta_2)-2\pi i| \leq \frac{4}{b-1}.$$

Let z_i, w_i denote the images of ζ_i, ω_i under f^{-1}. Then

$$h(z)=g\circ f(z)+\log\frac{z_0-w_1}{z_0-w_2}=\log\frac{z-w_1}{z-w_2}$$

is analytic in D and satisfies (4).

DEFINITION. *A set E in $\bar{\mathbf{C}}$ is said to be b-locally connected if for all z_0 and r, points in $E \cap \bar{B}(z_0, r)$ can be joined in $E \cap \bar{B}(z_0, br)$ and points in $E-B(z_0, r)$ can be joined in $E-B(z_0, r/b)$.*

See [5] and [6] for other applications of this concept.

LEMMA 3. *Suppose that D is a proper subdomain of $\mathbf{C}$. If D satisfies the Schwarzian univalence criterion for some constant a, then D is b-locally connected where*

$$b=\max\left(\frac{5}{a}+1, 3\right). \tag{11}$$

Proof. Suppose that D is not b-locally connected. Then there exist $z_0 \in \mathbf{C}$, $r \in (0, \infty)$ and two points in D for which the hypotheses of Lemma 1 or Lemma 2 hold. In either case, we obtain finite points z_1, $z_2 \in D$ and w_1, $w_2 \in C(D)$ such that

$$h(z) = \log \frac{z - w_1}{z - w_2}$$

is analytic in D and satisfies (4). Since $b \geq 3$, inequality (4) implies that

$$|h(z_1) - h(z_2)| \geq 2\pi - \frac{4}{b-1} > 4. \tag{12}$$

Now set

$$f(z) = \exp(ch(z)), \qquad c = \frac{2\pi i}{h(z_1) - h(z_2)}.$$

Then f is analytic with

$$S_f(z) = \frac{1-c^2}{2}\left(\frac{1}{z-w_1} - \frac{1}{z-w_2}\right)^2$$

in D. Next (4), (11) and (12) imply that

$$2|1-c^2| < \frac{5}{b-1} \leq a,$$

and hence that

$$|S_f(z)| \leq 2|1-c^2| \operatorname{dist}(z, \partial D)^{-2} \leq a \operatorname{dist}(z, \partial D)^{-2}$$

in D. Since D satisfies the univalence criterion, it follows that f must be univalent in D. But

$$\frac{f(z_1)}{f(z_2)} = \exp(c(h(z_1) - h(z_2))) = 1,$$

and we have a contradiction.

LEMMA 4. *Suppose that D is b-locally connected and that ∂D is connected and contains at least two points. Then ∂D is a K-quasiconformal circle where K depends only on b.*

Proof. Suppose that p is a point in $\bar{D}$. With each neighborhood U of p we associate a second neighborhood V as follows. If $p = z_0 \in \mathbf{C}$, choose $r \in (0, \infty)$ so that $\bar{B}(z_0, br) \subset U$ and let $V = B(z_0, r)$; if $p = \infty$ choose $r \in (0, \infty)$ so that $C(B(0, r/b)) \subset U$ and let $V = C(\bar{B}(0, r))$. In each case, the fact that D is b-locally connected implies that points in $D \cap V$ can be joined in $D \cap U$. Thus D is uniformly locally connected and ∂D is a Jordan curve γ by Theorem VI.16.2 in [11].

We show next that for any pair of finite points $z_1, z_2 \in \gamma$,

$$\min(\operatorname{dia}(\gamma_1), \operatorname{dia}(\gamma_2)) \le b^2 |z_1 - z_2|, \tag{13}$$

where γ_1, γ_2 denote the components of $\gamma - \{z_1, z_2\}$. By a theorem of Ahlfors, inequality (13) will then imply that γ is a K-quasiconformal circle, where K depends only on b, thus completing the proof. (See, for example, Theorem II.8.6 in [9].)

To this end fix $z_1, z_2 \in \gamma$, set

$$z_0 = \tfrac{1}{2}(z_1 + z_2), \qquad r = \tfrac{1}{2}|z_1 - z_2|,$$

and suppose that (13) does not hold. Then there exist $t \in (r, \infty)$ and finite points w_1, w_2 such that

$$w_i \in \gamma_i - B(z_0, b^2 t) \tag{14}$$

for $i = 1, 2$. Choose $s \in (r, t)$. Since $z_1, z_2 \in \gamma \cap B(z_0, s)$, we can find for $i = 1, 2$ an endcut α_i of D joining z_i to $z_i' \in D$ in $\bar{B}(z_0, s)$. Next since D is b-locally connected, we can find an arc α_3 joining z_1' to z_2' in $D \cap \bar{B}(z_0, bs)$. Then $\alpha_1 \cup \alpha_2 \cup \alpha_3$ contains a crosscut α of D from z_1 to z_2 with

$$\alpha \subset \bar{B}(z_0, bs). \tag{15}$$

By virtue of (14), the same argument can be applied to obtain a crosscut β of D from w_1 to w_2 with

$$\beta \subset C(B(z_0, bt)). \tag{16}$$

But (15) and (16) imply that $\alpha \cap \beta = \varnothing$, contradicting the fact that z_1 and z_2 separate w_1 and w_2 in γ. Thus (13) holds and the proof of Lemma 4 is complete.

