The Range of Atomless Group Valued Measures

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We prove the following results: (1) the range of an atomless group valued measure satisfying ccc is pathwise connected (Corollary 6; generalization of [2] Theorem 4); (2) the closure of the range of an atomless group valued measure is connected if it is compact (Theorem 3).

A *δ-ring* is a nonempty set $\mathfrak{R}$ such that for any sequence $(A_n)_{n\in\mathbf{N}}$ in $\mathfrak{R}$ we have $\bigcap_{n\in\mathbf{N}} A_n \in \mathfrak{R}$ and $A_0 \triangle A_1 \in \mathfrak{R}$. If moreover $\bigcup_{n\in\mathbf{N}} A_n \in \mathfrak{R}$ we call $\mathfrak{R}$ a *σ-ring*. A *semi-value* on a commutative group G is a map p of G into $\mathbf{R}_+$ such that

$$p(0)=0, \qquad p(x+y)\leq p(x)+p(y), \qquad p(-x)=p(x)$$

for any $x, y \in G$. Any family of semi-values on a commutative group G defines a group topology on G and any such topology is defined by the family of continuous semi-values.

Let $\mathfrak{R}$ be a δ-ring and let G be a Hausdorff topological commutative group. A *G-valued measure on* $\mathfrak{R}$ is a map μ of $\mathfrak{R}$ into G such that for any disjoint sequence $(A_n)_{n\in\mathbf{N}}$ in $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$ we have

$$\mu\left(\bigcup_{n\in\mathbf{N}} A_n\right)=\sum_{n\in\mathbf{N}} \mu(A_n).$$

We set

$$\mathfrak{N}(\mu):=\{A\in\mathfrak{R} \mid \forall B\in\mathfrak{R},\ B\subset A \Rightarrow \mu(B)=0\}.$$

We say that μ satisfies locally ccc if any disjoint family in $\mathfrak{R}\setminus\mathfrak{N}(\mu)$ is countable if its union is contained in a set of $\mathfrak{R}$. Let $\Lambda(\mu)$ be the set of subsets $\mathfrak{A}\neq\phi$ of $\mathfrak{R}\setminus\mathfrak{N}(\mu)$ such that the intersection of any countable family in $\mathfrak{A}$ belongs to $\mathfrak{A}$. The maximal elements of $\Lambda(\mu)$ (for the inclusion relation) will be called *atoms of* μ. Let $\mathfrak{A}$ be an atom of μ and let $\mathfrak{F}(\mathfrak{A})$ be the filter on $\mathfrak{R}$ generated by the filter base

$$\{\{B\in\mathfrak{A} \mid B\subset A\} \mid A\in\mathfrak{A}\}.$$

An atom $\mathfrak{A}$ of μ is called *improper* if $\mu(\mathfrak{F}(\mathfrak{A}))$ converges to 0; otherwise we call it *proper*. A measure possessing no proper atoms is called *atomless*.

Throughout this paper we shall denote by $\mathfrak{R}$ a δ-ring and by G a Hausdorff topological commutative group. We consider $\mathfrak{R}$ o dered by the inclusion relation and denote by Λ the set of lower directed nonempty subsets of $\mathfrak{R}\setminus\{\phi\}$. For any $\mathfrak{A}\in\Lambda$ we denote by $\mathfrak{F}(\mathfrak{A})$ the filter on $\mathfrak{R}$ generated by the filter base

$$\{\{B\in\mathfrak{A}\mid B\subset A\}\mid A\in\mathfrak{A}\}.$$

PROPOSITION 1. *Let μ be an atomless G-valued measure, let p be a continuous semi-value on G, and let u be the canonical map $G\to G/P^{-1}(0)$. Then $u\circ u$ is an atomless measure satisfying locally ccc.*

$p^{-1}(0)$ is a closed subgroup of G, $G/p^{-1}(0)$ is a Hausdorff topological commutative group, and $u\circ\mu$ is a measure on a δ-ring. Since $G/p^{-1}(0)$ possesses a coarser metrizable topology $u\circ\mu$ satisfies locally ccc. From $\mathfrak{N}(\mu)\subset\mathfrak{N}(u\circ\mu)$ we deduce by [1] Corollary 1.4 that $u\circ\mu$ is atomless. ■

