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Homotopy dimension and simple cohomological dimension of spaces

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Introduction

Throughout this paper all spaces will be assumed to be connected CW-complexes. We define the *homotopy dimension* of a space X, denoted ho dim X, by

$$\text{ho}\dim X = \min\ \{\text{dimension } Y\},$$

where Y ranges over all complexes homotopy equivalent to X. We define the *simple cohomological dimension* of X, denoted $\text{cd}_s X$, by

$$\text{cd}_s X = \sup\{i : H^i(X; A) \neq 0 \quad \text{for some abelian group } A\}.$$

In both cases we allow ∞ as a possible value.

Clearly $\text{cd}_s X \leq \text{ho}\dim X$, and it is well known that equality holds if X is 1-connected. The purpose of this paper is to prove that equality also holds for certain classes of non-1-connected spaces. In particular, our main result (Theorem 5.1) is that $\text{cd}_s X = \text{ho}\dim X$ if X is nilpotent and $\pi = \pi_1 X$ is finitely generated. This was conjectured (without the finite-generation hypothesis on π) by Mislin [12].

Our proof makes use of a result of Wall ([17, 18]) which relates ho dim X to the cohomological dimension of X with respect to local coefficient systems. More precisely, if we set

$$\text{cd}\, X = \sup\{i : H^i(X; M) \neq 0 \quad \text{for some } \pi_1 X\text{-module } M\},$$

then Wall proves that $\text{ho}\dim X = \text{cd}X$, provided $\text{cd}\, X > 2$. (If $\text{cd}\, X \leq 2$, Wall

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proves only that $\operatorname{ho\,dim} X \leq 3$.) This reduces us, essentially, to the algebraic problem of proving that $\operatorname{cd} X = \operatorname{cd}_s X$ under suitable hypotheses on X. For this purpose we study the behavior of $\operatorname{cd} X$ and $\operatorname{cd}_s X$ under passage to covering spaces; we then deduce the desired equality from the (trivial) fact that $\operatorname{cd} = \operatorname{cd}_s$ for 1-connected spaces.

The paper is organized as follows. §1 is primarily concerned with a finite regular covering $\tilde{X} \to X$ of finite-dimensional spaces. After a preliminary proposition, we prove a technical result about the transfer map (Prop. 1.2), which may be of independent interest, and we use it to study the relationship between $\operatorname{cd}_s \tilde{X}$ and $\operatorname{cd}_s X$. In §2 we study nilpotent spaces X with $\operatorname{cd} X \leq 2$. In particular, we show that $\operatorname{ho\,dim} X = \operatorname{cd} X$, so that the low-dimensional ambiguity mentioned above in connection with Wall's Theorem does not occur in the nilpotent case.

In §3 we use the results of §§1 and 2 to prove that $\operatorname{ho\,dim} X = \operatorname{cd}_s X$ for certain classes of spaces (including the nilpotent spaces) with finite fundamental group. In order to handle the infinite-dimensional case, we need two technical results, Propositions A and B, whose proofs are postponed to §4. We also give in §3 some examples of spaces such that $\operatorname{ho\,dim} X \neq \operatorname{cd}_s X$.

In §5 we extend the results of §3 to nilpotent spaces with finitely-generated fundamental group. As a bi-product of the proof, we obtain (Cor. 5.4) a lower bound on the dimension of a nilpotent space with given fundamental group.

In §6 we use the results of §5 to study the effect of localization on homotopy dimension. If X is a countable nilpotent complex and P is a set of primes, we prove that $\operatorname{ho\,dim} X_P \leq \operatorname{ho\,dim} X + 1$. This improves a result previously obtained by the second-named author (Notices Amer. Math. Soc., August 1975, p. A-529). For the proof, we need to work with homological dimension rather than cohomological dimension. The necessary results relating the two are given in Appendix A.

Finally, we include a second appendix, in which we indicate how some of the results of §1 can be extended to certain infinite covering spaces.

At this point, the second author would like to express his appreciation to the Mathematics Institute, University of Heidelberg, for its hospitality. Also, he would like to thank R. Strebel, Heidelberg, and G. Mislin, Zürich, for a number of helpful conversations and comments.

1. Cohomological dimension and simple cohomological dimension of finite covering spaces

We begin by recording some well-known facts concerning cohomological dimension:

PROPOSITION 1.1. (i) *Let C be the chain complex of the universal cover of X. Then C, regarded as a complex of free* $\mathbf{Z}[\pi_1 X]$*-modules, is homotopy equivalent to a complex* C' *of projective modules with* $C'_i = 0$ *for* $i > \mathrm{cd}\, X$.

(ii) *If* $\tilde{X}$ *is an arbitrary covering space of X, then* $\mathrm{cd}\, \tilde{X} \leq \mathrm{cd}\, X$.

(iii) *If* $\mathrm{cd}\, X < \infty$ *and* $\tilde{X} \to X$ *is a finite covering, then* $\mathrm{cd}\, \tilde{X} = \mathrm{cd}\, X$.

Proof. We may assume that $\mathrm{cd}\, X = n < \infty$. For (i) we can take C' to be the complex

$$\cdots \to 0 \to C_n/B_n \to C_{n-1} \to \cdots \to C_0,$$

where B_n is the module of n-dimensional boundaries of C, cf. [4], §1, proof of lemma. (ii) is an immediate consequence of (i). For (iii) one proves, exactly as in [15], 1.3, proof of Lemma 2, that the transfer map $\mathrm{tr}: H^n(\tilde{X}; M) \to H^n(X; M)$ is surjective for all π-modules M. Hence $\mathrm{cd}\, \tilde{X} \geq \mathrm{cd}\, X$, whence (iii). □

Now assume that $\tilde{X} \to X$ is a *regular* finite covering, with $\mathrm{cd}\, X < \infty$. In this case we will make more precise the result on the transfer map which we cited in the proof of (iii), and we shall use this to obtain results about $\mathrm{cd}_s \tilde{X}$.

