A remark on subgroups of infinite index in Poincaré duality groups.

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1. Introduction

1.1. Let R be a commutative ring with 1. A group G is a *Poincaré duality group of dimension n* (over R) if R admits an RG-projective resolution

$$0 \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \longrightarrow R \longrightarrow 0 \tag{1}$$

of length n with every P_j finitely generated, and if the "dual" sequence

$$0 \longrightarrow P_0^* \xrightarrow{\partial_1^*} P_1^* \longrightarrow \cdots \longrightarrow P_{n-1}^* \xrightarrow{\partial_n^*} P_n^* \longrightarrow H^n(G, RG) \longrightarrow 0$$

is exact and $H^n(G, RG)$ is R-isomorphic with R. In (1) R is by definition a trivial G-module, but $H^n(G, RG) \cong R$ need not be so.

1.2. Natural *examples* of Poincaré duality groups (over $\mathbf{Z}$) are the fundamental groups of closed aspherical manifolds. Poly-cyclic groups provide examples of solvable PD-groups over $\mathbf{Q}$. For them the cohomological dimension $cd_{\mathbf{Q}}$ coincides with the torsion-free rank ([4], p. 154 and [1], p. 391). This description of $cd_{\mathbf{Q}}$ makes it plain that a subgroup of infinite index in a poly-cyclic group with $cd_{\mathbf{Q}} = n$ has cohomological dimension at most $n-1$. We shall prove that every PD-group enjoys this property.

THEOREM. *A subgroup of infinite index in a Poincaré duality group of dimension n (over R) has cohomological dimension at most n − 1 (over R).*

1.3. The known PD-*groups of dimension* 2 over $\mathbf{Z}$ are the fundamental groups of the closed surfaces of positive genus. Any subgroup of infinite index in such a group is free. This is a classical result of the topology of surfaces (cf. [8], p. 143); it follows also from the subgroup theorems for Fuchsian groups (see [6]). Our result, in conjunction with the structure theorems of Stallings [9] and Swan [10], shows that all PD-groups of dimension 2 have this feature.

COROLLARY. *A torsion-free subgroup of infinite index in a PD-group of dimension* 2 (*over any commutative ring*) *is free.*

1.4. *Sources.* Our definition of PD-groups is in the spirit of [7], but is equivalent to the definition adopted in Bieri's set of lecture notes ([2], p. 156), which gives a detailed account on Poincaré duality groups and related topics. For the case $R=\mathbf{Z}_2$, the claim of Corollary 1.3 has been verified for finitely generated subgroups by Farrell, using topological methods [3] (p. 318, Theorem 2.2; Farrell's assumptions differ slightly from ours). The counterpart of Theorem 1.2, dealing with homological instead of cohomological dimension, is proved in [2], p. 175, Proposition 9.22.

2. Proof: Part I

2.1. The proof of 1.2 splits into two parts. In this section we use duality to transform the claim into a claim about a certain tensor product and show that this transformed claim need only be verified for subgroups of infinite index in free groups of finite rank. The verification is then carried out in Section 3.

2.2. Let R be a commutative ring with 1, and suppose $V \leqslant G$ is a subgroup of G. For any left-RV-module M, the R-module $\mathrm{Hom}_{RV}(RG, M)$ can be turned into a left-RG-module (using the right-G-structure on RG). One has then a canonical isomorphism

$$H^*(V, M) \cong H^*(G, \mathrm{Hom}_V(RG, M)).$$

If G is a Poincaré duality group of dimension n, this isomorphism, when combined with the duality isomorphism

$$H^*(G, ?) = \mathrm{Ext}^*_{RG}(R, ?) \cong \mathrm{Tor}^{RG}_{n-*}(\mathrm{Ext}^n_{RG}(R, RG), ?)$$

(see [2], p. 140, Theorem 9.2), gives in the top dimension n an isomorphism

$$H^n(V, M) \simeq H^n(G, RG) \otimes_{RG} \mathrm{Hom}_V(RG, M). \tag{2}$$

We have to prove that the left hand side of (2) vanishes, and shall do this by using the formulation given by the right hand side.

