Jordan-Hahn decomposition of signed weights on Finite orthogonality spaces.

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Jordan–Hahn decomposition of signed weights on finite orthogonality spaces

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1. Introduction

Let $(X, \#)$ be a finite orthogonality space with at least one weight and $\mathcal{V}(X, \#)$ the real vector space of signed weights on it. If Δ is a non-empty convex subset of weights on $(X, \#)$ then $(\text{lin}\,\Delta, \Delta)$ is a base normed space. There exists a unique (base norm continuous) linear functional e such that $e(\Delta)=\{1\}$. If we order $(\text{lin}\,\Delta)^*$ as follows: $f \leq g :\Leftrightarrow f(\omega) \leq g(\omega)$ for all $\omega \in \Delta$ $(f, g \in (\text{lin}\,\Delta)^*)$, then the triple $((\text{lin}\,\Delta)^*, \leq, e)$ becomes an order unit normed space.

Define $f_M(\nu) := \sum_{x \in C} \nu(x)$ where $\nu \in \text{lin}\,\Delta$, $M \in \mathcal{L}(X, \#)$ and $C \in \mathcal{O}(X, \#)$ such that $C^{**} = M$, then the elements of the logic of the orthogonality space are represented as linear functionals in the order-interval $[0, e]$. The interval $[0, e]$ is a convex subset of $(\text{lin}\,\Delta)^*$ and $(\text{ext}\,[0, e], \leq)$ is a subposet of $((\text{lin}\,\Delta)^*, \leq)$ with smallest element 0 and largest element e (for definitions see sections 2 and 3).

In this paper we are concerned with the problem of the logic $(\mathcal{L}(X, \#), \leq)$ of a finite orthogonality space $(X, \#)$ being order isomorphic to the poset $(\text{ext}\,[0, e], \leq)$. Key notions in these investigations are a generalized version of the Jordan–Hahn decomposition of signed measures and ultrafulness of the subset of weights under consideration. We study the interplay of these two notions and in these terms we give a necessary and sufficient condition for $(\mathcal{L}(X, \#), \leq)$ to be order isomorphic to the poset $(\text{ext}\,[0, e], \leq)$. If this condition holds then, among other interesting properties that follow from it, the orthogonality space will be a Dacey space and therefore its logic an orthomodular poset.

This study stretches into several branches of mathematics insofar as it relates graph theory and orthomodular structures to convex set theory and certain parts of functional analysis. However, the motivation for this research was originally derived from the foundations of quantum mechanics [10, 11] and empirical logic [4, 5, 12, 13]. Application of these results will be done elsewhere.

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2. Orthogonality spaces

By an *orthogonality space* we mean a pair $(X, \#)$ where X is a non-empty set and $\#$ a non-reflexive symmetric binary relation on X. An *orthogonal set* is defined to be a subset A of X such that $x \# y$ holds for all $x, y \in A$ with $x \neq y$. $\mathcal{O}(X, \#)$ denotes the set of all orthogonal subsets and $\mathcal{E}(X, \#)$ the set of all maximal orthogonal subsets of X. Note that for every $C \in \mathcal{O}(X, \#)$ there exists an $E \in \mathcal{E}(X, \#)$ such that $C \subseteq E$. The mapping $A \to A^* := \{x \in X \mid x \# y \text{ for all } y \in A\}$ for $A \subseteq X$ has the following properties: (i) $A \cap A^* = \emptyset$, (ii) $A \subseteq B$ implies $B^* \subseteq A^*$, (iii) $A \subseteq A^{**}$, (iv) $A^* = A^{***}$, (v) $\emptyset^* = X$ and $X^* = \emptyset$, (vi) $(A \cup B)^* = A^* \cap B^*$. If for two subsets $A, B \subseteq X$ $A \subseteq B^*$ holds, then we say that A is *orthogonal* to B and write $A \# B$.

Intuitively we may think of X as the outcome set for a collection of physical operations (experiments) and of $\#$ as an "operational rejection." The elements of $\mathcal{E}(X, \#)$ may be considered as the operations identified by their possible outcomes and the elements of $\mathcal{O}(X, \#)$ as the events. By a generalized proposition we mean a subset of X consisting of all the outcomes which reject all the outcomes that reject some event. Set-theoretical inclusion of propositions may then be interpreted as "logical implication."

Therefore we refer to $(\mathcal{L}(X, \#), \subseteq)$ where $\mathcal{L}(X, \#) := \{M \subseteq X \mid \text{there exists } C \in \mathcal{O}(X, \#) \text{ with } C^{**} = M\}$ as the *logic* of $(X, \#)$. Note that $X, \emptyset \in \mathcal{L}(X, \#)$ since $\emptyset^{**} = X^* = \emptyset$ and $E^{**} = \emptyset^* = X$ for $E \in \mathcal{E}(X, \#)$. If $M \# N$, $M, N \in \mathcal{L}(X, \#)$, then the supremum $M \vee N$ exists in the poset $(\mathcal{L}(X, \#), \subseteq)$ and is equal to $(C \cup D)^{**}$ where C, D are elements of $\mathcal{O}(X, \#)$ such that $M = C^{**}$ and $N = D^{**}$. Clearly, the mapping $M \to M^*$ is an orthocomplementation for the logic $(\mathcal{L}(X, \#), \subseteq)$ if and only if to every $M \in \mathcal{L}(X, \#)$ there exists a $C \in \mathcal{O}(X, \#)$ such that $M^* = C^{**}$. An orthogonality space is said to be a *Dacey space* [2] provided it has the following property: if x non-$\#$ y, then for every $E \in \mathcal{E}(X, \#)$ there exists a $z \in E$ such that z non-$\#$ x and z non-$\#$ y. Actually one can prove [2, 12]: $(X, \#)$ is a Dacey space if and only if $M \to M^*$ is an orthocomplementation that makes $(\mathcal{L}(X, \#), \subseteq)$ into an orthomodular poset; i.e.: (i) if $M \subseteq N^*$ then $M \vee N$ exists in $(\mathcal{L}(X, \#), \subseteq)$ and (ii) if $M \subseteq N$ then there exists an element $P \in \mathcal{L}(X, \#)$ such that $M \subseteq P^*$ and $M \vee P = N$ [3]. Every orthocomplete orthomodular poset arises from a (however not uniquely determined) orthogonality space.

An example of a Dacey space which is indeed prototypic for the mathematical foundations of quantum mechanics is given as follows: Let $\mathcal{N}$ be a von Neumann algebra. Denote with $\mathcal{P}$ the set of non-zero, orthogonal projections in $\mathcal{N}$ (i.e.: $0 \neq P = P \cdot P^* = P^*$). Standard arguments show that $(\mathcal{P}, \#)$ where $P_1 \# P_2 :\Leftrightarrow P_1 \cdot P_2 = 0$ is a Dacey space. Furthermore, the orthomodular poset $(\mathcal{L}(X, \#), \subseteq, {}^*)$ is ortho-order isomorphic to the projection lattice of $\mathcal{N}$ with $P \to 1 - P$ as orthocomplementation.

We will assume that the orthogonality spaces that appear in the sequel have *finite cardinality*.

By a *weight* ω on the orthogonality space $(X, \#)$ we mean a mapping $\omega: X \to [0, 1]$ such that $\sum_{x \in E} \omega(x) = 1$ for all $E \in \mathscr{E}(X, \#)$. A *signed weight* ν on $(X, \#)$ is a mapping $\nu: X \to \mathbf{R}$ which has the property that $\sum_{x \in E} \nu(x)$ $(E \in \mathscr{E}(X, \#))$ is independent of the particular choice of $E \in \mathscr{E}(X, \#)$. The set $\mathscr{V}(X, \#)$ of all signed weights on $(X, \#)$ is a real vector space (addition and scalar multiplication defined as usual), the *signed weight space* of $(X, \#)$. The set of all weights $\Omega(X, \#)$ forms a convex subset of $\mathscr{V}(X, \#)$. $\Omega(X, \#)$ is either empty, contains exactly one element or contains infinitely many elements [6, 7]. Let Δ be a convex subset of $\Omega(X, \#)$. A weight $\omega \in \Delta$ is called *pure* (*with respect to* Δ) if $\omega = t\omega_1 + (1-t)\omega_2$ $(\omega_1, \omega_2 \in \Delta, t \in (0, 1))$ implies that $\omega = \omega_1 = \omega_2$. A weight that is not pure is called a *mixture*. We can easily extend the signed weights to functions on $\mathscr{O}(X, \#)$ by putting $\nu(\varnothing) := 0$ and $\nu(C) := \sum_{x \in C} \nu(x)$ $(C \in \mathscr{O}(X, \#) \setminus \{\varnothing\})$.