Let $\tilde{\pi} = \pi_1 \tilde{X}$, and let $G = \pi/\tilde{\pi}$, the group of deck transformations. For any π-module M there is an action of G on $H^*(\tilde{X}; M)$, defined as follows. Let C be as in 1.1(i) and let π act on $\mathrm{Hom}_{\mathbf{Z}}(C, M)$ in the usual way: $(\gamma f)(c) = \gamma \cdot f(\gamma^{-1} c)$ for $\gamma \in \pi$, $f \in \mathrm{Hom}_{\mathbf{Z}}(C, M)$, $c \in C$. Then the submodule $\mathrm{Hom}_{\tilde{\pi}}(C, M) = C^*(\tilde{X}; M)$ inherits an action of $G = \pi/\tilde{\pi}$, and hence so does $H^*(\tilde{X}; M)$. The transfer map has the property that $\mathrm{tr}\,(g \cdot u) = \mathrm{tr}\,(u)$ for $g \in G$ and $u \in H^*(\tilde{X}; M)$, and so it induces by passage to the quotient a map $\tau: H^*(\tilde{X}; M)_G \to H^*(X; M)$. (Here, as usual, $(-)_G = H_0(G, -) = \mathbf{Z} \otimes_{\mathbf{Z}G} -$.)

PROPOSITION 1.2. *Let* $\tilde{X} \to X$ *be a regular finite covering with group G, and assume* $\mathrm{cd}\, X < \infty$. *Let M be a* π*-module and n an integer such that* $H^i(\tilde{X}; M) = 0$ *for* $i > n$. *Then* $H^i(X; M) = 0$ *for* $i > n$, *and* $\tau: H^n(\tilde{X}; M)_G \to H^n(X; M)$ *is an isomorphism.*

Specializing to the case where π acts trivially on M, and taking $n = \mathrm{cd}_s \tilde{X}$, we obtain:

THEOREM 1.3. *Let* $\tilde{X} \to X$ *be as in* 1.2. *Then* $\mathrm{cd}_s \tilde{X} \geq \mathrm{cd}_s X$, *with equality if and only if the G-module* $H^n(\tilde{X}; A)$ *is not perfect for some abelian group A, where* $n = \mathrm{cd}_s \tilde{X}$.

(Recall that a G-module L is *perfect* if $L_G = 0$.)

An important special case is that where G acts nilpotently on $H_*\tilde{X}$, i.e., each $H_i\tilde{X}$ has a finite filtration by G-submodules such that G acts trivially on the successive quotients. In this case we will say that $\tilde{X} \to X$ is *homologically nilpotent.* It follows easily from the universal coefficient theorem that $H^i(\tilde{X}; A)$ is a nilpotent G-module for any abelian group A if $\tilde{X} \to X$ is homologically nilpotent. Since it is clear that only the zero module can be both perfect and nilpotent, we obtain from 1.3:

COROLLARY 1.4. *Assume, in addition to the hypotheses of* 1.2, *that* $\tilde{X} \to X$ *is homologically nilpotent. Then* $\operatorname{cd}_s \tilde{X} = \operatorname{cd}_s X$.

The proof of Proposition 1.2 will use two lemmas, valid for any regular finite cover $\tilde{X} \to X$ with group G and any π-module M. Recall first that a G-module is said to be *induced* if it is isomorphic to $\mathbf{Z}G \otimes A$, for some abelian group A, and that induced modules are trivial for Tate cohomology: $\hat{H}^*(G, \mathbf{Z}G \otimes A) = 0$ (cf. [5], chap. XII, or [14], chap. VIII).

LEMMA 1.5. *The cochain module* $C^q = C^q(\tilde{X}; M)$, $q \geq 0$, *is an induced* G-*module. In particular,* $\hat{H}^*(G, C^q) = 0$.

Proof. We must show that $\operatorname{Hom}_{\tilde{\pi}}(F, M)$ is an induced G-module, where F is a free $\mathbf{Z}\pi$-module and M is an arbitrary π-module. It suffices to do this for $F = \mathbf{Z}\pi$. Let A be the underlying abelian group of M. Then there is G-isomorphism

$$\psi : \operatorname{Hom}_{\tilde{\pi}}(\mathbf{Z}\pi, M) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}G, A),$$

defined by $\psi(f)(g) = (g \cdot f)(1)$.[1] This is valid whether or not G is finite, and it shows that $\operatorname{Hom}_{\tilde{\pi}}(\mathbf{Z}\pi, M)$ is "co-induced." When G is finite, we clearly have $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}G, A) \approx \mathbf{Z}G \otimes A$, whence the lemma. □

LEMMA 1.6. *Let* Z^q *be the module of cocycles* $Z^q(\tilde{X}; M)$. *If* $\hat{H}^i(G, Z^q) = 0$ *for* $i = -1, 0$, *then* $\tau : H^q(\tilde{X}; M)_G \to H^q(X; M)$ *is an isomorphism.*

Proof. Clearly $C^*(X; M) = (C^*)^G$, where $C^* = C^*(\tilde{X}; M)$. Hence $Z^q(X; M) = (Z^q)^G$, and we have an exact sequence

$$(C^{q-1})^G \to (Z^q)^G \to H^q(X; M) \to 0. \qquad (*)$$

On the other hand, from $C^{q-1} \to Z^q \to H^q(\tilde{X}; M) \to 0$ we obtain

$$(C^{q-1})_G \to (Z^q)_G \to H^q(\tilde{X}; M)_G \to 0. \qquad (**)$$

[1] Here, as usual, G acts on $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}G, A)$ by $(g \cdot h)(g') = h(g'g)$, for $g, g' \in G$, $h \in \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}G, A)$.

