Homotopy Theory for the p-adic Special Linear Group.

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Homotopy Theory for the $\mathfrak{p}$-adic Special Linear Group

J. B. Wagoner [1])

If E is al ocal field, complete with finite residue field, the special linear group inherits a topology from E which makes it a locally compact, totally disconnected topological group and hence its homotopy groups in the usual sense are trivial. This paper proposes a definition for the higher homotopy groups of $SL(\mathfrak{l}+1, E)$ which on the one hand, agrees with the fundamental group computed by Moore [7] and by Matsumoto [5] from the viewpoint of universal topological central extensions and, on the other hand, is related to the algebraic K-theory of E viewed as an abstract field. The idea is to define groups $K_i^{\mathrm{top}}(E)$ which carry that part of the algebraic K-theory of a local field which comes from "continuous" invariants such as the continuous Steinberg symbols in the work of Matsumoto and Moore. We also define homotopy groups $K_i^{\mathrm{top}}(\mathfrak{O})$ for the special linear group over a compact discrete valuation ring $\mathfrak{O}$, and in a forthcoming joint paper of the author and R. J. Milgram we use the continuous cohomology of $SL(\mathfrak{l}+1, \mathfrak{O})$ to compute the rank of the free part of $K_i^{\mathrm{top}}(\mathfrak{O})$ considered as a module over the p-adic completion of the integers where p is the characteristic of the residue field of $\mathfrak{O}$. The theory as developed in this paper is closely connected to BN-pairs and buildings and in the last section we briefly discuss the relation of the spaces constructed in the first section to the $\mathfrak{p}$-adic building associated to $SL(\mathfrak{l}+1, E)$. We shall treat only the special linear group and the root system $A_{\mathfrak{l}}$; it seems likely that a similar program can be worked out for other simply connected algebraic groups. Useful background references are [1], [4], and [6].

Throughout this paper E will denote a field with a discrete valuation $v: E^* \to \mathbf{Z}$. Let $\mathfrak{O}$ be the valuation ring consisting of those $x \in E$ with $v(x) \geqslant 0$ and let $\mathfrak{p}$ be the maximal ideal consisting of those $x \in E$ with $v(x) > 0$. Let π be a generator for $\mathfrak{p}$.

§1. Definition of π_i^{ab} and π_i^{top}

In this section we define "abstract" homotopy groups $\pi_i^{ab} SL(\mathfrak{l}+1, E)$ and homotopy groups $\pi_i^{\mathrm{top}} SL(\mathfrak{l}+1, E)$ which take into account the topology on E. These definitions correspond respectively to the "linear" and "affine" BN-pair structures on $SL(\mathfrak{l}+1, E)$. For standard terminology in the theory of BN-pairs and root systems see [1] and [4, Chap. II].

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Let $\mathbf{R}^{l+1}$ be the set of all $(l+1)$-tuples $(x_0, x_1, \ldots, x_l)$ of real numbers. Let $e_i:\mathbf{R}^{l+1}\to\mathbf{R}$ be the ith coordinate function. The *linear stratification* of $\mathbf{R}^{l+1}$ is the decomposition of $\mathbf{R}^{l+1}$ into facettes determined by the hyperplanes $e_i-e_j=0$ of the linear root system of type A_l. As the fundamental chamber we take

$$C_0=\{x_0>x_1>\cdots>x_l\}$$

so that the positive linear roots are e_i-e_j with $i<j$ and the negative linear roots are e_i-e_j with $i>j$. The *affine stratification* of $\mathbf{R}^{l+1}$ is the decomposition into facettes determined by the hyperplanes $e_i-e_j+k=0$, $k\in\mathbf{Z}$, of the affine root system associated to A_l. The fundamental chamber is

$$C=\{x_l+1>x_0>\cdots>x_l\}.$$

If F and F' are both facettes in the linear stratification or in the affine stratification we write $F<F'$ to mean F is contained in the closure of F'. Actually, our terminology is not quite standard in that one usually speaks about the linear and affine stratifications of the subspace $V\subset\mathbf{R}^{l+1}$ given by the condition $x_0+\cdots+x_l=0$. However, the correspondence $F\to F\cap V$ is a bijection of facettes for both the linear and affine stratifications, and in the affine case the geometric realization of the nerve of the partially ordered set of facettes is precisely the first barycentric subdivision of the space V triangulated by the open rectilinear simplices $F\cap V$. The realization of the nerve of the linear stratification of $\mathbf{R}^{l+1}$-diagonal is S^{l-1}. In the present situation it is convenient to use $\mathbf{R}^{l+1}$ instead of V because then the natural stabilization map $\mathbf{R}^{l+1}\to\mathbf{R}^{l+2}$ given by

$$(x_0,\ldots, x_l)\to(x_0,\ldots, x_l, x_l)$$

takes facettes to facettes and preserves the "$<$" relation.