PROPOSITION 2. *Let μ be an atomless G-valued measure on $\mathfrak{R}$, let p be a continuous semi-value on G, and let $A\in\mathfrak{R}$. Then there exists an increasing map $B:[0, 1]\to\mathfrak{R}$ such that $B(0)=\phi$, $B(1)=A$ and such that $\mu\circ B$ is continuous with respect to the topology on G defined by p.*

By Proposition 1 and [3] Proposition 2 there exists for any $n\in\mathbf{N}$ a family $(A_{n,i})_{0<i\leqslant k_n}$ of pairwise disjoint sets of $\mathfrak{R}$ whose union is A and such that for any natural number $i\in]0, k_n]$ and for any $A'\in\mathfrak{R}$ contained in $A_{n,i}$ we have

$$p(\mu(A'))\leqslant\frac{1}{n}.$$

We may even assume $k_n\geqslant 2$ for any $n\in\mathbf{N}$. We set for any $n\in\mathbf{N}$

$$l_n := \prod_{m\leqslant n} k_m,$$

for any $i\in\mathbf{N}$, $0<i\leqslant l_0$,

$$A'_{0,i} := \bigcup_{j\leqslant i} A_{0,j},$$

and for any $n \in \mathbf{N}$, $A'_{n,0} := \phi$. We construct inductively for any $n \in \mathbf{N}\backslash\{0\}$ a family $(A'_{n,i})_{0<i\leq l_n}$ by setting for any $i \in \mathbf{N}$, $0 < i \leq l_n$,

$$A'_{n,i} := A'_{n-1,i'} \cup \left(A'_{n-1,i'+1} \cap \left(\bigcup_{j \leq i - i'k_n} A_{n,j}\right)\right),$$

where i' denotes the greatest natural number such that $i'k_n < i$. It can be shown inductively that the following properties hold for any $n \in \mathbf{N}$:

(a) $A'_{n,l_n} = A$;

(b) $0 < i \leq j \leq l_n \Rightarrow A'_{n,i} \subset A'_{n,j}$;

(c) $0 < i \leq l_n, \quad 0 < j \leq l_{n+1}, \quad \dfrac{i}{l_n} = \dfrac{j}{l_{n+1}} \Rightarrow A'_{n,i} = A'_{n+1,j}$;

(d) $0 < i \leq l_n, \quad A' \in \mathfrak{R}, \quad A' \subset A'_{n,i} \backslash A'_{n,i-1} \Rightarrow p(\mu(A')) \leq \dfrac{1}{n}$.

Let r be a rational number, $0 \leq r \leq 1$ for which there exists $n \in \mathbf{N}$ and $i \in \mathbf{N}$ such that $0 \leq i \leq l_n$ and $(i/l_n) = r$. By (c) we may set

$$B(r) := A'_{n,i}.$$

We have $B(0) = \phi$ and (by a)) $B(1) = A$. By b) $B(r) \subset B(r')$ for any $0 \leq r \leq r' \leq 1$. This last property allows us to extend the domain of B by setting for any $\alpha \in [0, 1]$

$$B(\alpha) := \bigcap_{r \geq \alpha} B(r) \in \mathfrak{R}.$$

By d) the map $\mu \circ B$ is continuous with respect to the topology on G defined by p. ■

THEOREM 3. *Let μ be an atomless measure on $\mathfrak{R}$ such that for any $A \in \mathfrak{R}$ the set $\{\mu(B) \mid B \in \mathfrak{R}, B \subset A\}$ is compact (resp. relatively compact). Then $\mu(\mathfrak{R})$ (resp. the closure of $\mu(\mathfrak{R})$) is connected.*

Let G be the target of μ, let $A \in \mathfrak{R}$. and let

$$\mathfrak{R}' := \{B \in \mathfrak{R} \mid B \subset A\}.$$

By Proposition 2 for any continuous semi-value p on G there exists a map

$$f:[0,1]\to\overline{\mu(\mathfrak{R}')}$$

continuous with respect to the topology on $\overline{\mu(\mathfrak{R}')}$ defined by p and such that $f(0)=0$, $f(1)=\mu(A)$. Hence $\mu(A)$ belongs to the connected component of 0 in $\overline{\mu(\mathfrak{R}')}$ (N. Bourbaki, nouvelle édition, TG II p. 32, Proposition 6). It follows that $\mu(A)$ belongs to the connected component of 0 in $\mu(\mathfrak{R})$ (resp. $\overline{\mu(\mathfrak{R})}$). Since A is arbitrary $\mu(\mathfrak{R})$ (resp. $\overline{\mu(\mathfrak{R})}$) is connected. ■