Now for any G-module L the norm operator $N=\sum_{g\in G} g$ induces a map $L_G \to L^G$, whose kernel and cokernel are $\hat{H}^{-1}(G, L)$ and $\hat{H}^0(G, L)$. In view of 1.5 and the hypothesis on Z^q, it follows that N induces an isomorphism of $(**)$ with $(*)$, and the lemma follows at once. □

Proof of Prop. 1.2. We may assume that X is a CW-complex of finite dimension d. For $n \le q \le d$ we have an exact sequence of G-modules

$$0 \to Z^q \to C^q \to C^{q+1} \to \cdots \to C^d \to 0,$$

where C^* and Z^q are as above. Using 1.5 we conclude that $\hat{H}^*(G, Z^q)=0$. By 1.6 $\tau: H^q(\tilde{X}: M)_G \to H^q(X; M)$ is an isomorphism, whence the proposition. □

Remark. There is a convergent fourth-quadrant spectral sequence which can be used to give an alternative proof of 1.2 and which yields further results relating $H^*(\tilde{X}; M)$ and $H^*(X; M)$:

$$E^2_{pq} = H_p(G, H^{-q}(\tilde{X}; M)) \Rightarrow H^{-(p+q)}(X; M).$$

2. Spaces of cohomological dimension ≤2

We will say that a space X is *homologically nilpotent* if its universal covering is homologically nilpotent, as defined in §1. If, in addition, $\pi_1 X$ is nilpotent, then X is called *nilpotent*. (This is equivalent to the standard definition, cf. [11], Prop. 2.1.) The purpose of this section is to prove:

THEOREM 2.1. *If X is nilpotent, or if X is homologically nilpotent and $\pi_1 X$ has torsion, then* $\operatorname{cd} X = \operatorname{ho\,dim} X$.

This is well known if $\operatorname{cd} X > 2$ (cf. Wall's theorem cited in the introduction). Hence the theorem follows from:

PROPOSITION 2.2. *Let X be a homologically nilpotent space with* $\operatorname{cd} X \le 2$. *Then $\pi_1 X$ is torsion-free, and either X is a wedge of two-spheres (up to homotopy type), or $X = K(\pi_1 X, 1)$. If, in addition, $\pi_1 X$ is nilpotent (i.e., X is a nilpotent space), then* (a) *$\pi_1 X$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ or to a subgroup of $\mathbf{Q}$, and* (b) $\operatorname{cd} X = \operatorname{ho\,dim} X$.

The proof uses the following three lemmas, which will be used again in later sections.

LEMMA 2.3. *Let $\tilde{X} \to X$ be a homologically nilpotent cover with deck-transformation group* **Z**. *If* $\operatorname{cd} \tilde{X} = \operatorname{cd}_s \tilde{X}$, *then* $\operatorname{cd} X = \operatorname{cd}_s X = \operatorname{cd} \tilde{X} + 1$.

Proof. Because $\operatorname{cd} \mathbf{Z} = 1$, the Cartan–Leray spectral sequence yields a short-exact sequence

$$0 \to H^1(\mathbf{Z}, H^{n-1}(\tilde{X}; M)) \to H^n(X; M) \to H^0(\mathbf{Z}, H^n(\tilde{X}; M)) \to 0, \qquad (*)$$

where M is an arbitrary $\pi_1 X$-module. It follows that

(a) $\operatorname{cd} X \le \operatorname{cd} \tilde{X} + 1$ and (b) $\operatorname{cd}_s X \le \operatorname{cd}_s \tilde{X} + 1$.

We claim that equality holds in (b). For if A is an abelian group and n an integer such that $H^{n-1}(\tilde{X}; A) \neq 0$, then $H^1(\mathbf{Z}, H^{n-1}(\tilde{X}; A)) = H^{n-1}(\tilde{X}; A)_{\mathbf{Z}} \neq 0$. Hence $H^n(X; A) \neq 0$ by $(*)$, whence the claim. We now have

$$\operatorname{cd}_s X \le \operatorname{cd} X \le \operatorname{cd} \tilde{X} + 1 = \operatorname{cd}_s \tilde{X} + 1 = \operatorname{cd}_s X,$$

and the lemma follows at once. □

LEMMA 2.4. *Let $\tilde{X} \to X$ be a finite, homologically nilpotent cover with group G, and let k be a field whose characteristic is zero or is prime to $|G|$. Then G acts trivially on $H_*(\tilde{X}; k)$ and $H_*(\tilde{X}; k) \approx H_*(X; k)$.*

Proof. The first assertion is immediate from the semi-simplicity of kG (Maschke's Theorem). The second assertion follows from the first and the well-known isomorphism $H_*(X; k) \approx H_*(\tilde{X}; k)_G$. □

If π is a group then we set, as usual, $\operatorname{cd} \pi = \operatorname{cd} K(\pi, 1)$ and $\operatorname{hd} \pi = \operatorname{hd} K(\pi, 1)$. (Here hd denotes homological dimension, cf. Appendix A). Recall that the *rank* (or *Hirsch number*) of a nilpotent group π is defined to be the sum of the ranks of the abelian groups $\Gamma_i \pi / \Gamma_{i+1} \pi$, where $(\Gamma_i \pi)$ is the lower central series of π.

LEMMA 2.5. *Let π be a torsion-free nilpotent group.*

(i) *If π is finitely generated, then* $\operatorname{hd} \pi = \operatorname{cd} \pi = \operatorname{rank} \pi$.
(ii) *If π is not finitely generated, then* $\operatorname{hd} \pi = \operatorname{rank} \pi$ *and* $\operatorname{cd} \pi = \operatorname{rank} \pi + 1$.

Proof. (i) is a well-known consequence of a theorem of Malcev [10], according to which $K(\pi, 1)$ has the homotopy type of a closed r-manifold, $r = \operatorname{rank} \pi$. (See also [7], §8.8.) The first equation of (ii) follows easily from (i) by a direct limit argument. The second equation is also immediate from (i) if rank $\pi=\infty$. If rank $\pi<\infty$, the second equation is proved in Gruenberg [7], §8.8. (Alternatively, one can again use a direct limit argument.) □

Proof of 2.2. When $\pi=\pi_1 X$ is trivial, it is easy to see that X is contractible or a wedge of two-spheres, so that in this case the result is immediate. Henceforth, we assume that π is non-trivial.

Let $\tilde{X}$ be the universal cover of X. Then, $\operatorname{cd}\tilde{X}\le\operatorname{cd}X\le 2$, so that $H_2\tilde{X}$ is free-abelian.