2.3. Choose a free group F which maps onto G, say $\pi: F \twoheadrightarrow G$, and let $\Pi: RF \to RG$ denote the associated projection between the group algebras. If Π is followed by an RV-linear map $RG \to M$ one obtains an RU-linear map $RF \to M$ where $U = \pi^{-1}(V)$ and M is viewed as an RU-module via $\pi_1: U \twoheadrightarrow V$. This process leads to an R-linear isomorphism

$$\Pi^*: \mathrm{Hom}_V(RG, M) \xrightarrow{\sim} \mathrm{Hom}_U(RF, M). \tag{3}$$

As both RG and RF carry a right-F-structure which is respected by Π^*, this map is actually an RF-isomorphism. If it is combined with (2) there results an R-isomorphism

$$H^n(V, M) \cong H^n(G, RG) \otimes_F \mathrm{Hom}_U(RF, M),$$

valid for any RV-module M. Remember $H^n(G, RG)$ is assumed to be R-isomorphic with R.

2.4. Since R is presupposed to admit an RG-projective resolution which is finitely generated in each dimension, G is, in particular, a finitely generated group. Hence F, introduced in the previous subsection, can be chosen to be finitely generated, and thus realized as the set of words on a finite alphabet X modulo the usual congruence relation. Note that the letters of X occur in pairs, a pair representing a generator of F and its inverse.

Let $\hat{R}$ be short for the right-RG-module $H^n(G, RG)$ viewed as an RF-module via Π. Since $\hat{R}$ is R-isomorphic with R, each letter x of the alphabet X is associated with a unit ρ_x^{-1} of R which describes the automorphism of $\hat{R}$ induced by x. For an arbitrary left-RF-module N, any element of $\hat{R} \otimes_F N$ can be written in the form $1 \otimes \alpha$. If α admits a representation

$$\alpha = \sum_{x \in X} (1 - \rho_x \cdot x)\alpha_x \qquad (\alpha_x \in N), \tag{4}$$

the tensor product $1 \otimes \alpha$ is zero; for we have:

$$1 \otimes \alpha = \sum_x 1 \otimes (1 - \rho_x \cdot x)\alpha_x = \sum_x (1 - \rho_x \cdot \rho_x^{-1}) \otimes \alpha_x = 0.$$

Note that the sum in (4) is finite, the alphabet X being finite.

2.5. As the foregoing discussion has made it clear it will do to establish the following

Claim. Let F be a finitely generated free group, realized as the set of congruence classes of words on the finite alphabet X. Associate with each letter x in X a unit ρ_x of the commutative ring R. If $U \leqslant F$ is a subgroup of infinite index and M an arbitrary left-RU-module, then any RU-linear map $\alpha: RF \to M$ can be written as a sum $\sum_x (1-\rho_x \cdot x)\alpha_x$ for suitable RU-linear maps $\alpha_x: RF \to M$ $(x \in X)$.

3. Proof: Part II

3.1. Let U, F and M be as in 2.5 and choose a left transversal T of U in F. (Then $F = \bigcup_{t \in T} Ut$ by definition.) An RU-linear map $\alpha: RF \to M$ is completely determined by its restriction to T, and every function $\alpha: T \to M$ arises in this way. Differently put, $\mathrm{Hom}_U(RF, M)$ is R-isomorphic with M^T, the R-module of all functions from T into M. This isomorphism can be used to endow M^T with an F-module structure. It is given by

$$(f\alpha)(t) = tf \cdot \overline{tf}^{\,-1} \cdot \alpha(\overline{tf}).$$

Here $\bar{\ }: F \to T$ denotes the function taking the element $f \in F$ to the member $\bar{f}$ of the transversal with $Uf = U\bar{f}$.

3.2. Now think of F as consisting of congruence classes of words on the alphabet X, and select a transversal which can be represented by a family S of freely reduced words satisfying the *Schreier condition*:

Any initial segment $x_1 \cdots x_k$ of a word

$x_1 \cdots x_k x_{k+1} \cdots x_m$ of S belongs itself to S.

(Such a choice is always possible, see e.g. [5], p. 95). The Schreier system carries a natural relation, defined by

$$s \leqslant s' \Leftrightarrow s \quad \text{is an initial segment of } s'.$$

This is a partial ordering with the following properties:

(i) $\langle S, \leqslant \rangle$ has a smallest element.
(ii) For each $s \in S$, the set $\{s_1 \in S \mid s_1 \leqslant s\}$ is finite and linearly ordered.
(iii) Every $s \in S$ has at most $|X|$-many immediate successors.
(iv) S is countably infinite, and $\langle S, \leqslant \rangle$ has chains of the order type of the natural numbers.