We can consider the weight functions as complete stochastic models for the experimental setting given by $(X, \#)$, in the sense that $\omega(x)$ is the "long run relative frequency" with which the outcome x occurs as a result of the execution of an operation for which x is an outcome.

We say that $\Delta \subseteq \Omega(X, \#)$ is a *full* (resp. *strong*) set of weights for the orthogonality space $(X, \#)$, if for all pairs $x, y \in X$ with x non-$\#$ y there exists an $\omega \in \Delta$ such that $1 < \omega(x) + \omega(y)$ (resp. $\omega(x) = 1$ and $\omega(y) \neq 0$). The subset Δ is said to be *unital* for $(X, \#)$ provided for every $x \in X$ there exists an $\omega \in \Delta$ such that $\omega(x) = 1$. Let $C \in \mathscr{O}(X, \#)$ and $\Delta \subseteq \Omega(X, \#)$, we define $C^1 := \{\omega \in \Delta \mid \omega(C) = 1\}$ and $C^0 := \{\omega \in \Delta \mid \omega(C) = 0\}$. The subset Δ is said to be an *ultrafull* set of weights for $(X, \#)$ if (i) Δ is a full set of weights for $(X, \#)$ and (ii) for all $C, D \in \mathscr{O}(X, \#)$, $C^1 \subseteq D^1$, $C^0 \subseteq D^0$ implies that $D^1 \subseteq C^1$, $D^0 \subseteq C^0$. We have the following implications: strong $\Rightarrow$ ultrafull $\Rightarrow$ full and strong $\Rightarrow$ unital.

We need the following lemma [4]:

LEMMA 2.1. *Suppose that* $(X, \#)$ *admits a full set of weights* Δ. *Then* $(X, \#)$ *is a Dacey space.*

Proof. Assume that for $x, y \in X$, $E \in \mathscr{E}(X, \#)$, $E \subseteq \{x\}^* \cup \{y\}^*$ holds. Denote $C := E \cap \{x\}^*$ and $D := E \setminus C$. Then $\omega(E) = 1 = \omega(C) + \omega(D)$ for all $\omega \in \Delta$. Clearly, $C \subseteq \{x\}^*$ and $D \subseteq \{y\}^*$. Let $F, G \in \mathscr{E}(X, \#)$ such that $C \cup \{x\} \subseteq F$ and $D \cup \{y\} \subseteq G$. Then $\omega(C) + \omega(x) \leq \omega(F) = 1$ and $\omega(D) + \omega(y) \leq \omega(G) = 1$. We add these inequalities and get $1 + \omega(x) + \omega(y) \leq 2$ or $\omega(x) + \omega(y) \leq 1$ for all $\omega \in \Delta$. Since Δ is full we conclude that $x \# y$.

Examples show that the converse of Lemma 2.1 is not true [7]. Again let $\Delta \subseteq \Omega(X, \#)$. We say $\nu \in \operatorname{lin} \Delta$ admits a *Jordan–Hahn decomposition* (*with respect* *to* Δ) [9, 16] provided there exist $D \in \mathcal{O}(X, \#)$, $\omega_1, \omega_2 \in \Delta$, $t_1, t_2 \geq 0$ such that $\omega_1(D) = 1$, $\omega_2(D) = 0$ and $\nu = t_1\omega_1 - t_2\omega_2$. The subset Δ is said to have the *Jordan–Hahn property* if each element $\nu \in \operatorname{lin} \Delta$, $\nu \neq t\omega$ $(t \in \mathbf{R}, \omega \in \Delta)$ admits a Jordan–Hahn decomposition.

A *signed state* μ on the logic $\mathcal{L}(X, \#)$ is a mapping $\mu : \mathcal{L}(X, \#) \to \mathbf{R}$ satisfying (i) $\mu(\emptyset) = 0$, (ii) if $M \# N$ then $\mu(M \vee N) = \mu(M) + \mu(N)$. A signed state μ for which $\mu(X) = 1$ and $\mu(\mathcal{L}(X, \#)) \subseteq [0, 1]$ is called a *state*. In section 5 we establish a connection between signed weights and signed states, resp. weights and states.

3. Base normed and order unit normed spaces

In the present section we give the definitions and the basic properties of base normed spaces and order unit normed spaces [1, 14, 15] and prove two theorems concerning the extreme points of the unit ball of an order unit normed space. We will end this section with a list of facts on convex sets in finite dimensional vector spaces. We rely on these results in the following sections.

Consider a pair (E, Δ) where E is a real vector space and Δ a non-empty convex subset of E such that $K := \{\nu \in E \mid \nu = t\omega,\ \omega \in \Delta \text{ and } t \geq 0\}$ is a generating cone for E with Δ as a base. Then $U := \operatorname{con}(\Delta \cup -\Delta)$ $(= \{t\omega_1 - (1-t)\omega_2 \mid \omega_1, \omega_2 \in \Delta \text{ and } t \in [0, 1]\})$ is convex, circled and absorbing and the corresponding Minkowski functional $\|\nu\|_B := \inf\{t \geq 0 \mid \nu \in t \cdot U\}$ becomes a seminorm. If $\|\cdot\|_B$ is indeed a norm for E, then we call (E, Δ) a *base normed space* and refer to $\|\cdot\|_B$ as the *base norm*. Let B_B denote the unit ball of the base normed space (E, Δ), then $B_B^0 \subseteq U \subseteq B_B$. Note that there is exactly one base norm continuous linear functional e such that $e(\Delta) = \{1\}$. Also note that $\|\omega\|_B = 1$ for all $\omega \in \Delta$.

Let $(F, \leq)$ be an ordered real vector space. An element $e \geq 0$ is called an *order unit* provided $\bigcup_{n=1}^{\infty} [-ne, ne] = F$. Clearly, $[-e, e]$ is convex, circled and absorbing. A triple $(F, \leq, e)$ where F is a real vector space, $\leq$ a partial order on F and $e \in F$ an order unit such that the Minkowski functional $\|\cdot\|_O$ on $[-e, e]$ becomes a norm is called an *order unit normed space*; $\|\nu\|_O$ is called the *order unit norm*. Similarly, $B_O^0 \subseteq [-e, e] \subseteq B_O$ where B_O denotes the unit ball in the order unit normed space $(F, \leq, e)$.

Let (E, Δ) be a base normed space, E^* the Banach dual (usual sup-norm) and e the unique base norm continuous linear functional such that $e(\Delta) = \{1\}$. Then $f \leq g :\Leftrightarrow f(\omega) \leq g(\omega)$ for all $\omega \in \Delta$ $(f, g \in E^*)$ is an ordering on E^* and e is an order unit of $(E^*, \leq)$ such that $(E^*, \leq, e)$ becomes an order unit normed space with $\|f\|_O = \sup_{\nu \in B_B} |f(\nu)|$. Note that $[-e, e] = B_O$.

Let $(F, \leq, e)$ be an order unit normed space. Clearly, $[0, e]$ is closed under the mapping $f \to f' := e - f$. A point in a convex subset P of a vector space is called *extreme* provided it is not properly contained in a line segment whose endpoints lie in this convex subset. The set of extreme points of P is denoted by $\operatorname{ext} P$.

LEMMA 3.1. *Let $(F, \leq, e)$ be an order unit normed space. If $f \in \operatorname{ext}[0, e]$ then $f' \in \operatorname{ext}[0, e]$. Moreover $0, e \in \operatorname{ext}[0, e]$.*

Proof. Let $f \in \operatorname{ext}[0, e]$. Assume that $f' = tg + (1-t)h$ where $g, h \in [0, e]$ and $t \in (0, 1)$. Then $f = e - f' = te + (1-t)e - (tg + (1-t)h) = t(e-g) + (1-t)(e-h) = tg' + (1-t)h'$. Since $g', h' \in [0, e]$, $t \in (0, 1)$ and $f \in \operatorname{ext}[0, e]$ we conclude that $f = g' = h'$, hence $f' = g = h$. Therefore $f' \in \operatorname{ext}[0, e]$.