Suppose π has torsion, and let G be a non-trivial, finite cyclic subgroup. By 2.4 applied to the cover $\tilde{X}\to\tilde{X}/G$, G acts trivially on $H_2(\tilde{X};\mathbf{Q})$, hence also on $H_2\tilde{X}$. The Cartan–Leray spectral sequence of $\tilde{X}\to\tilde{X}/G$, therefore, has $E^2_{pq}=H_pG\otimes H_q\tilde{X}$. Since $\operatorname{cd}(\tilde{X}/G)\le\operatorname{cd}X\le 2$, the (only non-zero) differential d^3 gives isomorphisms $H_pG\approx H_{p-3}G\otimes H_2\tilde{X}$ for $p\ge 4$. But this is impossible because, when p is odd, $H_pG\ne 0$ and $H_{p-3}G=0$. Thus, π is torsion-free.

Since π is non-trivial, it must contain an infinite-cyclic subgroup C. By 2.3 applied to $\tilde{X}\to\tilde{X}/C$, we have $\operatorname{cd}\tilde{X}=\operatorname{cd}(\tilde{X}/C)-1\le 1$. Hence, $\tilde{X}$ is contractible and $X=K(\pi,1)$.

Finally, suppose that π is nilpotent. Then, by 2.5, either π is finitely generated and of rank 2, or rank $\pi\le 1$; (a) follows easily. To obtain (b), we proceed by cases. If π is finitely-generated (non-trivial), then $\pi=\mathbf{Z}$ or $\mathbf{Z}\oplus\mathbf{Z}$, and $X=K(\pi,1)\simeq S^1$ or $S^1\times S^1$, respectively, and the result is immediate. Otherwise, π may be obtained as the direct limit of a sequence of self-maps of $\mathbf{Z}$, and so X may be obtained as the infinite mapping telescope of the corresponding sequence of self-maps of S^1. Thus, $\operatorname{ho\,dim}X\le 2$. But $\operatorname{cd}\pi=2$ by 2.5, and so $\operatorname{cd}X=2=\operatorname{ho\,dim}X$. □

3. Homotopy dimension of spaces with finite fundamental group

Throughout this section X will be a space whose fundamental group π is finite, and $\tilde{X}$ will be the universal cover of X.

Our main result is:

THEOREM 3.1. *If X is nilpotent then* $\operatorname{ho\,dim}X=\operatorname{cd}_s X$.

Our proof of Theorem 3.1 will also show that $\operatorname{ho\,dim}X=\operatorname{cd}_s X$ for certain classes of non-nilpotent spaces. (See 3.2, 3.3, and 3.4.)

We consider first the case $\operatorname{cd}X<\infty$ (equivalently, $\operatorname{ho\,dim}X<\infty$), in which case the theorem follows from the following:

PROPOSITION 3.2. *Let* $\operatorname{cd} X = d < \infty$. *Then* $\operatorname{cd} X = \operatorname{cd}_s X$ *if and only if the* π*-module* $H^d(\tilde{X}; A)$ *is not perfect for some abelian group* A. *In particular,* $\operatorname{cd} X = \operatorname{cd}_s X$ *if* X *is homologically nilpotent, in which case we also have* $\operatorname{ho\,dim} X = \operatorname{cd}_s X$.

Proof. We have $\operatorname{cd} X = \operatorname{cd} \tilde{X}$ by 1.1(iii), and $\operatorname{cd} \tilde{X} = \operatorname{cd}_s \tilde{X}$ trivially; hence $\operatorname{cd} X = \operatorname{cd}_s X$ if and only if $\operatorname{cd}_s \tilde{X} = \operatorname{cd}_s X$. The proposition now follows from 1.3, 1.4, and 2.1. □

Remark. If $H_* X$ is countable, then $\operatorname{cd}_s X$ can be computed from $H^*(X; \mathbf{Z})$, and the conclusion of the proposition can be replaced by the simpler statement, $\operatorname{cd} X = \operatorname{cd}_s X$ if and only if $H^d(\tilde{X}; \mathbf{Z})$ is not perfect.

In order to treat the case $\operatorname{cd} X = \infty$, we will need two results about spaces with a (finite) nilpotent fundamental group:

PROPOSITION A. *Suppose that* π *is nilpotent. For any abelian group* A, $H^i(\tilde{X}; A)$ *is a perfect* π*-module for* $i > \operatorname{cd}_s X$.

PROPOSITION B. *Suppose that* π *is a* p*-group for some prime* p *and that* $\operatorname{cd} \tilde{X} < \infty$. *Then* $\operatorname{cd} X < \infty$ *if and only if* $d(X; \mathbf{Z}/p) < \infty$, *where* $d(X; -) = \sup\{i : H^i(X; -) \neq 0\}$.

The proofs will be given in the next section.

It follows from Proposition A that if X is nilpotent and $\operatorname{cd} \tilde{X} = \infty$, then $\operatorname{cd}_s X = \infty$, so that the conclusion of 3.1 holds in this case. Hence the only remaining case of 3.1 is that where $\operatorname{cd} X = \infty$ and $\operatorname{cd} \tilde{X} < \infty$. The theorem in this case follows from:

PROPOSITION 3.3. *Suppose* $\operatorname{cd} X = \infty$ *and* $\operatorname{cd} \tilde{X} < \infty$. *If* X *is nilpotent, or if* $H_* \tilde{X}$ *is finitely generated, then* $\operatorname{cd}_s X = \infty$.

Proof. For each prime p dividing $|\pi|$, let $\pi(p)$ be a p-Sylow subgroup of π, and let $X(p)$ be the corresponding covering space of X. A standard transfer argument (cf. [14], chap IX, §2, or [5], chap XII, §10) shows that there is an injection

$$H^*(X; M) \hookrightarrow \bigoplus_{p \mid |\pi|} H^*(X(p); M)$$

for any π-module M. Since $\operatorname{cd} X = \infty$, it follows that $\operatorname{cd} X(p) = \infty$ for some p, and hence, by Prop. B, that $d(X(p); \mathbf{Z}/p) = \infty$. We claim that this implies that $d(X; \mathbf{Z}/p) = \infty$, whence the proposition.

In case X is nilpotent, the claim follows from the fact that the map $X(p) \to X$ becomes a homotopy equivalence when localized at p. Alternatively, one can use 2.4 and 5.3 to prove that $H_*(X(p); \mathbf{Z}/p) \approx H_*(X; \mathbf{Z}/p)$.