By (i) and (ii) the poset $\langle S, \leqslant \rangle$ is a tree. (Henceforth we shall, by abuse of notation, write S both for the set S and the poset $\langle S, \leqslant \rangle$.) A chain of S is called a branch if it contains with an element all the smaller ones. An *ω-branch* is a branch of the order type of the natural numbers. By (ii) and (iv) S has ω-branches. The union of all ω-branches will be denoted by S_ω.

3.3. We now come to the first phase of construction. In it we define, given $\alpha : S \leftrightarrow T \to M$, functions $\beta_x : S \to M$ such that $\sum_x (1 - \rho_x \cdot x)\beta_x$ agrees with α outside S_ω. This can be achieved as follows. For $s \in S \backslash S_\omega$ and $s_1 \in S$, set

$$\rho(s, s_1) = \begin{cases} \rho_{x_1} \cdot \rho_{x_2} \cdot \cdots \cdot \rho_{x_k} & \text{if } s < s_1 \quad \text{and} \quad s_1 = s x_1 \cdots x_k \\ 1 & \text{if } \quad s = s_1 \\ 0 & \text{otherwise} \end{cases}$$

If $\alpha : S \to M$ is the given function, define for each $x \in X$ a function $\beta_x : S \to M$ by the formula

$$\beta_x(s) = \begin{cases} \sum_{s_1 \in S} \rho(s, s_1)\alpha(s_1) & \text{if } \quad s \in S \backslash S_\omega \quad \text{and if } s \text{ ends with } x \\ 0 & \text{otherwise} \end{cases}$$

(Since each $s \in S \backslash S_\omega$ has only finitely many elements above it, the sum if finite.) For $s \in S \backslash S_\omega$ one gets

$$\left(\sum_x \beta_x \right)(s) = \sum_{s_1} \rho(s, s_1) \cdot \alpha(s_1) \tag{5}$$

For the computation of $\sum_x \rho_x \cdot x \cdot \beta_x$ one has to take into account the following fact. If the representative $\overline{sx}$ ends in x then $\overline{sx} = s_1 x$ for some $s_1 \in S$ and

$Usx = U\overline{sx} = Us_1x$. Since s and s_1 are both members of a transversal of U in F, they must be equal and $sx = \overline{sx}$. A glance at the definition of β_x then reveals that $sx \cdot \overline{sx}^{-1}\beta_x(\overline{sx}) = \beta_x(\overline{sx})$, whether $\overline{sx}$ ends with x or not.

Bearing this in mind one gets:

$$\left(\sum_x \rho_x \cdot x \cdot \beta_x\right)(s) = \sum_x \rho_x \cdot \beta_x(\overline{sx}) = \sum \left\{\rho_x \sum_{s_1} \rho(\overline{sx}, s_1) \cdot \alpha(s_1) \,\middle|\, \overline{sx} = sx\right\}$$

$$= \sum_{s_1} \rho(s, s_1)\alpha(s_1) - \alpha(s). \tag{6}$$

It suffices to compare (5) and (6) to see that $\sum_x (1-\rho_x \cdot x)\beta_x$ agrees with the given α outside S_ω.

3.4. The number of ω-branches of S_ω is countable, say $B_1, B_2, B_3, \ldots$. Define a decomposition of S_ω by setting

$$C_1 = B_1$$

$$C_k = B_k \setminus (B_1 \cup B_2 \cup \cdots \cup B_{k-1}) \qquad (1 < k < \omega).$$

Each component C_k is order isomorphic with the natural numbers. Its first element will be denoted by σ_k. For $s, s_1 \in S$ set

$$\rho'(s_1, s) = \begin{cases} \rho_{x_1}^{-1} \cdots \rho_{x_m}^{-1} & \text{if } s_1 \text{ and } s \text{ belong to the same component,} \\ & s_1 < s \text{ and } s = s_1x_1 \cdots x_m. \\ 0 & \text{otherwise} \end{cases}$$

If $\alpha : S \to M$ is a function vanishing outside S_ω, define for every $x \in X$ a function $\beta'_x : S \to M$ by the formula:

$$\beta'_x(s) = \begin{cases} -\sum_{\sigma_k \le s_1 < s} \rho'(s_1, s)\alpha(s_1) & \text{if } s \text{ is in } C_k \text{ and ends with } x \\ 0 & \text{otherwise.} \end{cases}$$