Assume now that $e \notin \operatorname{ext}[0, e]$. Then there exist $f, g \in [0, e]$, $f \neq g$ such that $e = \frac{1}{2}f + \frac{1}{2}g$, hence $f = 2e - g$. But $e \leq 2e - g = f \leq e$, hence $f = e$ and similarly $g = e$ which is a contradiction. Hence $e \in \operatorname{ext}[0, e]$, and thus $0 = e' \in \operatorname{ext}[0, e]$.

LEMMA 3.2. *Let $(F, \leq, e)$ be an order unit normed space such that $[-e, e] = B_O$. If $f \in \operatorname{ext}[0, e]$ and $f \neq 0$ then $\|f\|_O = 1$.*

Proof. Let $f \in [0, e]$, $f \neq 0$ and suppose that $\|f\|_O \neq 1$. Note that $f/\|f\|_O \in B_O = [-e, e]$ and $0 \leq f$, hence $f/\|f\|_O \in [0, e]$. Since $[0, e] \subseteq [-e, e] = B_O$ we get $\|f\|_O < 1$. Now $f = \|f\|_O(f/\|f\|_O) + (1 - \|f\|_O)0$ with $f/\|f\|_O \neq 0$ and $\|f\|_O \in (0, 1)$. Therefore $f \notin \operatorname{ext}[0, e]$.

THEOREM 3.3. *Let $(F, \leq, e)$ be an order unit normed space such that $[-e, e] = B_O$. Then $(\operatorname{ext}[0, e], \leq)$ is an orthocomplemented poset with $f \to f'$ as orthocomplementation.*

Proof. $(\operatorname{ext}[0, e], \leq)$ is a poset with smallest element 0 and greatest element e; furthermore $f \to f'$ is an involution on this poset. It remains to show that for all $f \in \operatorname{ext}[0, e]$, $f \wedge f'$ exists in $(\operatorname{ext}[0, e], \leq)$ and is equal to 0. We have $0 \leq f, f'$. Assume that there is an element $g \in [0, e]$, $g \neq 0$ such that $g \leq f, f'$ then $0 < g \leq \frac{1}{2}f + \frac{1}{2}f' = \frac{1}{2}(f + f') = \frac{1}{2}e$. Therefore $0 \neq \|g\|_O \leq \frac{1}{2}$. Hence $g \notin \operatorname{ext}[0, e]$ by lemma 3.2. This proves the theorem.

LEMMA 3.4. *Let $(F, \leq, e)$ be an order unit normed space. The mapping $\Phi : F \to F$, given by $\Phi(f) := f - f'$, is an order isomorphism from $[0, e]$ onto $[-e, e]$ and $\Phi^{-1}(g) = (g + e)/2$. Moreover, the mappings Φ and Φ^{-1} preserve convex combinations and $\Phi(f') = -\Phi(f)$.*

Proof. Note that $\Phi(f) = f - f' = 2f - e$.

Under the mapping $f \to -f$ $([-e, e], \leq)$ becomes an involution poset with smallest element $-e$ and greatest element e. The foregoing results lead immediately to

THEOREM 3.5. *Let* $(F, \leq, e)$ *be an order unit normed space such that* $[-e, e] = B_O$. *Then* $f \to -f$ *makes* $(\mathrm{ext}[-e, e], \leq)$ *into an orthocomplemented poset* (e: *greatest element,* $-e$: *smallest element*) *ortho-order isomorphic to the orthocomplemented poset* $(\mathrm{ext}[0, e], \leq, ')$. *The ortho-order isomorphism is given by* Φ.

In view of the results in the last section of this paper, we would like to point out that examples of order unit normed spaces for which $(\mathrm{ext}[0, e], \leq, ')$ is not an orthomodular poset are easily constructed.

We now give some basic facts on finite dimensional convex subsets and their facial structure (see e.g. [8]).

Let E be a finite dimensional real vector space and P a convex subset of E. A convex subset F of P is said to be a *face* of P provided $t\nu_1 + (1-t)\nu_2 \in F$, $\nu_1, \nu_2 \in P$, $t \in (0, 1)$ implies that $\nu_1, \nu_2 \in F$. Note that $\varnothing$ and P are faces. A face $F \neq \varnothing, P$ is called *proper*. If F_1 is a face of F_2 and F_2 is a face of P, then F_1 is a face of P. Clearly, if ν is an extreme point of P then $\{\nu\}$ is a face of P. A face that is maximal in the poset $(\{F \mid F \text{ face of } P, F \neq P\}, \subseteq)$ is called a *facet* of P. Assume that for $f \in E^*$ there exists $s \in \mathbf{R}$ such that $f(P) \subseteq (-\infty, s]$ then $f^{-1}(s) \cap P$ is a face of P. A face that arises in this manner is called *exposed*. Correspondingly, $\nu \in P$ is called an *exposed point* of P if $\{\nu\}$ is an exposed face; $\exp P$ denotes the set of exposed points of P.

Let P be a non-empty compact convex subset of E. Then $\varnothing \neq \exp P \subseteq \mathrm{ext}\, P$ and $P = \mathrm{con\,ext}\, P = \mathrm{cl\ con}\ \exp P$ (Theorem of Minkowski-Carathéodory and theorem of Straszewicz). P is said to be a *polytope* provided $\mathrm{ext}\, P$ is finite. Note that in the case of a polytope every face is an exposed face: in particular $\mathrm{ext}\, P = \exp P$. A polytope may be equivalently defined as the convex hull of a finite, non-empty set or as a bounded, non-empty set which is the intersection of finitely many closed half spaces. Since the convex hull of a compact set is closed we can say a non-empty compact convex subset P of E is a polytope if and only if $\exp P$ is finite. Note that a proper face of a polytope is a polytope in its own right. A face of a polytope is a facet if and only if $\dim F = \dim P - 1$ (affine dimensions).

4. The signed weight space

For the remainder of this paper we assume that the (finite) orthogonality spaces $(X, \#)$ under consideration possess at least one weight. We are going to study the base normed and order unit normed spaces "generated" by suitable subsets of $\Omega(X, \#)$.

With every $x \in X$ we can associate a linear functional f_x on $\mathscr{V}(X, \#)$ by defining $f_x(\nu) := \nu(x)$, $\nu \in \mathscr{V}(X, \#)$. The linear functional $e := \sum_{x \in E} f_x$ is independent of the particular choice of $E \in \mathscr{E}(X, \#)$, by definition of a signed weight, and $e(\omega) = 1$ for all $\omega \in \Omega(X, \#)$. Clearly, $f_x(\nu) = 0$, all $x \in X$, implies that $\nu = 0$, hence $\{f_x \mid x \in X\}$ is a finite total set of linear functionals on $\mathscr{V}(X, \#)$ and therefore $\mathscr{V}^*(X, \#)$ and finally $\mathscr{V}(X, \#)$ is finite dimensional. A local base for the unique compatible Hausdorff topology on $\mathscr{V}(X, \#)$ (e.g., Euclidean topology) is given by the sets $\{N(\varepsilon, x) \mid x \in X,\ \varepsilon \geq 0\}$ where $N(\varepsilon, x) = \{\nu \in \mathscr{V}(X, \#) \mid |f_x(\nu)| < \varepsilon\}$ together with their finite intersections. Note that $\nu_n \to \nu$ in the Euclidean topology if and only if $\nu_n(x) \to \nu(x)$ for all $x \in X$. This shows that $\Omega(X, \#)$ is a closed subset of $\mathscr{V}(X, \#)$.

THEOREM 4.1. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$. Then* $(\operatorname{lin} \Delta, \Delta)$ *is a base normed space.*

Proof. The set $K := \{\nu \in \operatorname{lin} \Delta \mid \nu = t\omega,\ t \geq 0,\ \omega \in \Delta\}$ is a generating cone for $\operatorname{lin} \Delta$ since Δ is convex and $0 \notin \Delta$. To show that Δ is a base for K, assume that $t_1\omega_1 = t_2\omega_2$ where $t_1, t_2 > 0$ and $\omega_1, \omega_2 \in \Delta$. Then $t_1 = e(t_1\omega_1) = e(t_2\omega_2) = t_2$ since $e(\omega_1) = e(\omega_2) = 1$. Thus $\omega_1 = \omega_2$.