In case $H_*\tilde{X}$ is finitely generated, we apply the finiteness theorem of equivariant cohomology theory ([13], Cor. 2.3) to the inclusion $(\pi(p), \tilde{X}) \hookrightarrow (\pi, \tilde{X})$. We conclude that $H^*(X(p); \mathbf{Z}/p)$ is finitely generated as a module over the ring $H^*(X; \mathbf{Z}/p)$, and the claim follows at once. □

COROLLARY 3.4. *If* X *is homologically nilpotent and* $\oplus_i H_i\tilde{X}$ *is finitely generated, then* $\operatorname{ho\,dim} X = \operatorname{cd}_s X$.

Proof. This follows from 3.2 if $\operatorname{cd} X < \infty$ and 3.3 if $\operatorname{cd} X = \infty$. □

We close this section by briefly discussing some examples of spaces X such that $\operatorname{ho\,dim} X \neq \operatorname{cd}_s X$.

EXAMPLES 3.5. (a) Let π be a non-trivial group with $H_1(\pi) = H_2(\pi) = 0$. According to Dror [6] there exists an acyclic space X with fundamental group π, such that π acts trivially on the higher homotopy groups of X. Then X is homologically nilpotent (cf. [8], II, proof of 2.18), but $\operatorname{cd}_s X = 0$ and $\operatorname{ho\,dim} X \neq 0$.

This shows that the word "nilpotent" cannot be replaced by "homologically nilpotent" in 3.1. (Of course, 3.2 and 3.3 show that this replacement *can* be made, provided one adds suitable finiteness assumptions.)

(b) For any non-trivial group π, one can construct a finite-dimensional space X with fundamental group π, such that there is an arbitrarily large gap between $\operatorname{ho\,dim} X$ and $\operatorname{cd}_s X$. If π is finite, X can be taken to be finite. (In this case, by 3.2, the top-dimensional cohomology group of $\tilde{X}$ will necessarily be a perfect π-module. If π is nilpotent, the same applies to $H^i\tilde{X}$ for $i > \operatorname{cd}_s X$, by Prop. A.)

To construct X we start with an arbitrary finite-dimensional complex Y with fundamental group π. Choose $n > \dim Y$. Using a construction of Bousfield and Dror ([3], Lemma 6.2) one can attach to Y cells of dimension n, $n+1$, and $n+2$, with the n-cells attached trivially, to obtain a space X such that $H_*(X, Y) = 0$ but $\pi_*(X, Y) \neq 0$. Applying the relative Hurewicz Theorem to $(\tilde{X}, \tilde{Y})$, it follows that $H_i\tilde{X} \neq 0$ for some $i \geq n$, hence $\operatorname{ho\,dim} X \geq n$. But $\operatorname{cd}_s X = \operatorname{cd}_s Y < n$.

4. Proofs of Propositions A and B

See §3 for the statements of Propositions A and B.

LEMMA 4.1. *Let* $\tilde{X} \to X$ *be a regular covering whose group* G *is a finite* p*-group for some prime* p. *Assume that* $d(X; \mathbf{Z}/p) < \infty$.

(a) *If M is a G-module such that $pM=0$, then $d(X;M)=d(X;\mathbf{Z}/p)$.*

(b) $d(\tilde{X};\mathbf{Z}/p)=d(X;\mathbf{Z}/p)$.

(In (a) *M is regarded as π_1X-module via the canonical surjection $\pi_1X\to G$.)*

Proof. Since the augmentation ideal of $\mathbf{F}_pG$ is nilpotent (cf. [14], chap. IX, §1), any $\mathbf{F}_pG$-module M has a finite filtration $0=M_0\subset M_1\subset\cdots\subset M_n=M$ such that G acts trivially on M_i/M_{i-1}. One now proves (a) by induction on n, using the fact that $d(X;M_i/M_{i-1})=d(X;\mathbf{Z}/p)$. For (b), apply (a) with $M=\mathbf{F}_pG$. □

LEMMA 4.2. *Let π be a finite nilpotent group and L a π-module such that $pL=0$ for some prime p. Then L is perfect if and only if $L^\pi=0$. Consequently, if L is perfect, then any submodule of L is perfect.*

Proof. The second statement follows from the first, since the condition $L^\pi=0$ is inherited by any submodule. To prove the first statement, let π' be the (normal) subgroup of π consisting of elements of order prime to p. Then $L_\pi=(L_{\pi'})_{\pi/\pi'}$ and $L^\pi=(L^{\pi'})^{\pi/\pi'}$. Since π/π' is a p-group, it follows that $L_\pi=0\Leftrightarrow L_{\pi'}=0$ and that $L^\pi=0\Leftrightarrow L^{\pi'}=0$, cf. [14], chap. IX, §4, Lemme 4. Since $|\pi'|$ is prime to p, the norm operator induces $L_{\pi'}\approx L^{\pi'}$, whence the lemma. □

LEMMA 4.3. *Let $f:\tilde{X}\to X$ be a finite regular covering map with group G and let M be a π_1X-module.*

(a) *If $H^i(\tilde{X};M)_G=0$ then $H^i(X;M)$ is annihilated by $|G|$.*

(b) *If $H^i(X;M)=0$ then $H^i(\tilde{X};M)_G$ is annihilated by $|G|$.*

Proof. Let $\tau:H^i(\tilde{X};M)_G\to H^i(X;M)$ be the map induced by the transfer map, as in §1. Let $\rho:H^i(X;M)\to H^i(\tilde{X};M)_G$ be f^* followed by the canonical surjection $H^i(\tilde{X};M)\to H^i(\tilde{X};M)_G$. It is well-known that $\mathrm{tr}\circ f^*=|G|\cdot\mathrm{id}$ on $H^i(X;M)$ and that $f^*\circ\mathrm{tr}$ is the norm operator on $H^i(\tilde{X};M)$. It follows that $\tau\rho=|G|\cdot\mathrm{id}$ on $H^i(X;M)$ and that $\rho\tau=|G|\cdot\mathrm{id}$ on $H^i(\tilde{X};M)_G$, whence the lemma. □

Proof of Proposition A. We may assume that $\mathrm{cd}_s X=n<\infty$. Assume first that $pA=0$ for some prime p. Let π' be as in the proof of 4.2 and let X' be the corresponding covering space of X. Using 4.1(b) (applied to $X'\to X$), we find $d(X';A)=d(X';\mathbf{Z}/p)=d(X;\mathbf{Z}/p)\le n$. Since $H^*(X';A)\approx H^*(\tilde{X};A)_{\pi'}$, it follows that $H^i(\tilde{X};A)$ is perfect as π'-module, hence also as π-module, for $i\ge n$.