Then each β'_x vanishes outside S_ω and also on the initial elements σ. It follows without pain that $\sum_x (1-\rho_x \cdot x)\beta'_x$ vanishes outside S_ω. For $s \in C_k$ one gets

$$\left(\sum_x \beta'_x\right)(s) = -\sum_{\sigma_k \le s_1 < s} \rho'(s_1, s) \cdot \alpha(s_1) \tag{6}$$

For the computation of $\sum \rho_x \cdot x \cdot \beta_x'$ we use again that $sx \cdot \overline{sx}^{-1} \cdot \beta_x(\overline{sx}) = \beta_x(\overline{sx})$. For an $s \in C_k$ one then gets:

$$\Big(\sum_x \rho_x \cdot x \cdot \beta_x'\Big)(s) = \sum_x \rho_x \cdot \beta'(\overline{sx})$$

$$= -\sum \left\{\rho_x \cdot \sum_{\sigma_k \leq s_1 < \overline{sx}} \rho'(s_1, \overline{sx}) \cdot \alpha(s_1) \,\middle|\, \overline{sx} = sx\right\}.$$

Now $\rho'(s_1, \overline{sx})$ is zero if s_1 and $\overline{sx}$ lie in different components (or if $\overline{sx}$ lies outside S_ω). Thus only the summand with $sx \in C_k$ makes a contribution. Say this happens for x_1. Therefore

$$\sum_x \rho_x \cdot x \cdot \beta_x)(s) = -\sum_{\sigma_k \leq s_1 \leq s} \rho_{x_1} \cdot \rho'(s_1, sx_1) \cdot \alpha(s_1). \tag{7}$$

From (6), (7) and the definition of ρ', one sees that $(\sum (1 - \rho_x \cdot x)\beta_x)\,(s)$ equals $\alpha(s)$. This settles claim 2.5, and with it Theorem 1.2.

4. Comments

There are several directions in which the assumptions of 1.2 or of 2.5 could be weakened that deserve to be mentioned.

4.1. First the concept of a PD-group can be generalized. We required, inter alia, $H^n(G, RG)$ to be a free cyclic R-module. If this condition is replaced, say, by R-free and RG-cyclic, everything else remaining unchanged, the conclusion of 1.2 is no longer true. The class of torsion-free 1-relator groups abounds in counterexamples, e.g.

$$\langle a, t : t^{-1}at = a^m \rangle \qquad (m \in \mathbf{Z}\backslash\{-1, 0, 1\})$$

(cf. [2], Section 9.8, pp. 165–167).

4.2. Being interested in minimalizing the assumptions, one could require G to be a finitely generated group of finite cohomological dimension n, R to admit an RG-projective resolution which is finitely generated in the top dimension n and $H^n(G, RG)$ to be a free cyclic R-module. Then the proof carries over verbatim (cf. [2], p. 138, Exercise). However, I know of no example which satisfies these assumptions without being a PD-group.

4.3. Second a word on 2.5. Our proof relies heavily on the assumption that F be finitely generated. This assumption is essential, as we now indicate. By 2.3 it will suffice to prove that $R \otimes_{RG} \mathrm{Hom}_R(RG, R)$ is non-trivial for a suitable group G.

For simplicity assume $R = \mathbf{Z}$. Let G be an infinite, locally finite group and consider the augmentation $\varepsilon: \mathbf{Z}G \to \mathbf{Z}$ taking every $g \in G$ onto 1. For every finitely generated subgroup $H \leq G$ we have an isomorphism ,

$$\mathbf{Z} \underset{\mathbf{Z}H}{\otimes} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}G, \mathbf{Z}) \cong \mathbf{Z} \underset{\mathbf{Z}H}{\otimes} \mathbf{Z}^G \cong \mathbf{Z}^{G/H}$$

Under this isomorphism $1 \otimes_H \varepsilon$ corresponds to the function $|H| \cdot \varepsilon: G/H \to \mathbf{Z}$ taking gH to $|H|$. It follows that $1 \otimes \varepsilon$ is non-zero in $\mathbf{Z} \otimes_{\mathbf{Z}G} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}G, \mathbf{Z})$. (The above example can easily be adapted to an arbitrary commutative ring R).

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