To prove that the Minkowski functional on $U = \operatorname{con}(\Delta \cup -\Delta)$ is indeed a norm, it is enough to show that $\|\nu\|_B = 0$ implies that $\nu = 0$. So assume that $\|\nu\|_B = 0$. Then $\|t\nu\|_B = t\|\nu\|_B = 0$ for all $t \in \mathbf{R}$. But U is circled, hence $t\nu \in 1 \cdot U$. Therefore $|f_x(t\nu)| \leq 1$ or $|f_x(\nu)| \leq 1/t$ for all $t > 0$ and $x \in X$. Thus $f_x(\nu) = 0$ for all $x \in X$. The set $\{f_x \mid x \in X\}$ being total, we conclude that $\nu = 0$.

The linear functional e takes the value 1 on Δ and therefore, in its restriction to $\operatorname{lin} \Delta$, serves as an order unit in $((\operatorname{lin} \Delta)^*, \leq)$ that makes $((\operatorname{lin} \Delta)^*, \leq, e)$ into the order unit normed space corresponding to $(\operatorname{lin} \Delta, \Delta)$ (see section 3).

If Δ is closed (e.g., $\omega_n(x) \to \nu(x)$ for all $x \in X$, $\omega_n \in \Delta$, implies that $\nu \in \Delta$) then $U = \operatorname{con}(\Delta \cup -\Delta)$ is closed and, by the introductory remarks in section 3, $U = B_B$. It should be noted that $B_B (\subseteq \operatorname{lin} \Delta)$ and $B_O = [-e, e] (\subseteq (\operatorname{lin} \Delta)^*)$ are compact since they are unit balls of finite dimensional normed vector spaces. The map Φ^{-1} is affine, thus $[0, e] = \Phi^{-1}[-e, e]$ is compact too. Also note that since $e(\Delta) = \{1\}$ we have $0 \notin \operatorname{aff} \Delta$ and therefore $\dim U = \dim \Delta + 1$.

Now we are going to investigate the relation between exposed linear functionals of $[-e, e]$ and a certain class of faces of U. As we will see in the next section, this relationship in presence of the Jordan–Hahn property, plays a fundamental role.

LEMMA 4.2. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$. Then $f \in [-e, e]$ is an exposed point of $[-e, e]$ if and only if there exists an element $\nu \in \operatorname{lin} \Delta$ with $\|\nu\|_B = 1$ such that $\{f\} = \{g \in [-e, e] \mid g(\nu) = 1\}$.*

Proof. Assume that $f \in \exp[-e, e]$. By definition there exists $\hat{\nu} \in (\operatorname{lin} \Delta)^{**}$ and $s \in \mathbf{R}$ such that $\hat{\nu}([-e, e]) \subseteq (-\infty, s]$ and $\{f\} = \{g \in [-e, e] \mid \hat{\nu}(g) = s\}$. Clearly, $\sup_{h \in [-e, e]} \hat{\nu}(h) = s$. Note that $\hat{\nu} \neq 0$ since $\{f\} \neq [-e, e]$. Recall that $(\operatorname{lin} \Delta)^{**}$ in the sup-norm is isometric to $\operatorname{lin} \Delta$ in the base norm under the evaluation map. Therefore there exists a (unique) $\nu \in \operatorname{lin} \Delta$, $\nu \neq 0$ such that $\hat{\nu}(h) = h(\nu)$ for all $h \in (\operatorname{lin} \Delta)^*$. Since $-[-e, e] = [-e, e]$ we get $\|\nu\|_B = \sup_{h \in [-e,e]} |\hat{\nu}(h)| = \sup_{h \in [-e,e]} \hat{\nu}(h) = s$. Now $\nu_0 := \nu/s$ is the desired element since $\|\nu_0\|_B = 1$, $f(\nu_0) = f(\nu)/s = \hat{\nu}(f)/s = s/s = 1$ and if $g(\nu_0) = 1$, $g \in [-e, e]$ then $\hat{\nu}(g) = g(\nu) = s$ hence $g = f$. The converse follows immediately.

LEMMA 4.3. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$ and $f \in [0, e]$. Then*

(i) *$f^{-1}(1) \cap \Delta$, $f^{-1}(0) \cap \Delta$ are exposed faces of Δ and U; furthermore $f^{-1}(1) \cap \Delta = \Phi(f)^{-1}(1) \cap \Delta$ and $f^{-1}(0) \cap \Delta = -[\Phi(f)^{-1}(1) \cap -\Delta]$.*

(ii) *$\Phi(f)^{-1}(1) \cap U$, $\Phi(f)^{-1}(-1) \cap U$ are exposed faces of U different from U; furthermore*
$\Phi(f)^{-1}(1) \cap U = \operatorname{con}[(f^{-1}(1) \cap \Delta) \cup -(f^{-1}(0) \cap \Delta)]$ and $\Phi(f)^{-1}(-1) \cap U = -[\Phi(f)^{-1}(1) \cap U]$.

The proof of Lemma 4.3 is straightforward and is omitted.

THEOREM 4.4. *Let Δ be a non-empty closed convex subset of $\Omega(X, \#)$. If $f \in \exp[-e, e]$ then $f^{-1}(1) \cap U$ is a maximal proper exposed face of U.*

Proof. Let $f \in \exp[-e, e]$ and $\nu \in \operatorname{lin} \Delta$ as in Lemma 4.2. Then $\nu \in f^{-1}(1) \cap U$ since $U = B_B$. Note that $f^{-1}(1) \cap U$ is an exposed face of U different from U, by Lemma 4.3. Assume now that $f^{-1}(1) \cap U \subseteq F$ where F is an exposed face $\neq U$. Then there exists an element $g \in (\operatorname{lin} \Delta)^*$ and $s \in \mathbf{R}$ such that $g(U) \subseteq (-\infty, s]$ and $g^{-1}(s) \cap U = F$. Note that $g(\nu) = s$. Then $\|g\|_O = \sup_{\nu \in U} |g(\nu)| = \sup_{\nu \in U} g(\nu) = s$ since $U = -U$. Clearly, $s \neq 0$, else $g = 0$ and $F = U$. Now $g/s \in [-e, e]$ and $(g/s)(\nu) = 1$ hence $g/s = f$ since $f \in \exp[-e, e]$. Therefore, $f^{-1}(1) \cap U = (g/s)^{-1}(1) \cap U = g^{-1}(s) \cap U = F$.

5. The logic of an orthogonality space

Assume again that Δ is a non-empty convex subset of $\Omega(X, \#)$. Each element of $\mathcal{O}(X, \#)$ can be represented as a linear functional on $\operatorname{lin} \Delta$ by defining $f_C := \sum_{x \in C} f_x$, $C \in \mathcal{O}(X, \#) \backslash \{\phi\}$ and $f_\phi = 0$. Clearly, $f_C(\nu) = \nu(C)$ and $f_E = e$ for all $E \in \mathscr{E}(X, \#)$. Since $0 \leq \omega(C) \leq 1$ for all $\omega \in \Delta$ it follows that each f_C is contained in the order interval $[0, e]$ of the order unit normed space $((\operatorname{lin} \Delta)^*, \leq, e)$. Note that

$C^1 = f_C^{-1}(1) \cap \Delta$ and $C^0 = f_C^{-1}(0) \cap \Delta$. Therefore, by Lemma 4.3, C^1 and C^0 are faces of Δ.

Next we are going to define an order morphism from the logic $(\mathscr{L}(X, \#), \subseteq)$ into the poset $([0, e], \leq)$. To do so we need the following lemma.

LEMMA 5.1. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$ and $C, D \in \mathcal{O}(X, \#)$. If $C^{**} \subseteq D^{**}$ then $f_C \leq f_D$.*

Proof. Since $C^{**} \subseteq D^{**}$ we get $D^* = D^{***} \subseteq C^{***} = C^*$. Let $E \in \mathscr{E}(X, \#)$ such that $D \subseteq E$. Then $E \setminus D \subseteq D^*$ hence $(E \setminus D) \# C$. Thus there exists an element $F \in \mathscr{E}(X, \#)$ such that $(E \setminus D) \cup C \subseteq F$. Let $\omega \in \Delta$ then $\omega[(E \setminus D) \cup C] = \omega(E \setminus D) + \omega(C) = \omega(E) - \omega(D) + \omega(C) \leq \omega(F)$. Since $\omega(E) = \omega(F) = 1$ we get $\omega(C) \leq \omega(D)$ or equivalently $f_C(\omega) \leq f_D(\omega)$ for all $\omega \in \Delta$. Hence $f_C \leq f_D$.