The "linear" BN structure on $SL(l+1, E)$ has

B = upper triangular matrices
N = matrices with exactly one non-zero entry in each row and each column.

The linear Weyl group $W_0=N/B\cap N$ is isomorphic to S_{l+1}, the symmetric group on $l+1$ letters generated by reflections in the hyperplanes $e_i-e_j=0$. W_0 acts on $\mathbf{R}^{l+1}$ by permuting the coordinates. The "affine" BN structure on $SL(l+1, E)$ has

B = subgroup of $SL(l+1,\mathfrak{O})$ consisting of matrices (m_{ij}) with $v(m_{ij})>0$ whenever $i>j$.
N = same as for the linear case.

The affine Weyl group $W = N/B \cap N$ fits into an exact sequence

$$1 \to T \to W \to W_0 \to 1$$

where T is a free abelian group of rank $\mathfrak{l}$. W is isomorphic to the group generated by the reflections in the planes $e_i - e_j + k = 0$. An element w of W acts on $\mathbf{R}^{\mathfrak{l}+1}$ as follows: Choose a representative for w in N of the form $\sigma \cdot d$ where σ is a permutation matrix and d is a diagonal matrix with entries $\pm\pi^{n_0}, \ldots, \pm\pi^{n_\mathfrak{l}}$ such that $n_0 + \cdots + n_\mathfrak{l} = 0$. Then w acts as translation by $(-n_0, \ldots, -n_\mathfrak{l})$ followed by permutation of coordinates according to σ.

If F is a facette in the linear stratification, let $U_F \subset SL(\mathfrak{l}+1, E)$ denote the subgroup generated by the elementary matrices $e_{ij}(\lambda)$ where $e_i - e_j > 0$ on F and $\lambda \in E$. If F is a facette of the affine stratification and n is a positive integer, let $k(F, e_i - e_j)_n$ be the least integer $k \in n \cdot Z$ such that $e_i - e_j + k > 0$ on F. Let $U_F^n \subset SL(\mathfrak{l}+1, E)$ denote the subgroup generated by the $e_{ij}(\lambda)$ where $v(\lambda) \geqslant k(F, e_i - e_j)_n$ and by the elements of the subgroup H^n consisting of those diagonal matrices with entries in the subgroup of units $1 + \mathfrak{p}^n$ in $\mathfrak{O}^*$.

LEMMA 1. (A) *If $F < F'$ in the linear stratification, then $U_F \subset U_{F'}$.*
(B) *If $F < F'$ in the affine stratification, then $U_F^n \subset U_{F'}^n$.*
Furthermore, if m divides n, then $U_F^n \subset U_F^m$.

The proof of this lemma will be given later on in this section.

For any two cosets $\alpha \cdot U_F$ and $\beta \cdot U_{F'}$ where $\alpha, \beta \in SL(\mathfrak{l}+1, E)$ define $\alpha \cdot U_F < \beta \cdot U_{F'}$ to mean that $F < F'$ and $\alpha \cdot U_F \subset \beta \cdot U_{F'}$. Similarly, define $\alpha \cdot U_F^n < \beta \cdot U_{F'}^n$ to mean $F < F'$ and $\alpha \cdot U_F^n \subset \beta \cdot U_{F'}^n$.

Now let $SL^{ab}(\mathfrak{l}+1, E)$ be the geometric realization of the simplicial set which has as its k-simplices $(k+1)$-tuples

$$(\alpha_0 \cdot U_{F_0} < \alpha_1 \cdot U_{F_1} < \cdots < \alpha_k \cdot U_{F_k})$$

where the faces F_i belong to the linear stratification of $\mathbf{R}^{\mathfrak{l}+1} - \Delta$. Here Δ denotes the diagonal. The group $SL(\mathfrak{l}+1, E)$ acts on $SL^{ab}(\mathfrak{l}+1, E)$ by the formula

$$\alpha \cdot (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k}) = (\alpha\alpha_0 \cdot U_{F_0} < \cdots < \alpha\alpha_k \cdot U_{F_k}).$$