Now let A be arbitrary. For any prime p, let ${}_pA$ (resp. A_p) be the kernel (resp. the cokernel) of multiplication by p in A. By the previous paragraph the π-modules $H^i(\tilde{X};{}_pA)$ and $H^i(\tilde{X};A_p)$ are perfect for $i>n$, hence the same is true of any submodule by 4.2. Using the long exact cohomology sequences associated to the coefficient sequences

$$0 \to {}_pA \hookrightarrow A \xrightarrow{p} pA \to 0 \qquad \text{and} \qquad 0 \to pA \hookrightarrow A \to A_p \to 0,$$

we conclude that the maps $H^i(\tilde{X}; A) \to H^i(\tilde{X}; pA)$ and $H^i(\tilde{X}; pA) \to H^i(\tilde{X}; A)$ have perfect cokernels, hence they induce surjections $H^i(\tilde{X}; A)_\pi \to H^i(\tilde{X}; pA)_\pi$ and $H^i(\tilde{X}; pA)_\pi \to H^i(\tilde{X}; A)_\pi$. Thus $H^i(\tilde{X}; A)_\pi$ is p-divisible for all primes p. On the other hand, $H^i(\tilde{X}; A)_\pi$ is annihilated by $|\pi|$ for $i > n$, by 4.3(b). Hence it is zero. □

Proof of Proposition B. It suffices to prove that if $d(X; \mathbf{Z}/p) < \infty$ then $\operatorname{cd} X < \infty$. Let M be an arbitrary π-module. Then $d(X; {}_pM) = d(X; M_p) = d(X; \mathbf{Z}/p)$ by 4.1(a). Using cohomology exact sequences as in the proof of Proposition A, we conclude that $H^i(X; M)$ is p-divisible for $i > d(X; \mathbf{Z}/p)$. On the other hand, if also $i > \operatorname{cd} \tilde{X}$, then 4.3(a) shows that $H^i(X; M)$ is annihilated by a power of p, and hence it is zero. Thus $\operatorname{cd} X < \infty$. □

5. Homotopy dimension of nilpotent spaces

The following theorem, which generalizes Theorem 3.1, is the main result of this paper:

THEOREM 5.1. *If X is a nilpotent space with finitely-generated fundamental group, then* $\operatorname{ho\,dim} X = \operatorname{cd}_s X$.

Out proof of 5.1 will also show:

THEOREM 5.2. *If X is as in 5.1 and $\pi = \pi_1 X$, then* $\operatorname{ho\,dim} X = \operatorname{rank} \pi + \operatorname{ho\,dim} X^{\text{tor}}$, *where X^{tor} is the covering space of X corresponding to the torsion subgroup π^{tor} of π.*

(See §2 for the definition of *rank.*)

We shall need one lemma. We call a regular covering map $\tilde{X} \to X$ *nilpotent* if it is homologically nilpotent and the group of deck transformations is nilpotent.

LEMMA 5.3. *Let $\tilde{X} \to X$ be a nilpotent cover with group G. If N is a normal subgroup of G, then the cover $\tilde{X}/N \to X$ is nilpotent.*

Proof. If N is central in G, then one proves the lemma by considering the action of G/N on the spectral sequence

$$E^2_{pq} = H_p(N, H_q\tilde{X}) \Rightarrow H_{p+q}(\tilde{X}/N).$$

In the general case we can find a central series $\{1\} = N_0 \subset N_1 \subset \cdots \subset N_k = G$, one of whose terms is N. Using the special case treated above, it follows inductively that $\tilde{X}/N_i \to X$ is nilpotent, whence the lemma. □

Proof of 5.1 *and* 5.2. By 2.1, we may replace ho dim by cd in the statements of the theorems. We now argue by induction on rank π. If rank $\pi = 0$ then 5.1 follows from 3.1 and 5.2 is vacuous. If rank $\pi > 0$, let π' be a normal subgroup such that $\pi/\pi' \approx \mathbf{Z}$, and let X' be the corresponding covering space of X. Then rank $\pi' =$ rank $\pi - 1$, so we have, by the inductive hypothesis, $\operatorname{cd} X' = \operatorname{cd}_s X' = \operatorname{cd} X^{\text{tor}} + \operatorname{rank} \pi - 1$. The covering $X' \to X$ is nilpotent by 5.3, so that Lemma 2.3 yields $\operatorname{cd} X = \operatorname{cd}_s X = \operatorname{cd} X^{\text{tor}} + \operatorname{rank} \pi$. □

As a corollary of 5.2, we obtain the following lower bound on the dimension of a nilpotent space with a given fundamental group:

COROLLARY 5.4. *Let X be a nilpotent space and let $\pi = \pi_1 X$.*

(a) ho dim $X \geq$ rank π.

(b) *If $X \neq K(\pi, 1)$, then* ho dim $X \geq$ rank $\pi + 2$.

(c) *If π has torsion, then* ho dim $X \geq$ rank $\pi + 3$.

Proof. If π is finitely-generated, this follows from 5.2, together with the observation that ho dim $X^{\text{tor}} \geq 2$ if $X \neq K(\pi, 1)$ and ho dim $X^{\text{tor}} \geq 3$ if $\pi^{\text{tor}} \neq 1$ (see 2.2). In the general case, we apply what we have just proved to the covering spaces X' of X with $\pi' = \pi_1 X'$ finitely generated. Since ho dim $X \geq$ ho dim X' and rank $\pi = \sup\{\operatorname{rank} \pi'\}$, the result follows. □

This corollary suggests the following problem: Given a nilpotent group π, find the minimum dimension of a nilpotent complex with fundamental group π.