Now let $M \in \mathscr{L}(X, \#)$ and define $f_M := f_C$ where $C \in \mathcal{O}(X, \#)$ such that $M = C^{**}$. Due to Lemma 5.1, $M \to f_M$ is a mapping. Indeed, it is an order morphism from the logic $(\mathscr{L}(X, \#), \subseteq)$ into the poset $([0, e], \leq)$; the image is denoted by $\mathscr{L}_f(X, \#)$. We have $f_X = f_E = e$.

LEMMA 5.2. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$ and let $M, N \in \mathscr{L}(X, \#)$. Then*

(i) *if $M \# N$ then $f_{M \vee N} = f_M + f_N$ and $f_M \leq f'_N$;*
(ii) *if $\mathscr{L}(X, \#)$ is *-closed then $f'_M = f_{M^*}$.*

Proof. (i) There exist $C, D \in \mathcal{O}(X, \#)$ such that $M = C^{**}$ and $N = D^{**}$. Now $C^{**} \# D^{**}$ or $C \subseteq C^{**} \subseteq D^{***} = D^*$. Hence $C \cap D = \varnothing$ and $C \cup D \in \mathcal{O}(X, \#)$. $f_{C \cup D}(\omega) = \sum_{x \in C \cup D} \omega(x) = \sum_{x \in C} \omega(x) + \sum_{x \in D} \omega(x) = f_C(\omega) + f_D(\omega) = (f_C + f_D)(\omega)$ for all $\omega \in \Delta$. Therefore $f_{C \cup D} = f_C + f_D$ and finally, since $(C \cup D)^{**} = M \vee N$, we get $f_{M \vee N} = f_M + f_N$. But $f_{M \vee N} \leq e$, hence $f_M \leq f'_N$.

(ii) If $\mathscr{L}(X, \#)$ is *-closed, then $M \to M^*$ is an orthocomplementation (see Section 2). Now $M \# M^*$ and thus $M \vee M^* = X$. By (i) $e = f_X = f_{M \vee M^*} = f_M + f_{M^*}$. Thus $f_{M^*} = e - f_M = f'_M$.

Using Lemma 5.2 (i), one easily verifies that $\mu_\nu(M) := f_M(\nu)$ defines a signed state on $\mathscr{L}(X, \#)$. Furthermore μ_ν $(\nu \in \operatorname{lin} \Delta)$ is a state on $\mathscr{L}(X, \#)$ if and only if $\nu \in \Delta$.

THEOREM 5.3. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$. The mapping $f: \mathscr{L}(X, \#) \to \mathscr{L}_f(X, \#)$ is an order isomorphism if and only if Δ is a full set of weights for $(X, \#)$.*

Proof. Assume that $f: \mathscr{L}(X, \#) \to \mathscr{L}_f(X, \#)$ is an order isomorphism. If $\omega(x) + \omega(y) \leq 1$ for all $\omega \in \Delta (x, y \in X)$, then $\omega(x) + \omega(y) \leq 1 = \omega(x) + \omega(E - x)$ where $E \in \mathscr{E}(X, \#)$ with $x \in E$. Hence $\omega(y) \leq \omega(E - x)$ for all $\omega \in \Delta$, thus $f_y \leq f_{E-x}$ or $f_{\{y\}^{**}} \leq f_{(E-x)^{**}}$. The mapping in question being an order isomorphism, we get $\{y\}^{**} \subseteq (E-x)^{**}$ or equivalently $(E-x)^* \subseteq \{y\}^*$, thus $x \# y$. This proves that Δ is a full set of weights for $(X, \#)$.

Conversely, it remains to show that $f_M \leq f_N$ implies that $M \subseteq N$. By assumption Δ is full, thus, by Lemma 2.1, $(X, \#)$ is a Dacey space and therefore $\mathscr{L}(X, \#)$ is *-closed. Now let $f_M \leq f_N$. Then $f_N = f_{N^{**}} = f'_{N^*}$, by Lemma 5.2 (ii). Thus $f_M + f_{N^*} \leq e$. Now let $C, D \in \mathcal{O}(X, \#)$ be such that $M = C^{**}$ and $N^* = D^{**}$, then $f_C + f_D \leq e$ or equivalently $\omega(C) + \omega(D) \leq 1$, for all $\omega \in \Delta$. Therefore $x \# y$ for all $x \in C$, $y \in D$. Thus $C \# D$ and therefore $M = C^{**} \subseteq D^* = N^{**} = N$.

The next two theorems serve us as a key for the main results. They give an equivalent for the Jordan–Hahn property of a non-empty closed convex subset Δ of $\Omega(X, \#)$ in terms of the extremal linear functionals in $[0, e]$.

THEOREM 5.4. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$. If Δ has the Jordan–Hahn property then* $\operatorname{ext}[0, e] \subseteq \mathscr{L}_f(X, \#)$ *and the sets Δ, U, $[0, e]$, $[-e, e]$ are polytopes.*

Proof. Let $g \in \exp[-e, e]$. By Lemma 4.2 there exists $\nu \in \operatorname{lin} \Delta$ with $\|\nu\|_B = 1$ and $\{g\} = \{h \in [-e, e] \mid h(\nu) = 1\}$. If $\nu = t\omega$ $(t \in \mathbf{R}, \omega \in \Delta)$ then $\nu = \pm \Delta$, hence $g = \pm e \in \mathscr{L}_f(X, \#)$. So assume that $\nu \neq t\omega$. Since Δ has the Jordan–Hahn property, there exist $D \in \mathcal{O}(X, \#)$, $\omega_1, \omega_2 \in \Delta$, $t_1, t_2 \geq 0$ such that $\nu = t_1\omega_2 - t_2\omega_2$ and $f_D(\omega_1) = 1$, $f_D(\omega_2) = 0$. Recall that $(\operatorname{lin} \Delta)^{**}$ is isometric to $\operatorname{lin} \Delta$ (sup-norm, base norm) and note that $-[-e, e] = [-e, e]$, then $t_1 + t_2 = \Phi(f_D)(\nu) \leq \sup_{h \in [-e, e]} h(\nu) = \|\nu\|_B = 1 \leq t_1\|\nu_1\|_B + t_2\|\nu_2\|_B = t_1 + t_2$. Hence $\Phi(f_D)(\nu) = 1$. Since $g \in \exp[-e, e]$ we conclude that $\Phi(f_D) = g$. Therefore $\exp[-e, e] \subseteq \Phi(\mathscr{L}_f(X, \#))$. The set on the right-hand side being finite entails that the compact convex set $[-e, e]$ has finitely many exposed points, hence is a polytope and $\operatorname{ext}[-e, e] = \exp[-e, e]$. By Theorem 3.5, $\operatorname{ext}[0, e] = \Phi^{-1}(\operatorname{ext}[-e, e])$ thus $[0, e]$ is also a polytope and finally $\operatorname{ext}[0, e] \subseteq \mathscr{L}_f(X, \#)$.

We have also shown that any $\nu \in \operatorname{lin} \Delta$ with $\|\nu\|_B = 1$ can be represented as $\nu = t\omega_1 - (1-t)\omega_2$ where $\omega_1, \omega_2 \in \Delta$ and $t \in [0, 1]$. Since $B_B^0 \subseteq U = \operatorname{con}(\Delta \cup -\Delta) \subseteq B_B$ we conclude that $U = B_B$.