Define

$$\pi_i^{ab} SL(\mathfrak{l}+1, E) = \pi_i SL^{ab}(\mathfrak{l}+1, E).$$

The stabilization map $\mathbf{R}^{\mathfrak{l}+1} - \Delta \to \mathbf{R}^{\mathfrak{l}+2} - \Delta$ induces a simplicial map

$$SL^{ab}(\mathfrak{l}+1, E) \to SL^{ab}(\mathfrak{l}+2, E)$$

and for $i \geqslant 1$ we let

$$K_{i+1}^{ab}(E) = \pi_i^{ab} SL(E) = \lim_{\substack{\rightarrow \\ \mathrm{I}}} \pi_i SL^{ab}(\mathrm{I}+1, E)$$

The definition of K_i^{ab} given here is essentially the same as that of the groups denoted K_i^{BN} in [10]. See also [13]. It is valid for any associative ring with unit E and it is a theorem [11] that $K_i^{ab}(E) = K_i^{Q}(E)$, the algebraic K-theory groups of Quillen [8]. When E is a field, it follows from [6, Cor. 11.2] and [10, Prop. 2] that for $\mathrm{I} \geqslant 3$

$$K_2(E) = K_2(\mathrm{I}+1, E) = \pi_1 SL^{ab}(\mathrm{I}+1, E).$$

Actually it is possible to show this for $\mathrm{I} \geqslant 2$ provided $SL^{ab}(\mathrm{I}+1, E)$ is defined using all linear facettes of $R^{\mathrm{I}+1}$. The reason why facettes in $\mathbf{R}^{\mathrm{I}+1} - \Delta$ are used is to get a space which maps to the building corresponding to the linear BN-pair structure on $SL(\mathrm{I}+1, E)$. See [10, §2]. As far as algebraic K-theory is concerned it is immaterial which method is used to define $\pi_i^{ab} SL(E)$ because they both are the same in the limit.

Fix a positive integer n. Let $SL_n^{\mathrm{top}}(\mathrm{I}+1, E)$ denote the geometric realization of the simplicial set which has as its k-simplices the $(k+1)$-tuples

$$(\alpha_0 \cdot U_{F_0}^n < \alpha_1 \cdot U_{F_1}^n < \cdots < \alpha_k \cdot U_{F_k}^n)$$

The group $SL(\mathrm{I}+1, E)$ acts on $SL_n^{\mathrm{top}}(\mathrm{I}+1, E)$ in the same way it acts on $SL^{ab}(\mathrm{I}+1, E)$. Whenever m divides n, $\alpha \cdot U_F^n < \beta \cdot U_{F'}^n$ implies $\alpha \cdot U_F^m < \beta \cdot U_{F'}^m$ so the correspondence $\alpha \cdot U_F^n \to \alpha \cdot U_F^m$ induces a simplicial map

$$SL_m^{\mathrm{top}}(\mathrm{I}+1, E) \leftarrow SL_n^{\mathrm{top}}(\mathrm{I}+1, E). \tag{$*$}$$

We define

$$\pi_i^{\mathrm{top}} SL(\mathrm{I}+1, E) = \lim_{\substack{\leftarrow \\ m \mid n}} \pi_i SL_n^{\mathrm{top}}(\mathrm{I}+1, E).$$

The stabilization map induces a simplicial map

$$SL_n^{\mathrm{top}}(\mathrm{I}+1, E) \to SL_n^{\mathrm{top}}(\mathrm{I}+2, E)$$

of inverse systems and we therefore can set

$$\pi_i^{\mathrm{top}} SL(E) = \lim_{\substack{\rightarrow \\ \mathrm{I}}} \pi_i^{\mathrm{top}} SL(\mathrm{I}+1, E).$$

By analogy with algebraic K-theory we could propose for $i \geqslant 2$ the definition

$$K_i^{\mathrm{top}}(E) = \pi_{i-1}^{\mathrm{top}} SL(E).$$

In the next section we will construct a homomorphism $\pi_i^{ab} \to \pi_i^{top}$, and in the third section we will use this to prove

THEOREM A. *If E is a local field (i.e. E is complete with finite residue field) and $\mathfrak{l} \geqslant 2$, then there is an isomorphism*

$$\pi_1^{top} SL(\mathfrak{l}+1, E) \simeq \mu(E)$$

where $\mu(E)$ is the group of roots of unity in E.

This result indicates that $\pi_i^{top} SL(\mathfrak{l}+1, E)$ may give the "correct" value for the higher homotopy of $SL(\mathfrak{l}+1, E)$ only in the stable range; that is, where i is somewhat smaller than $\mathfrak{l}+1$. A word about the motivation for the definition of K_i^{top}: the definition of K_i^{ab} using the linear BN-pair structure on $SL(\mathfrak{l}+1, E)$ comes directly from the geometry of Morse functions on manifolds as is explained in [10]. The group K_i^{top} came about as an attempt to see if the affine BN-structure on $SL(\mathfrak{l}+1, E)$ could be used in an analogous way.