6. Homotopy dimension and localization

Let X be a nilpotent space, P a set of primes, and X_P the localization of X at P (cf. [16]). If X is a sphere, then X_P may be constructed as the infinite mapping telescope of a sequence of self-maps of X; if X is 1-connected, then X_P may be constructed inductively by localizing the attaching maps for the cells of X. (See

[16], §2.) From these two constructions, we can deduce that the relation

$$\text{ho dim } X_P \leq \text{ho dim } X + 1 \tag{$*$}$$

is valid for all 1-connected X. In fact, easy examples show that $(*)$ is essentially the only relation between these quantities for 1-connected X.

For non-1-connected nilpotent spaces, the second construction described above breaks down, since cell-attaching destroys nilpotence. Moreover, the standard method of localizing non-1-connected nilpotent spaces – namely, inductively, by means of principal homotopy decompositions – destroys all dimension information. Thus, $(*)$ cannot be obtained in general via known constructions. By applying the results of §§3 and 5, however, together with Appendix A, we shall show that $(*)$ does hold for a large class of nilpotent spaces.

THEOREM 6.1. *Suppose that X is nilpotent and that either H_*X is countable or $\pi_1 X$ is finite. Then,*

$$\text{ho dim } X_P \leq \text{ho dim } X + 1.$$

Remark. If we assume only that X is nilpotent and $\pi_1 X$ is countable, then we can prove the weaker inequality $\text{ho dim } X_P \leq \text{ho dim } X + 2$. The methods are similar to those of the following proof, but they involve cohomology instead of homology.

Proof of 6.1. If $\pi_1 X$ is finite, then so is $\pi_1(X_P) = (\pi_1 X)_P$. Now, because of the Universal Coefficient Theorem and the effect of localization on ordinary homology, it is clear that

$$\text{cd}_s X_P \leq \text{cd}_s X + 1,$$

and so the result in this case follows from Theorem 3.1.

If H_*X is countable, then H_*X_P is countable, and hence $\pi_* X_P$ is countable (cf. [8], II, 2.16). Thus X_P has the homotopy type of a countable complex, and we may apply A1(iii) to conclude that $\text{cd } X_P \leq \text{hd } X_P + 1$, where hd is *homological dimension*, as described in Appendix A. Since $\text{cd } X_P = \text{ho dim } X_P$ (Theorem 2.1), the desired inequality will follow if we prove

$$\text{hd } X_P \leq \text{ho dim } X. \tag{1}$$

We first prove (1) when $\pi_1 X$ is finite. Let $\tilde{X}$ be the universal cover of X. Then, its localization $\tilde{X}_P$ has the homotopy type of the universal cover of X_P. We may assume that $\operatorname{ho\,dim} X<\infty$. By the first paragraph, it follows that $\operatorname{ho\,dim} X_P<\infty$. Thus, we may apply the homology analogue of Proposition 1.1(iii) to the universal covers, obtaining $\operatorname{hd} X=\operatorname{hd} \tilde{X}$ and $\operatorname{hd} X_P=\operatorname{hd} \tilde{X}_P$. But, clearly $\operatorname{hd} \tilde{X}_P\leq \operatorname{hd} \tilde{X}$, by the main property of localization, and so we have $\operatorname{hd} X_P\leq \operatorname{hd} X\leq \operatorname{ho\,dim} X$, as required.

Next, suppose that $\pi=\pi_1 X$ is finitely-generated. Consider the covering $(X_P)^{\mathrm{tor}}\to X_P$. Since $\operatorname{hd}(\pi_P/\pi_P^{\mathrm{tor}})=\operatorname{rank}(\pi_P/\pi_P^{\mathrm{tor}})=\operatorname{rank}\pi$ by Lemma 2.5, the homology spectral sequence of the cover yields

$$\operatorname{hd} X_P\leq \operatorname{hd}(X_P)^{\mathrm{tor}}+\operatorname{rank}\pi. \tag{2}$$

Now, it is easy to see that $(X_P)^{\mathrm{tor}}\simeq (X^{\mathrm{tor}})_P$, and so we have

$$\operatorname{hd}(X_P)^{\mathrm{tor}}\leq \operatorname{ho\,dim} X^{\mathrm{tor}}, \tag{3}$$

by the previous paragraph. Combining (2) and (3) with the equality $\operatorname{ho\,dim} X= \operatorname{ho\,dim} X^{\mathrm{tor}}+\operatorname{rank}\pi$ from 5.2, we obtain (1) as desired.

In general, write $\pi_1 X$ as a union of an increasing sequence of finitely-generated subgroups, and let

$$X_1\to X_2\to\cdots$$

be the corresponding sequence of covering spaces of X. Clearly $X\simeq \operatorname{ho-}\varinjlim X_i$, the homotopy direct limit (or mapping telescope) of $\{X_i\}$. Hence, $X_P\simeq \operatorname{ho-}\varinjlim (X_i)_P$ and $\operatorname{hd} X_P\leq \sup\{\operatorname{hd}(X_i)_P\}$. But, by the previous paragraph, $\operatorname{hd}(X_i)_P\leq \operatorname{ho\,dim} X_i\leq \operatorname{ho\,dim} X$, from which (1) follows immediately. □

Appendix A. Homological dimension

We define the *homological dimension* of a space X by

$$\operatorname{hd} X=\sup\{i: H_i(X;M)\neq 0 \quad \text{for some } \pi_1 X\text{-module } M\}.$$

The following theorem relates $\operatorname{hd} X$ to $\operatorname{cd} X$.

THEOREM A1. (*cf.* [1], *Prop.* 2.4). (i) $\operatorname{hd} X\leq \operatorname{cd} X$.
(ii) *If X has only finitely many cells in each dimension, then* $\operatorname{hd} X=\operatorname{cd} X$.
(iii) *If X has only countably many cells, then* $\operatorname{cd} X\leq \operatorname{hd} X+1$.

Proof. (i) follows immediately from Prop. 1.1(i). For (ii) and (iii) we may assume that $\operatorname{hd} X = n < \infty$. Then, in particular, $H_i\tilde{X} = H_i(X; \mathbf{Z}\pi) = 0$ for $i > n$, where $\tilde{X}$ is the universal cover of X and $\pi = \pi_1 X$. Hence the complex $C = C_*(\tilde{X})$ gives us a free resolution

$$\cdots \to C_{n+1} \to C_n \to C_n/B_n \to 0,$$

where B_n is the module of n-boundaries of C. If M is a π-module and $k > 0$, it follows that $\operatorname{Tor}_k^{\mathbf{Z}\pi}(C_n/B_n, M) = H_{n+k}(C \otimes_\pi M) = H_{n+k}(X; M) = 0$, hence C_n/B_n is a flat $\mathbf{Z}\pi$-module.