Next we show that Δ and U are polytopes. Since $\|\nu\|_B = \sup_{f \in [-e,e]} f(\nu)$ and $f(B_B) \subseteq [-1, 1]$ for all $f \in [-e, e]$ we get $U = \bigcap_{f \in [-e,e]} f^{-1}[-1, 1]$. Since $[-e, e]$ is a polytope we have $[-e, e] = \operatorname{con}\operatorname{ext}[-e, e]$ which implies that $U = \bigcap_{f \in \operatorname{ext}[-e,e]} f^{-1}[-1, 1]$. Obviously, $\Delta = e^{-1}(1) \cap \operatorname{con}(\Delta \cup -\Delta) = e^{-1}(1) \cap U$. Thus $\Delta = \bigcap_{f \in \operatorname{ext}[-e,e]} f^{-1}[-1, 1] \cap e^{-1}(-\infty, 1] \cap e^{-1}[1, +\infty)$. Thus the non-empty, bounded sets Δ and U, both are equal to the intersection of finitely many closed half spaces. Therefore Δ and U are polytopes.

THEOREM 5.5. *Let Δ be a non-empty closed convex subset of $\Omega(X, \#)$. If* $\operatorname{ext}[0, e] \subseteq \mathcal{L}_f(X, \#)$ *then Δ has the Jordan–Hahn property.*

Proof. Let $\nu \in \operatorname{lin}\Delta$, $\nu \neq t\omega$ $(t \in \mathbf{R},\ \omega \in \Delta)$. We have $0 \neq \|\nu\|_B = \sup_{f \in [-e,e]} f(\nu)$. Since $[-e, e]$ is compact, the supremum is attained at an extreme point. Considering that $\operatorname{ext}[-e, e] \subseteq \Phi(\mathcal{L}_f(X, \#))$, there exists an element $C \in \mathcal{O}(X, \#)$ such that $\Phi(f_C)(\nu) = \|\nu\|_B$. The set Δ being closed implies that $B_B = \operatorname{con}(\Delta \cup -\Delta)$. Therefore there exist $\omega_1, \omega_2 \in \Delta$ and $t \in (0, 1)$ such that $\nu/\|\nu\|_B = t\omega_1 - (1-t)\omega_2$. Now $1 = \Phi(f_D)(\nu/\|\nu\|_B) = t\Phi(f_D)(\omega_1) + (1-t)\Phi(f_D)(-\omega_2)$ and since $\Phi(f_D)(B_B) \subseteq [-1, 1]$ we conclude that $\Phi(f_D)(\omega_1) = 1$ and $\Phi(f_D)(\omega_2) = -1$. Thus $f_D(\omega_1) = [(\Phi(f_D) + e)/2](\omega_1) = 1$ and similarly $f_D(\omega_2) = 0$. Now $\omega_1(D) = 1$, $\omega_2(D) = 0$ $(\omega_1, \omega_2 \in \Delta)$,

$$\nu = t\,\|\nu\|_B\omega_1 - (1-t)\,\|\nu\|_B\omega_2 \quad \text{and} \quad t\,\|\nu\|_B, \qquad (1-t)\,\|\nu\|_B \geq 0.$$

This establishes a Jordan–Hahn decomposition for ν.

EXAMPLE 1. We give a simple example of an orthogonality space $(X, \#)$ with a non-empty closed convex subset of weights that is strong but fails to have the Jordan–Hahn property.

Let $X := \{x_1, x_2, x_3, x_4\}$ with $x_1 \# x_2$ and $x_3 \# x_4$ (Convention: for pairs not mentioned here, the relation $\#$ fails to hold). Define $\omega_1(x) := (1$ for $x = x_1$, 0 for $x = x_2$, $\frac{1}{2}$ for $x = x_3$, $\frac{1}{2}$ for $x = x_4)$, $\omega_2(x) := (0, 1, \frac{1}{2}, \frac{1}{2})$, $\omega_3(x) := (\frac{1}{2}, \frac{1}{2}, 1, 0)$, $\omega_4(x) := (\frac{1}{2}, \frac{1}{2}, 0, 1)$. ω_1, ω_2, ω_3, ω_4 are weights for $(X, \#)$ and $\Delta := \{t_1\omega_1 + t_2\omega_2 + t_3\omega_3 + t_4\omega_4 \mid \sum_{i=1}^4 t_i = 1,\ t_i \geq 0\}$ is a non-empty closed convex subset of $\Omega(X, \#)$. Using ω_i $(i = 1, 2, 3, 4)$ one immediately shows that Δ is a strong set of weights for $(X, \#)$. One easily checks that the signed weight ν defined by $\nu(x) := (\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4})$ is an element of $\operatorname{lin}\Delta$, but does not admit a Jordan–Hahn decomposition with respect to Δ.

THEOREM 5.6. *Let Δ be a non-empty convex subset of $\Omega(X, \#)$. We consider the following statements*:

(i) a) *Δ has the Jordan–Hahn property and*

b) *$C^1 \subseteq D^1$ and $C^0 \subseteq D^0$ implies $D^1 \subseteq C^1$ and $D^0 \subseteq C^0$* $(C, D \in \mathcal{O}(X, \#))$;

(ii) $\operatorname{ext}[0, e] = \mathcal{L}_f(X, \#)$.

Then (i) $\Rightarrow$ (ii). *If Δ is assumed to be closed then* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii): If Δ has the Jordan–Hahn property then $\operatorname{ext}[0, e] \subseteq \mathcal{L}_f(X, \#)$ and $[-e, e]$ is a polytope, by Theorem 5.4. Let $C \in \mathcal{O}(X, \#)$ then $\Phi(f_C)(C^1) = \{1\}$

and $\Phi(f_C)(C^0)=\{-1\}$. Consider the set $F:=\{g\in[-e,e]\mid g(C^1)=\{1\}$ and $g(-C^0)=\{1\}\}$. Note that $\Phi(f_C)\in F$. One is easily convinced that F is a face of the polytope $[-e,e]$ thus is a polytope in its own right and therefore compact. Thus $\text{ext}\, F\neq\varnothing$. Since $\text{ext}\, F\subseteq \text{ext}[-e,e]\subseteq \Phi(\mathscr{L}_f(X,\#))$ there exists $D\in\mathcal{O}(X,\#)$ such that $\Phi(f_D)\in \text{ext}\, F$. Then clearly $f_D(C^1)=[(\Phi(f_D)+e)/2](C^1)=\{1\}$ and $f_D(C^0)=[(\Phi(f_D)+e)/2](C^0)=\{0\}$. Therefore $C^1\subseteq f_D^{-1}(1)\cap\Delta=D^1$ and $C^0\subseteq f_D^{-1}(0)\cap\Delta=D^0$. By (i)b) we get $C^1=D^1$ and $C^0=D^0$, hence $f_C^{-1}(1)\cap\Delta=f_D^{-1}(1)\cap\Delta$ and $f_C^{-1}(0)\cap\Delta=f_D^{-1}(0)\cap\Delta$. Lemma 4.3(ii) then shows that $\Phi(f_C)^{-1}(1)\cap U=\Phi(f_D)^{-1}(1)\cap U$.

Now $\Phi(f_D)\in\exp[-e,e]=\text{ext}[-e,e]$, $[-e,e]$ being a polytope. By Lemma 4.2, there exists $\nu\in\text{lin}\,\Delta$ with $\|\nu\|_B=1$ and $\{\Phi(f_D)\}=\{g\in[-e,e]\mid g(\nu)=1\}$. Thus $\nu\in\Phi(f_D)^{-1}(1)\cap U$ since $B_B=U$. Therefore $\Phi(f_C)(\nu)=1$ which in turn implies that $\Phi(f_C)=\Phi(f_D)$. Hence $f_C\in\text{ext}[0,e]$.

Finally we show that if Δ is closed then (ii) $\Rightarrow$ (i). By Theorem 5.5, Δ has the Jordan–Hahn property. Let $C,D\in\mathcal{O}(X,\#)$ such that $C^1\subseteq D^1$ and $C^0\subseteq D^0$. Since $C^1=f_C^{-1}(1)\cap\Delta$, $C^0=f_C^{-1}(0)\cap\Delta$ and similar for D, we get, by Lemma 4.3(ii), $\Phi(f_C)^{-1}(1)\cap U\subseteq\Phi(f_D)^{-1}(1)\cap U\neq U$. Now $\Phi(f_C)\in\text{ext}[-e,e]=\exp[-e,e]$ thus by Theorem 4.4, $\Phi(f_C)^{-1}(1)\cap U$ is a maximal proper exposed face. Therefore $\Phi(f_D)^{-1}(1)\cap U\subseteq\Phi(f_C)^{-1}(1)\cap U$. By Lemma 4.3(i), $D^1=f_D^{-1}(1)\cap\Delta=\Phi(f_D)^{-1}(1)\cap U\cap\Delta\subseteq\Phi(f_C)^{-1}(1)\cap U\cap\Delta=f_C^{-1}(1)\cap\Delta=C^1$ and $D^0=f_D^{-1}(0)\cap\Delta=-(\Phi(f_D)^{-1}(1)\cap U\cap-\Delta)\subseteq-(\Phi(f_C)^{-1}(1)\cap U\cap-\Delta)=f_C^{-1}(0)\cap\Delta=C^0$.