PROPOSITION 4. *Let μ be an atomless G-valued measure on $\mathfrak{R}$, let A be an increasing map of* $[0,1]$ *into $\mathfrak{R}$, and let p be a continuous semi-value on G. Then there exists an increasing map B of* $[0,1]$ *into $\mathfrak{R}$ such that*

$$A([0,1])\subset B([0,1])$$

and such that $\mu\circ B$ is continuous with respect to the topology on G defined by p.

Let G_p be the group G endowed with the topology defined by p, let M be the topological group $G_p/p^{-1}(0)$ and let u be the canonical map $G\to M$. By Proposition 1 $u\circ\mu$ is an atomless measure satisfying locally ccc. Let T be the set of $\alpha\in[0,1]$ at which $u\circ\mu\circ A$ is not continuous from the left. For any $\alpha\in T$ we have

$$A(\alpha)\setminus\bigcup_{\beta<\alpha}A(\beta)\notin\mathfrak{R}(u\circ\mu).$$

It follows that T is countable. Let $\alpha\in T$. By Proposition 2 there exists for any $\alpha\in T$ an increasing map A_α of $[0,1]$ into $\mathfrak{R}$ such that

$$A_\alpha(0)=\phi,\qquad A_\alpha(1)=A(\alpha)\setminus\bigcup_{\beta<\alpha}A(\beta),$$

and such that $\mu\circ A_\alpha$ is continuous as a map in G_p. Let us endow the set

$$C:=\{(\alpha,\beta)\in[0,1]\times[0,1]\mid \alpha\in T\quad\text{or}\quad\beta=0\}$$

with the lexicographical order relation. It is easy to see that C is order complete and contains a countable infinite subset which is dense in order. Moreover for any $a,b\in C$ with $a<b$ there exists $c\in C$ with $a<c<b$. From these properties we

deduce that there exists a bijective map $\psi:[0,1]\to C$ which is an isomorphism of ordered sets. Let $t\in[0,1]$ and let $(\alpha,\beta)=\psi(t)$. If $\alpha\notin T$ we set

$$B(t):=A(\alpha);$$

if $\alpha\in T$ we set

$$B(t):=A_\alpha(\beta)\cup\Big(\bigcup_{\gamma<\alpha}A(\gamma)\Big).$$

Then B is an increasing map of $[0,1]$ into $\mathfrak{R}$ such that

$$A([0,1])\subset B([0,1])$$

and such that $\mu\circ B$ is continuous from the left as a map in G_p. Moreover if A is continuous from the right then B is continuous from the right.

If we repeat the same construction starting with B instead of A and replacing the continuity from the left by the continuity from the right we get a map with the required properties. ■

THEOREM 5. *Let μ be an atomless G-valued measure on $\mathfrak{R}$ satisfying locally ccc and let $A\in\mathfrak{R}$. Then there exists an increasing map $B:[0,1]\to\mathfrak{R}$ such that $B(0)=\phi$, $B(1)=A$ and such that $\mu\circ B$ is continuous.*

Assume the contrary and let ω_1 be the first uncountable ordinal. We construct inductively a family $(p_\xi)_{\xi<\omega_1}$ of continuous semi-values on G and a family $(B_\xi)_{\xi<\omega_1}$ of increasing maps of $[0,1]$ into $\mathfrak{R}$ such that we have for any $\xi<\omega_1$:

(a) $B_\xi(0)=\phi$, $\quad B_\xi(1)=A$;

(b) $\mu\circ B_\xi$ is continuous with respect to the topology on G defined by $\{p_\eta\mid\eta<\xi\}$ and it is not continuous with respect to the topology on G defined by p_η;

(c) $\bigcup_{\eta<\xi}B_\eta([0,1])\subset B_\xi([0,1])$.