The proof of Theorem A in §3 together with the partial computation of $K_2(E)$ known from the work of Moore [7; also 6, A. 14] gives

COROLLARY. *$K_2^{ab}(E) = K_2^{top}(E) \oplus D$ where D is infinitely divisible.*

The remainder of the section proves some lemmas which will be needed later on.

Proof of Lemma 1. The definition of a facette [1, p. 58] implies that for any facette F in the linear stratification and any linear root $e_i - e_j$ there are three mutually distinct possibilities: $e_i - e_j > 0$ on F, $e_i - e_j = 0$ on F, or $e_i - e_j < 0$ on F. The same is true of any facette F in the affine stratification and any affine root $e_i - e_j + k$.

Now let $F < F'$ in the linear stratification. To show $U_F \subset U_{F'}$ it must be verified that $e_i - e_j > 0$ on F implies $e_i - e_j > 0$ on F'. Since F is contained in the closure of F' there is some $x \in F'$ sufficiently close to F such that $e_i - e_j > 0$ on x. Hence $e_i - e_j > 0$ on all of F'.

Let $F < F'$ in the affine stratification. To show $U_F^n \subset U_{F'}^n$ we must show that $k = k(F, e_i - e_j)_n$ is greater than or equal to $k' = k(F', e_i - e_j)_n$ for each linear root $e_i - e_j$. As in the linear case $e_i - e_j + k > 0$ on F implies $e_i - e_j + k > 0$ on F'. Hence $k \geqslant k'$.

Finally, $U_F^n \subset U_F^m$ whenever m divides n because then $n \cdot Z \subset m \cdot Z$. q.e.d.

Now let F be a facette in the linear stratification. Let ${}^+U_F$ be the subgroup of U_F generated by the $e_{ij}(\lambda)$ with $i<j$, and let ${}^-U_F$ be generated by the $e_{ij}(\lambda)$ with $i>j$. Similarly for any facette F in the affine stratification let ${}^+U_F^n$ be the subgroup of U_F^n generated by the $e_{ij}(\lambda)$ with $i<j$ and $v(\lambda) \geqslant k(F, e_i - e_j)_n$. Let ${}^-U_F^n$ be the subgroup generated by the $e_{ij}(\lambda)$ with $i>j$ and $v(\lambda) \geqslant k(F, e_i - e_j)_n$.

LEMMA 2. (A) $U_F = {}^+U_F \cdot {}^-U_F = {}^-U_F \cdot {}^+U_F$

(B) $U_F^n = {}^+U_F^n \cdot H^n \cdot {}^-U_F^n = {}^-U_F^n \cdot H^n \cdot {}^+U_F^n$.

Proof of 2, Part (A). We will show that $U_F={}^+U_F\cdot{}^-U_F$; the argument for $U_F={}^-U_F\cdot{}^+U_F$ is similar. To simplify notation let $U=U_F$ and $U_p=U\cap SL(p,E)$ for $2\leqslant p\leqslant \mathfrak{l}+1$. The claim (A) is true for $p=2$ because then $U=\{e_{01}(\lambda)\mid\lambda\in E\}$ or $U=\{e_{10}(\lambda)\mid\lambda\in E\}$. Now assume inductively that $U_{\mathfrak{l}}={}^+U_{\mathfrak{l}}\cdot{}^-U_{\mathfrak{l}}$. Let $V^+\subset U_{\mathfrak{l}}$ be the subgroup generated by the $e_{i,\mathfrak{l}}(\lambda)$ with $e_i-e_{\mathfrak{l}}>0$ on F and V^- be the subgroup generated by the $e_{\mathfrak{l},i}(\lambda)$ with $e_{\mathfrak{l}}-e_i>0$ on F. Then the Steinberg relations show

(i) $U_{\mathfrak{l}}\cdot V^{\pm}=V^{\pm}\cdot U_{\mathfrak{l}}$ and ${}^-U_{\mathfrak{l}}\cdot V^+=V^+\cdot{}^-U_{\mathfrak{l}}$.

For any $0\leqslant i\leqslant\mathfrak{l}-1$ we cannot have both $e_i-e_{\mathfrak{l}}>0$ and $e_{\mathfrak{l}}-e_i>0$ on F; in other words for $0\leqslant i\leqslant\mathfrak{l}-1$ we do not have both $e_{i,\mathfrak{l}}(\lambda)\in V^+$ and $e_{\mathfrak{l},i}(\lambda)\in V^-$. Thus the Steinberg relations imply

(ii) given $v_1\in V^+$ and $v_2\in V^-$, there are elements $u\in U_{\mathfrak{l}}$, $w_1\in V^+$, and $w_2\in V^-$ such that $v_2\cdot v_1=u\cdot w_1\cdot w_2$.