EXAMPLE 2. We give an example of an orthogonality space $(X,\#)$ with a non-empty closed convex subset Δ of weights that is full for $(X,\#)$ and has the Jordan–Hahn property but $\text{ext}[0,e]\neq\mathscr{L}_f(X,\#)$. Let $X:=\{x_1,x_2,x_3,x_4,x_5\}$ with $x_1\#\, x_2\#\, x_3\#\, x_1$ and $x_4\#\, x_5$ (same convention as in Example 1). Note that $(X,\#)$ is a Dacey space (see Fig. 1). The logic $(\mathscr{L}(X,\#),\subseteq)$ is depicted in Fig. 2. We have put: $A:=\{x_1\}$, $B:=\{x_2\}$, $C:=\{x_3\}$, $D:=\{x_4\}$ hence $A^*=\{x_2,x_3\}$, $B^*=\{x_1,x_3\}$, $C^*=\{x_1,x_2\}$ and $D^*=\{x_5\}$. Define the weights $\omega_1(x):=(1,0,0,\frac{1}{4},\frac{3}{4})$, $\omega_2(x):=(0,1,0,\frac{1}{2},\frac{1}{2})$ and $\omega_3(x):=(0,0,1,\frac{3}{4},\frac{1}{4})$. Then $\Delta:=\{\sum_{i=1}^3 t_i\omega_i\mid\sum_{i=1}^3 t_i=1;\ t_i\geq 0\}$ is clearly a non-empty closed convex subset of $\Omega(X,\#)$ (see Fig. 3). Using standard criteria one finds that Δ is a full set of weights for $(X,\#)$. Note that $\dim\Delta=2$, $\dim\text{lin}\,\Delta=3$ and $\{\omega_1,\omega_2,\omega_3\}$ is a base for $\text{lin}\,\Delta$. One easily finds that $\text{ext}[0,e]=\{0,f_1,f_2,f_3,f_1',f_2',f_3',e\}$ where f_i $(i=1,2,3)$ are given by $f_i(\omega_k)=\delta_{i,k}$ $(i,k=1,2,3)$. Then $f_1=f_A$, $f_1'=f_{A^*}$, $f_2=f_B$, $f_2'=f_{B^*}$, $f_3=f_C$, $f_3'=f_{C^*}$, $0=f_\varnothing$ and $e=f_X$. Hence $\text{ext}[0,e]\subseteq\mathscr{L}_f(X,\#)$, thus, by Theorem 5.5, Δ has the Jordan–Hahn property. Since $0\neq\omega(x_4)\neq 1$ and $0\neq\omega(x_5)\neq 1$ for all $\omega\in\Delta$, we have $f_D, f_{D^*}\notin\text{ext}[0,e]$. This is also an example of an orthogonality space with a full but not ultrafull set of weights for $(X,\#)$.

If Δ is a non-empty closed convex subset of $\Omega(X,\#)$ and $\text{ext}[0,e]=\mathscr{L}_f(X,\#)$

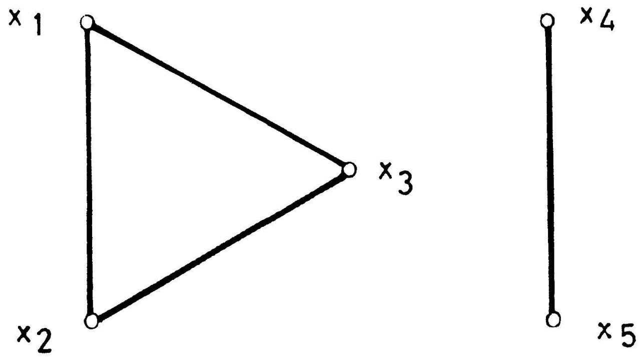

Figure 1

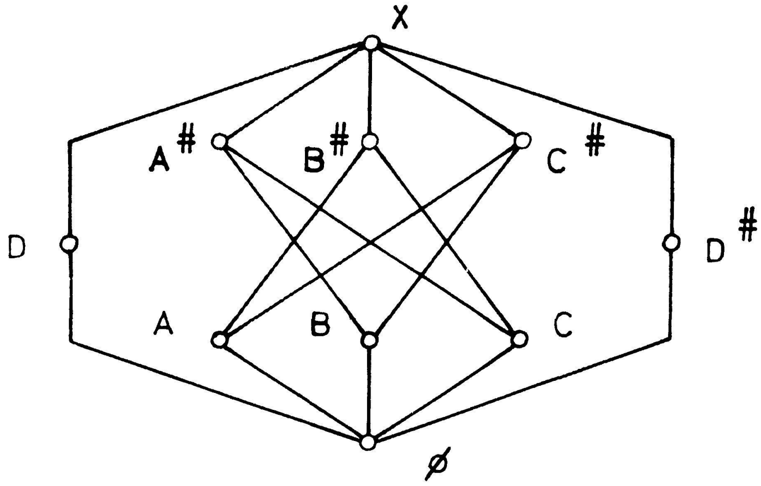

Figure 2

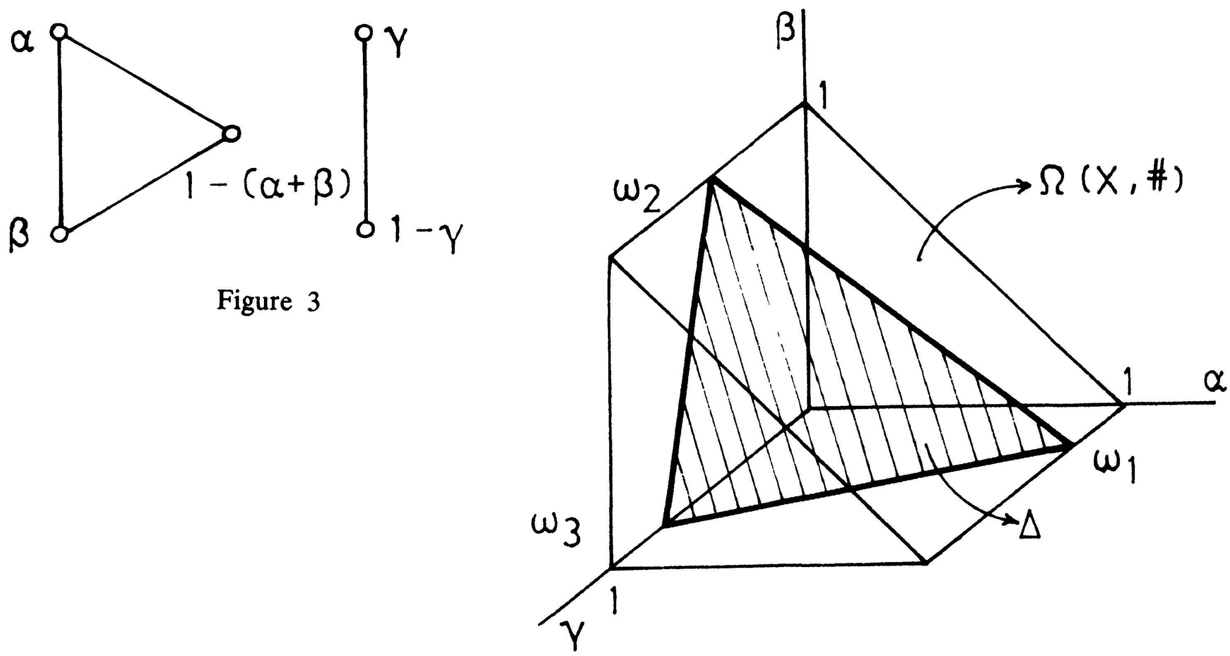

Figure 3

then Δ is not necessarily a full set of weights for $(X, \#)$ as is shown in the following example.