Under the hypothesis of (ii), each C_i is a finitely-generated $\mathbf{Z}\pi$-module; thus C_n/B_n is finitely-presented and hence is projective ([9], Cor. 1.4). It follows that C has the homotopy type of the n-dimensional complex

$$\cdots \to 0 \to C_n/B_n \to C_{n-1} \to \cdots \to C_0,$$

hence $\operatorname{cd} X \leq n$. Under the hypothesis of (iii), C_n/B_n is countably presented, hence has projective dimension at most 1 ([9], Theorem 3.2). Thus B_n is projective and C has the homotopy type of the $(n+1)$-dimensional complex

$$\cdots \to 0 \to B_n \to C_n \to C_{n-1} \to \cdots \to C_0,$$

hence $\operatorname{cd} X \leq n+1$. □

Appendix B. Dimension of infinite covering spaces

In this appendix we show how to extend some of the results of §§1 and 3 to a certain class of infinite covering spaces. In particular, we shall consider a regular coversing $\tilde{X} \to X$ with group G, and we shall make the following assumptions:

(a) X is a finite-dimensional complex.

(b) G is of type (VFP), i.e., G has a subgroup G' of finite index such that $\mathbf{Z}$ admits a finite projective resolution over $\mathbf{Z}G'$. (Such a subgroup G' is said to be of type (FP).)

We will denote by n the virtual cohomological dimension of G (i.e., $n = \operatorname{cd} G'$), and we denote by D the right $\mathbf{Z}G$-module $H^n(G, \mathbf{Z}G)$. In order to compare the results of this appendix with those of §§1 and 3, the reader should keep in mind that if G is finite then $n = 0$ and $D = \mathbf{Z}$ (with trivial G-action).

THEOREM B1. (a) $\operatorname{cd} X - n \leq \operatorname{cd} \tilde{X} \leq \operatorname{cd} X$.

(b) *Assume that* $\tilde{X}$ *is dominated by a finite complex and that* D_0, *the underlying abelian group of* D, *has the property that* $D_0 \otimes A \neq 0$ *for every non-zero abelian group* A. *Then* $\operatorname{cd} X - n = \operatorname{cd} \tilde{X}$.

Remark. The hypothesis on D_0 holds, for example, if G is finitely generated and nilpotent (or, more generally, if G is polycyclic). In this case $n = \operatorname{rank} G$ and $D_0 \approx \mathbf{Z}$.

*Proof of B*1. In view of 1.1(iii) and the fact that $H^n(G, \mathbf{Z}G) \approx H^n(G', \mathbf{Z}G')$ if $(G:G') < \infty$, we may replace X by a finite covering space and thereby reduce to the case where G is of type (FP). In this case we have $H^n(G, L) \approx D \otimes_G L$ for any G-module L (cf. [2], Thm. 4.2).

Consider now the Cartan-Leray spectral sequence,

$$E_2^{pq} = H^p(G, H^q(\tilde{X}; M)) \Rightarrow H^{p+q}(X; M),$$

where M is an arbitrary π-module. Then $E_2^{pq} = 0$ unless $p \leq n$ and $q \leq d = \operatorname{cd} \tilde{X}$, so $\operatorname{cd} X \leq d + n$. This proves the first inequality of (a), and the second is given by 1.1(ii).

By a "corner argument" in the spectral sequence, we deduce further that $H^{d+n}(X; M) \approx H^n(G, H^d(\tilde{X}; M)) \approx D \otimes_G H^d(\tilde{X}; M)$. In particular, let $M = \mathbf{Z}\pi \otimes_{\tilde{\pi}} N$, where N is a $\tilde{\pi}$-module. Then one can verify (using the finiteness assumption on $\tilde{X}$) that $H^d(\tilde{X}; M) \approx \mathbf{Z}G \otimes H^d(\tilde{X}; N)$; hence $H^{d+n}(X; M) \approx D_0 \otimes H^d(\tilde{X}; N)$, and (b) follows easily. □

In order to study $\operatorname{cd}_s \tilde{X}$, we will need the following generalization of 1.2:

PROPOSITION B2. *Let* M *be a* π-*module and* k *an integer such that* $H^i(\tilde{X}; M) = 0$ *for* $i > k$. *Then* $H^i(X; M) = 0$ *for* $i > k + n$ *and* $H^{k+n}(X; M) \approx D \otimes_G H^k(\tilde{X}; M)$.

Proof. If G is of type (FP), this is essentially proved in the proof of B1. The general case can easily be deduced by passing to a finite cover and using 1.2. The details are left to the reader. □

Specializing to the case where π acts trivially on M, we obtain:

THEOREM B3. *One has* $\operatorname{cd}_s \tilde{X} \geq \operatorname{cd}_s X - n$, *with equality if and only if* $D \otimes_G H^k(\tilde{X}; A) \neq 0$ *for some abelian group* A, *where* $k = \operatorname{cd}_s \tilde{X}$.

From this we deduce the following generalization of the "if" part of Proposition 3.2:

COROLLARY B4. *Suppose that* $\tilde{X} \to X$ *is a universal covering and that* $D \otimes_G H^k(\tilde{X}; A) \neq 0$ *for some* A, *where* $k = \mathrm{cd}_s \tilde{X} = \mathrm{cd}\, \tilde{X}$. *Then* $\mathrm{cd}\, X = \mathrm{cd}_s X$.

Proof. We have $\mathrm{cd}\, X \leq \mathrm{cd}\, \tilde{X} + n$ by B1(a) and $\mathrm{cd}\, \tilde{X} + n = \mathrm{cd}_s \tilde{X} + n = \mathrm{cd}_s X$ by B3. Thus $\mathrm{cd}\, X \leq \mathrm{cd}_s X$, whence the corollary. □

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