Using (i), (ii), and the induction hypothesis we have

$$U_{\mathfrak{l}+1}={}^+U_{\mathfrak{l}}\cdot{}^-U_{\mathfrak{l}}\cdot V^+\cdot V^-={}^+U_{\mathfrak{l}}\cdot V^+\cdot{}^-U_{\mathfrak{l}}\cdot V^-={}^+U_{\mathfrak{l}+1}\cdot{}^-U_{\mathfrak{l}+1}.$$

Proof of 2. Part (B). The argument that $U_F^n={}^+U_F^n\cdot H^n\cdot{}^-U_F^n$ is essentially the same as Proposition (2.6.4.) of [4, p. 29]. The only minor difference is that here the groups U_F^n are defined using strict inequalities while in (2.6.4) similar groups $P_{(S)}$ are defined using weak inequalities and it is assumed that S has a non-empty interior in order to invoke (III) on p. 27 of [4]. Condition (III) is what allows one to reverse the order of $e_{ji}(\mu)\cdot e_{ij}(\lambda)$ whenever $i<j$. To make the proof of (2.6.4) work here we only need the following statement analogous to (III): Fix a pair of indices $i<j$. Let $\alpha=e_i-e_j+k$ where $k=k(F,e_i-e_j)_n$ and $\beta=e_j-e_i+k'$ where $k'=k(F,e_j-e_i)_n$. Let U_α be the subgroup generated by $e_{ij}(\lambda)$ where $v(\lambda)\geqslant k$ and U_β be the subgroup generated by $e_{ji}(\lambda)$ where $v(\lambda)\geqslant k'$. Then the subgroup generated by U_α, U_β, and H^n is

$$U_\alpha\cdot H^n\cdot U_\beta=U_\beta\cdot H^n\cdot U_\alpha.$$

The proof of this is essentially the matrix identity

$$\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}\begin{pmatrix}d&0\\0&d^{-1}\end{pmatrix}\begin{pmatrix}1&0\\\mu&1\end{pmatrix}=\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}\begin{pmatrix}1&0\\d^{-1}\mu z&1\end{pmatrix}\begin{pmatrix}1&z^{-1}\lambda d^{-1}\\0&1\end{pmatrix}$$

where $z=d+d^{-1}\lambda\mu$ and we must check that $z\in 1+\mathfrak{p}^n$. First note that since $e_i-e_j+k>0$ on F, $e_j-e_i-k<0$ on F and so $k'=r\cdot n-k$ for some $r>0$. Hence

$$v(\lambda\mu)=v(\lambda)+v(\mu)\geqslant k+k'=k+r\cdot n-k\geqslant n.$$

Now write $d=1+x$ and $d^{-1}=1+y$ where $x,y\in\mathfrak{p}^n$. Then $z=1+x+\lambda\mu+y\lambda\mu$ and $v(x+\lambda\mu+y\lambda\mu)\geqslant\min(n,n,2n)\geqslant n$. q.e.d.

LEMMA 3. (cf. (I) on p. 27 of [4]). (A) *If F is a facette of the linear stratification and $w \in W_0$ then $w \cdot U_F \cdot w^{-1} = U_{w \cdot F}$.*

(B) *Let F be a facette of the affine stratification and w be a Weyl group element of the form $\sigma \cdot t$ where $\sigma \in W_0$ and t is the diagonal matrix $(\pm\pi^{n_0}, \ldots, \pm\pi^{n_l})$ with $n_i \equiv 0 \bmod n$. Then*

$$w \cdot U_F^n \cdot w^{-1} = U_{w \cdot F}^n.$$

Proof. (A) is left as an exercise. Here is the proof of (B): Let $e_{ij}(\lambda)$ be a generator of U_F^n. Then $\alpha = e_i - e_j + v(\lambda)$ is positive on F, $v(\lambda) \geqslant k(F, e_i - e_j)_n$, and $\alpha \circ w^{-1} = e_{\sigma(i)} - e_{\sigma(j)} + (n_i - n_j) + v(\lambda)$ is positive on $w \cdot F$. Now $w \cdot e_{ij}(\lambda) \cdot w^{-1} = e_{\sigma(i)\sigma(j)}(\lambda')$ where $\lambda' = \pm\lambda\pi^{n_i - n_j}$ and so $v(\lambda') = v(\lambda) + n_i - n_j$. Since n divides $n_i - n_j$,

$$v(\lambda') \geqslant k(F, e_i - e_j)_n + n_i - n_j = k(w \cdot F, e_{\sigma(i)} - e_{\sigma(j)})_n.$$

Hence $w \cdot e_{ij}(\lambda) \cdot w^{-1} \in U_{w \cdot F}^n$. q.e.d.