LEMMA 5. *Suppose that D is b-locally connected. Then each component of ∂D is either a point or a K-quasiconformal circle where K depends only on b.*

Proof. Let B_0 be a component of ∂D, let C_0 denote the component of $C(D)$ which contains B_0, and let $D_0 = C(C_0)$. Then D_0 is a domain with $\partial D_0 = B_0$. (See, for example, the proof of Theorem VI.16.3 in [11].) To complete the proof we need only show that D_0 is b-locally connected. For then by Lemma 4, ∂D_0 will be a point or a K-quasiconformal circle where $K = K(b)$.

Fix $z_0 \in \mathbf{C}$ and $r \in (0, \infty)$. Given $z_1, z_2 \in D_0 \cap \bar{B}(z_0, r)$ we must find an arc γ joining these points in $D_0 \cap \bar{B}(z_0, br)$. For this let α be any arc joining z_1 and z_2 in $\bar{B}(z_0, r)$. If $\alpha \subset D_0$, we may take $\gamma = \alpha$. Suppose that $\alpha \not\subset D_0$ and for $i = 1, 2$ let α_i denote the component of $\alpha \cap D_0$ which contains z_i. Then for each i there exists a point w_i such that

$$w_i \in \alpha_i \cap D. \tag{17}$$

If $z_i \in D$, we may take $w_i = z_i$; otherwise $z_i \in C_i$, a component of $C(D)$ different from C_0, and the fact that

$$\bar{\alpha}_i \cap C_0 \neq \emptyset, \qquad \alpha_i \cap C_i \neq \emptyset$$

implies that α_i must meet D and hence contain a point w_i satisfying (17). Since D is b-locally connected and since

$$w_1, w_2 \in \alpha \cap D \subset D \cap \bar{B}(z_0, r),$$

we can join w_1 and w_2 by an arc β in $D \cap \bar{B}(z_0, br)$. Then $\alpha_1 \cup \beta \cup \alpha_2$ will contain an arc γ joining z_1 and z_2 in $D_0 \cap \bar{B}(z_0, br)$.

Next the same argument shows that each pair of points in $D_0 - B(z_0, r)$ can be joined in $D_0 - B(z_0, r/b)$. Hence D_0 is b-locally connected and the proof is complete.

Proof of Theorem 6. Suppose that D is a proper subdomain of $\mathbf{C}$ which satisfies the Schwarzian univalence criterion with constant a. Lemma 3 implies that D is b-locally connected, where b is as in (11). Then Lemma 5 implies that each component of ∂D is either a point or a K-quasiconformal circle, where K depends only on b, and hence only on a.

4. Universal Teichmüller space

We conclude this paper with an application of Theorem 5 to Teichmüller theory.

Let $B_2 = B_2(L, 1)$ denote the Banach space of functions φ analytic in the lower half plane L with norm

$$\|\varphi\| = \sup_{z \in L} \rho_L(z)^{-2} |\varphi(z)| < \infty,$$

where $\rho_L(z) = \frac{1}{2}|y|^{-1}$ is the hyperbolic metric in L. Next let S denote the family of $\varphi = S_g$ where g is conformal in L, and let $T = T(1)$ denote the subfamily of those $\varphi = S_g$ for which g has a quasiconformal extension to $\bar{\mathbf{C}}$. From Theorem 1 it follows that $\|\varphi\| \le 6$ for all $\varphi \in S$ and hence that $T \subset S \subset B_2$. The set T is the universal Teichmüller space. (See, for example, [4].)

Suppose that $\varphi \in \operatorname{int}(S)$. Then $\varphi = S_g$ where g maps L conformally onto a simply connected subdomain D of $\mathbf{C}$. In addition, there exists a constant $a > 0$ such that $\psi \in S$ whenever $\|\psi - \varphi\| \le a$. If f is analytic with

$$|S_f(z)| \le a\rho_D(z)^2$$

in D, then $\psi = S_{f \circ g}$ is analytic in L, $\|\psi - \varphi\| \le a$, and hence f is univalent in D. Thus ∂D is a quasiconformal circle by Theorem 5, g has a quasiconformal extension to $\bar{\mathbf{C}}$, and $\varphi \in T$. Hence

$$\operatorname{int}(S) \subset T. \tag{18}$$

Next using the Remark following Theorem 4, Ahlfors showed in [1] that

$$T = \operatorname{int}(T). \tag{19}$$

Combining (18) and (19) we obtain the following result.

COROLLARY 3. *T is the interior of S in B_2.*

Unfortunately Corollary 3 neither implies nor is implied by the truth of the following interesting conjecture due to Bers. (See, for example, [4].)

CONJECTURE. *S is the closure of T in B_2.*

Lehto observed in [8] that one would settle the Bers conjecture in the negative if one could find a Jordan domain D and a positive constant a such that ∂D is not a quasiconformal circle and such that f has a quasiconformal extension to $\bar{\mathbf{C}}$

whenever f is analytic with

$$|S_f(z)| \leq a\rho_D(z)^2$$

in D. Theorem 5 shows, however, that no such domain D exists.

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University of Michigan
Ann Arbor, Michigan

University of Paris
Orsay, France

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