EXAMPLE 3. Let $(X, \#)$ be the orthogonality space of Example 2. Consider the weights $\omega_1(x) := (1, 0, 0, 1, 0)$, $\omega_2(x) := (0, 1, 0, 0, 1)$ and $\omega_3(x) := (0, 0, 1, 1, 0)$. Then $\Delta := \{\sum_{i=1}^3 t_i\omega_i \mid \sum_{i=1}^3 t_i = 1;\ t_i \geq 0\}$ is a non-empty closed convex subset of $\Omega(X, \#)$. But Δ is not full since x_2 non-# x_4 and $\omega(x_2) + \omega(x_4) = t_1 + t_2 + t_3 = 1$, $\omega \in \Delta$. Again $\dim \Delta = 2$, $\dim \operatorname{lin} \Delta = 3$ and $\{\omega_1, \omega_2, \omega_3\}$ is a base for $\operatorname{lin} \Delta$. $\operatorname{ext}[0, e] = \{0, f_1, f_2, f_3, f_1', f_2', f_3', e\}$ where $f_i(x_k) = \delta_{i,k}$ $(i, k = 1, 2, 3)$. Now $f_\varnothing = 0$, $f_A = f_1$, $f_B = f_{D^*} = f_2$, $f_C = f_3$, $f_{A^*} = f_1'$, $f_{B^*} = f_D = f_2'$, $f_{C^*} = f_3'$ and $f_X = e$. Therefore $\operatorname{ext}[0, e] = \mathcal{L}_f(X, \#)$.

Now the main results of this paper.

THEOREM 5.7. *Let* $(X, \#)$ *be a finite orthogonality space with* $\Omega(X, \#) \neq \varnothing$ *and let* Δ *be a non-empty convex subset of* $\Omega(X, \#)$. *Consider the following statements*:

(i) Δ *is an ultrafull set of weights for* $(X, \#)$ *and has the Jordan–Hahn property*;
(ii) *the mapping* $M \in \mathcal{L}(X, \#) \rightarrow f_M \in [0, e]$ *is an order isomorphism from the logic* $(\mathcal{L}(X, \#), \subseteq)$ *onto the poset* $(\operatorname{ext}[0, e], \leq)$.

Then (i) $\Rightarrow$ (ii). *If* Δ *is assumed to be closed then* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii): Assume that Δ is ultrafull for $(X, \#)$ and has the Jordan–Hahn property. Since Δ is also full we get, by Theorem 5.3, $M \rightarrow f_M$ is an order isomorphism from $\mathcal{L}(X, \#)$ onto $\mathcal{L}_f(X, \#)$. By Theorem 5.6, $\operatorname{ext}[0, e] = \mathcal{L}_f(X, \#)$.

Now we are going to show that (ii) $\Rightarrow$ (i) under the assumption that Δ is closed. Assume that $M \rightarrow f_M$ is an order isomorphism from $(\mathcal{L}(X, \#), \subseteq)$ onto $(\operatorname{ext}[0, e], \leq)$. By Theorem 5.3, Δ is a full set of weights for $(X, \#)$. Considering that $\mathcal{L}_f(X, \#) = \operatorname{ext}[0, e]$, we conclude, using Theorem 5.6, that Δ is ultrafull for $(X, \#)$ and has the Jordan–Hahn property. This completes the proof of the theorem.

THEOREM 5.8. *Let* $(X, \#)$ *be a finite orthogonality space with* $\Omega(X, \#) \neq \varnothing$ *and* Δ *a non-empty convex subset of* $\Omega(X, \#)$.

If Δ *is ultrafull and has the Jordan-Hahn property then*

(i) $(X, \#)$ *is a Dacey space and* $(\mathcal{L}(X, \#), \subseteq, ^*)$ *is an orthomodular poset,*
(ii) $(\operatorname{ext}[0, e], \leq, ')$ *is an orthomodular poset ortho-order isomorphic to* $(\mathcal{L}(X, \#), \subseteq, ^*)$,
(iii) Δ *has finitely many weights that are pure with respect to* Δ,

(iv) *for any* $C \in \mathcal{O}(X, \#)$

$\Delta \subseteq \operatorname{aff}\{\omega \in \Delta \mid \omega$ *pure with respect to* Δ, $\omega(C) = 1$ *or* $\omega(C) = 0\}$,

(v) Δ *is unital for* $(X, \#)$.

Proof.

(i): Δ is also full. By Lemma 2.1, $(X, \#)$ is a Dacey space and finally $(\mathscr{L}(X, \#), \subseteq, ^*)$ is an orthomodular poset.

(ii): By Theorem 5.7, (i) of this theorem and Lemma 5.2(ii), $M \to f_M$ is an ortho-order isomorphism between the orthomodular poset $(\mathscr{L}(X, \#), \subseteq, ^*)$ and the orthocomplemented poset $(\operatorname{ext}[0, e], \leq, ')$.

(iii): Clear, since Δ is a polytope (Theorem 5.4).

(iv): Let $C \in \mathcal{O}(X, \#)$. Since $[-e, e]$ is a polytope (Theorem 5.4) we have $\Phi(f_C) \in \exp[-e, e]$. By theorem 4.4, $\Phi(f_C)^{-1}(1) \cap U$ is a maximal proper exposed face, hence a facet of the polytope U (Theorem 5.4). Therefore $\dim[\Phi(f_C)^{-1}(1) \cap U] = \dim U - 1 = \dim \Delta$. Clearly, $\operatorname{ext}(f_C^{-1}(1) \cap \Delta)$, $\operatorname{ext}(-(f_C^{-1}(0) \cap \Delta)) \subseteq \operatorname{ext}[\Phi(f_C)^{-1}(1) \cap U]$. Since $\Phi(f_C)^{-1}(1) \cap U = \operatorname{con}[(f_C^{-1}(1) \cap \Delta) \cup -(f_C^{-1}(0) \cap \Delta)]$ (Lemma 4.3ii) and $\Delta \cap -\Delta = \emptyset$ we conclude that $\operatorname{ext}[\Phi(f_C)^{-1}(1) \cap U] = \operatorname{ext}(f_C^{-1}(1) \cap \Delta) \cup \operatorname{ext}(-(f_C^{-1}(0) \cap \Delta)) = (f_C^{-1}(1) \cap \operatorname{ext} \Delta) \cup (f_C^{-1}(0) \cap -\operatorname{ext} \Delta)$. This set contains an affine base for

$$\operatorname{aff}[\Phi(f_C)^{-1}(1) \cap U],$$

say $\{\omega_1, \omega_2, \ldots, \omega_m, -\omega_{m+1}, \ldots, -\omega_{\dim \Delta+1}\}$ where $\{\omega_1, \ldots, \omega_m\} \subseteq f_C^{-1}(1) \cap \operatorname{ext} \Delta$ and $\{-\omega_{m+1}, \ldots, -\omega_{\dim \Delta+1}\} \subseteq f_C^{-1}(0) \cap -\operatorname{ext} \Delta$. One is easily convinced that $\{\omega_1, \omega_2, \ldots, \omega_m, \omega_{m+1}, \ldots, \omega_{\dim \Delta+1}\} \subseteq \operatorname{ext} \Delta$ is affinely independent. Hence $\operatorname{aff}\{\omega_1, \ldots, \omega_{\dim \Delta+1}\} \supseteq \Delta$. Now $\omega_i(C) = f_C(\omega_i) = 1$ for $1 \leq i \leq m$ and $\omega_k(C) = 0$ for $m + 1 \leq k \leq \dim \Delta + 1$.

(v): Assume that there exists $x \in X$ such that $\{x\}^1 = \emptyset$. By (iv), $\Delta \subseteq \operatorname{aff}\{\omega \in \Delta \mid \omega(x) = 0\}$. Thus $f'_x(\Delta) = (e - f_x)(\Delta) = \{1\}$. Thus $e = f'_x$ or $f'_{\{x\}^{**}} = f_{\{x\}^*} = f_X$, by (i) and Lemma 5.2(ii). Due to the order isomorphism $M \to f_M$ we get $\{x\}^* = X$ or $\{x\}^{**} = X^* = \emptyset$. Which is a contradiction since $x \in \{x\}^{**}$.

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