LEMMA 4. *Let $u_1, \ldots, u_s \in U_F$ where F is a facette of the linear stratification of $\mathbf{R}^{l+1}$. Also fix $n > 0$. Then*

(A) *there is some facette G such that each $u_i \in U_G^n$ and*

(B) *the union of all such facettes is a convex subset of $\mathbf{R}^{l+1}$.*

Proof of (A). Choose a linear chamber D with $F < D$ and let σ be a permutation such that $\sigma^{-1} \cdot D = C_0$, the fundamental linear chamber. Then $U_F \subset U_D$ and $\sigma^{-1} \cdot U_D \cdot \sigma = U_{C_0}$. Let $v_\alpha = \sigma^{-1} u_\alpha \sigma \in U_{C_0}$ for $1 \leqslant \alpha \leqslant s$. Write each v_σ uniquely as a product of $e_{ij}(\lambda)$'s, $i < j$, ordered lexicographically. For $i < j$ let k_{ij} be the minimum of the $v(\lambda)$'s where $e_{ij}(\lambda)$ appears in at least one of the product expressions. For each $0 \leqslant i \leqslant l-1$ choose $s_i \in n \cdot Z$ in such a way that, setting $f_{ij} = s_i + \cdots + s_{j-1}$ whenever $i < j$, we have $f_{ij} \leqslant k_{ij} - n$. Then any affine facette G of maximal possible dimension in

$$\bigcap_{i<j} \{e_i - e_j + f_{ij} = 0\}$$

is non-empty and any $e_{ij}(\lambda)$ in the product expression for any v_α lies in U_G^n. Hence each v_α is in U_G^n. Finally

$$u_\alpha = \sigma v_\alpha \sigma^{-1} \in \sigma \cdot U_G^n \cdot \sigma^{-1} = U_{\sigma \cdot G}^n.$$

Proof of (B). Any $u \in U_F$ can be written uniquely as a product

$$u = u_+ \cdot u_- = \prod_{\alpha=1}^{r} e_{i_\alpha j_\alpha}(\lambda_\alpha) \cdot \prod_{\alpha=r+1}^{r+s} e_{i_\alpha j_\alpha}(\lambda_\alpha) \tag{**}$$

where $i_\alpha < j_\alpha$ for $1 \leqslant \alpha \leqslant r$, $i_\alpha > j_\alpha$ for $r+1 \leqslant \alpha \leqslant r+s$, each $e_{i_\alpha} - e_{j_\alpha} > 0$ on F, and the terms in u_+ and u_- are arranged lexicographically.

CLAIM. *$u \in U_G^n$ iff for each such $e_{i_\alpha j_\alpha}(\lambda_\alpha)$ we have* $v(\lambda_\alpha) \geqslant k(G, e_{i_\alpha} - e_{j_\alpha})_n$.

To prove this write $u = v_+ \cdot h \cdot v_-$ as in Lemma 2(B). Then $u_+ \cdot u_- = v_+ \cdot h \cdot v_-$ or $v_+^{-1} \cdot u_+ = h \cdot v_- \cdot u_-^{-1}$. Hence $h = 1$, $u_+ = v_+$, and $u_- = v_-$. Now v_+ can be written as a product in lexicographical order of terms $e_{ij}(\lambda)$ where $i < j$ and $v(\lambda) \geqslant k(G, e_i - e_j)_n$. Since this lexicographically ordered product is unique for elements in the subgroup of upper triangular matrices the expressions for u_+ and v_+ are the same. This means $v(\lambda_\alpha) \geqslant k(G, e_{i_\alpha} - e_{j_\alpha})_n$ for $1 \leqslant \alpha \leqslant r$. A similar argument works for $r+1 \leqslant \alpha \leqslant r+s$.