Let $\xi<\omega_1$ and assume the families were constructed for all ordinals strictly smaller than ξ. The set

$$C=\bigcup_{\eta<\xi}B_\eta([0,1])$$

is linearly ordered with respect to the inclusion relation and contains a countable subset which is dense in order. Hence there exists a subset M of $[0,1]$ and a bijection $\psi: M \to C$ which is an isomorphism of ordered sets. We may easily extend ψ to an increasing map of $[0, 1]$ to $\mathfrak{R}$. By Proposition 4 there exists an increasing map B_ξ of $[0, 1]$ into $\mathfrak{R}$ such that

$$\psi([0, 1]) \subset B_\xi([0, 1])$$

and such that $\mu \circ B_\xi$ is continuous with respect to the topology on G defined by $\{p_\eta \mid \eta < \xi\}$. Since ϕ, $A \in \psi$ $([0, 1])$ we may assume $B_\xi(0) = \phi$ and $B_\xi(1) = A$. Hence B_ξ fulfills a) and b). By the hypothesis of the proof $\mu \circ B_\xi$ is not continuous. Hence there exists a continuous semi-value p_ξ on G such that $\mu \circ B_\xi$ is not continuous with respect to the topology on G defined by p_ξ.

We set for any $\xi < \omega_1$ and for any $\alpha \in [0, 1]$

$$\bar{B}_\xi(\alpha) := A \cap \left(\bigcap_{\beta > \alpha} B_\xi(\beta) \right) \setminus \left(\bigcup_{\gamma < \alpha} B_\xi(\gamma) \right),$$

$$M_\xi := G/p_\xi^{-1}(0),$$

and denote by φ_ξ the canonical map $G \to M_\xi$. By c) two sets of the type $\bar{B}_\xi(\alpha)$ either are disjoint or one of them is included in the other one. By b) there exists for any $\xi < \omega_1$ an $\alpha(\xi) \in [0, 1]$ such that $\bar{B}_\xi(\alpha(\xi)) \notin \mathfrak{N}(\varphi_\xi \circ \mu)$. By b) for any $\eta < \xi$ we have $\bar{B}_\xi(\alpha(\xi)) \in \mathfrak{N}(\varphi_\eta \circ \mu)$. Let us denote by M_0 (resp. M_1) the set of $\xi < \omega_1$ for which the set

$$\{\eta < \omega_1 \mid \bar{B}_\eta(\alpha(\eta)) \subset \bar{B}_\xi(\alpha(\xi))\}$$

is countable (resp. uncountable). We set for any $\xi \in M_0$

$$C_\xi := \bar{B}_\xi(\alpha(\xi)) \setminus \bigcup_{\substack{\eta < \omega_1 \\ \eta > \xi}} \bar{B}_\eta(\alpha(\eta)).$$

Since $(C_\xi)_{\xi \in M_0}$ is a family of pairwise disjoint sets of $\mathfrak{R} \setminus \mathfrak{N}(\mu)$ and since μ satisfies locally ccc M_0 is countable. We may therefore construct a strictly increasing family $(\zeta(\xi))_{\xi < \omega_1}$ of elements of M_1 such that

$$\bar{B}_{\zeta(\eta)}(\alpha(\zeta(\eta))) \subset \bar{B}_{\zeta(\xi)}(\alpha(\zeta(\xi)))$$

for any ξ, η such that $\xi < \eta < \omega_1$. Then

$$(\bar{B}_{\zeta(\xi)}(\alpha(\zeta(\xi))) \setminus \bar{B}_{\zeta(\xi+1)}(\alpha(\zeta(\xi+1))))_{\xi < \omega_1}$$

is a family of pairwise disjoint sets of $\mathfrak{R}\setminus\mathfrak{N}(\mu)$ contained in A and this contradicts the hypothesis that μ satisfies locally ccc. ■

COROLLARY 6. *If μ is an atomless measure on $\mathfrak{R}$ satisfying locally ccc then $\mu(\mathfrak{R})$ is pathwise connected.* ■

Remark. D. Landers ([2] Theorem 4) showed that $\mu(\mathfrak{R})$ is pathwise connected if there exists an atomless submeasure $\lambda:\mathfrak{R}\to[0,\infty[$ dominating μ. In this case μ is atomless and satisfies locally ccc (since λ satisfies locally ccc and $\mathfrak{N}(\lambda)\subset\mathfrak{N}(\mu)$).

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