Now we complete the proof of (B). Let $u_1, \ldots, u_s \in U_G^n$ and $U_{G'}^n$. Choose points $x \in G$ and $x' \in G'$ and let $x(t) = (1-t)x + tx'$ for $0 \leqslant t \leqslant 1$. Let $G(t)$ be the unique facette containing $x(t)$. We shall show each u_α lies in $U_{G(t)}^n$ by showing that any $e_{ij}(\lambda)$ which appears in the product expression $(**)$ for any u_α belongs to $U_{G(t)}^n$. This amounts to showing that $v(\lambda) \geqslant k(G(t), e_i - e_j)_n$ for $0 \leqslant t \leqslant 1$. Let $k = k(G, e_i - e_j)_n$ and $k' = k(G', e_i - e_j)_n$. Let $x(t)_i$ be the ith coordinate of the vector $x(t)$. Then by the claim

$$x(0)_i - x(0)_j + k > 0$$

and

$$x(1)_i - x(1)_j + k' > 0.$$

Hence

$$x(t)_i - x(t)_j + k_t > 0.$$

where $k_t = (1-t) \cdot k + t \cdot k'$. Since $v(\lambda) \geqslant k$ and k',

$$\begin{aligned} v(\lambda) &\geqslant \text{smallest integer divisible by } n \text{ that is at least as big as } k_t \\ &\geqslant k(G(t), e_i - e_j)_n \end{aligned}$$

q.e.d.

§2. The Homomorphism $\pi_i^{ab} \to \pi_i^{\text{top}}$

This section defines a sequence of maps

$$\phi_n : SL^{ab}(I+1, E) \to SL_n^{\text{top}}(I+1, E)$$

which are compatible up to base point preserving homotopy with the maps in the inverse system $(*)$ used to define π_i^{top} and therefore induce a homomorphism

$$\phi : \pi_i^{ab} SL(I+1, E) \to \pi_i^{\text{top}} SL(I+1, E).$$

These in turn are compatible with stabilization and induce a homomorphism

$$\Phi : K_{i+1}^{ab}(E) \to K_{i+1}^{\text{top}}(E).$$

To simplify notation let $X=SL^{ab}(\mathfrak{l}+1, E)$ and $Y_n=SL_n^{top}(\mathfrak{l}+1, E)$. The base points of X and Y_n respectively are U_{C_0} and U_C^n where C_0 and C are the fundamental linear and affine chambers. Both X and Y_n are finite dimensional simplicial complexes of dimension $\mathfrak{l}-1$ and $\mathfrak{l}$ respectively. The map $\phi_n: X \to Y_n$ will be constructed inductively over the dual skeletons X_r^* of the r-skeletons X_r of X. Recall that X_r^* is the subcomplex of the first barycentric subdivision of X consisting of the union of the duals σ^* of simplices σ of X of dimension at least r where σ^* is all simplices $\sigma_1 < \cdots < \sigma_s$ such that $\sigma < \sigma_1$. For each vertex $v=\alpha \cdot U_F$ of X choose an element $g_v \in \alpha \cdot U_F$. If $v=U_{C_0}$ let $g_v=\mathrm{id}$. For each simplex σ of X let V_σ be the finite set of elements $\{g_v\}$ where v is a vertex of σ. The collection of V_σ satisfies

(a) $\sigma < \tau$ implies $V_\sigma \subset V_\tau$

(b) if $\sigma=(\alpha_0 \cdot U_{F_0} < \cdots < \alpha_p \cdot U_{F_p})$, then $g^{-1} \cdot h \in U_{F_p}$ for $g, h \in V_\sigma$.

In view of Lemma 4, (b) implies

(c) for each σ there is a facette F of the affine stratification such that $g^{-1} \cdot h \in U_F^n$ whenever $g, h \in V_\sigma$.

Using (a), (b), (c) we shall associate to each simplex σ of X a contractible set $C_\sigma \subset Y_n$ such that $\sigma < \tau$ implies $C_\sigma \supset C_\tau$. Let $A \subset Y_n$ denote the "standard" apartment consisting of simplices $(U_{F_0}^n < \cdots < U_{F_s}^n)$ and recall from the first section that A is the first barycentric subdivision of $V \subset \mathbf{R}^{\mathfrak{l}+1}$. For each simplex σ of X let $D_\sigma \subset A$ denote the union of all simplices $F \cap V$ where F satisfies (c). D_σ is convex by Lemma 4. Finally let $C_\sigma = g \cdot D_\sigma$ where g is any element of V_σ; this is well defined by (c).

Now we can construct ϕ_n. Map each vertex v_σ of X^* corresponding to a top dimensional simplex σ of X to any point in C_σ. Assume inductively that ϕ_n has been constructed over X_r^* in such a way that

$$\phi_n(\sigma^*) \subset C_\sigma \quad \text{for each simplex } \sigma \text{ of } X \text{ of dimension at least } r. \tag{†}$$

We shall show how to extend ϕ_n over X_{r-1}^* so that (†) remains satisfied. Let τ be an $(r-1)$-simplex of X and let $\tau'=(\sigma_1 < \cdots < \sigma_s)$ be a simplex in $\partial\tau^*$. Thus τ is a proper face of σ_1 and τ' is a simplex in σ_1^*. By (†) $\phi_n(\tau') \subset C_{\sigma_1} \subset C_\tau$. Since $\partial\tau^*$ is the union of such simplices τ', we have $\phi_n(\partial\tau^*) \subset C_\tau$ and hence ϕ_n can be extended over τ^* so that $\phi_n(\tau^*) \subset C_\tau$. Continuing this procedure gives the desired map $\phi: X \to Y_n$. To get the base points right note that if σ is the vertex U_{C_0}, then $C_\sigma = A$; hence we can map U_{C_0} to U_C^n in the last induction step.

It remains to show that ϕ_n is independent of the choice of elements $\{g_v\}$. Let $\{V'_\sigma\}$ and $\{V''_\sigma\}$ be two collections satisfying (a), (b), (c) coming from two choices of the g_v. Let $\{C'_\sigma\}$ and $\{C''_\sigma\}$ be the corresponding collections of contractible sets and note that $C'_\sigma \cup C''_\sigma$ is contractible. For let $C'_\sigma = g \cdot D'_\sigma$ and $C''_\sigma = h \cdot D''_\sigma$. Then

$$C'_\sigma \cap C''_\sigma = g \cdot (D'_\sigma \cap D''_\sigma) = h \cdot (D'_\sigma \cap D''_\sigma)$$

and $D'_\sigma \cap D''_\sigma$ is convex.

Now let ϕ_n' and ϕ_n'' be the two maps constructed using the collections $\{V_\sigma'\}$ and $\{V_\sigma''\}$. For each vertex v_σ of X^* define the homotopy $H: v_\sigma \times I \to Y_n$ by joining $\phi_n'(v_\sigma) \in C_\sigma'$ to $\phi_n''(v_\sigma) \in C_\sigma''$ with a path in $C_\sigma' \cup C_\sigma''$ and assume inductively that the homotopy $H: X_r^* \times I \to Y_n$ has been defined between $\phi_n' \mid X_r^*$ and $\phi_n'' \mid X_r^*$ so that

$$H(\sigma^* \times I) \subset C_\sigma' \cup C_\sigma'' \text{ for any simplex } \sigma \text{ of } X \text{ of dimension at least } r. \qquad (\dagger\dagger)$$

We will extend H over $X_{r-1}^* \times I$ so that the condition $(\dagger\dagger)$ still holds. Let τ be any $(r-1)$-simplex of X and $\tau' = (\sigma_1 < \cdots < \sigma_p)$ be any simplex in $\partial\tau^*$. Then $H(\tau' \times I) \subset C_{\sigma_1}' \cup C_{\sigma_1}'' \subset C_\tau' \cup C_\tau''$ and hence $H(\partial\tau^* \times I) \subset C_\tau' \cup C_\tau''$. Since $\phi_n'(\tau^* \times 0) \subset C_\tau'$ and $\phi_n''(\tau^* \times 1) \subset C_\tau''$, we can extend H to $\tau^* \times I$ so that $H(\tau^* \times I) \subset C_\tau' \cup C_\tau''$.

This completes the construction of ϕ_n.

§3. Computation of π_1^{top}

This section proves Theorem A which says that whenever E is complete with finite residue field

$$\pi_1^{\text{top}} SL(\mathfrak{l}+1, E) \simeq \mu(E)$$

for $\mathfrak{l} \geqslant 2$. The proof is based on the following information about Milnor's group $K_2(E)$ coming from the work of Moore [7; also 6, A.14], Dennis-Stein [2, §4], and Stein [9, Th. 2.5 and Th. 3.1]:

(i) $K_2(E) \simeq \mu(E) \oplus D$

where D is infinitely divisible and there is some $n_0 \geqslant 1$ (depending on E) such that

(ii) for each $n \geqslant n_0$ the group D is generated by $\{u, v\}$ with $u \in 1 + \mathfrak{p}^n$ and $v \in \mathfrak{O}^*$; furthermore D is the kernel of the map $K_2(\mathfrak{O}) \to K_2(\mathfrak{O}/\mathfrak{p}^n)$

The plan of the proof is to construct homomorphisms

$$\phi_n : K_2(E) \simeq K_2(\mathfrak{l}+1, E) \to \pi_1 SL_n^{\text{top}}(\mathfrak{l}+1, E)$$

for all $n \geqslant 1$ and

$$\psi_n : \pi_1 SL_n^{\text{top}}(\mathfrak{l}+1, E) \to K_2(E)/D$$

for n sufficiently large such that

(1) ϕ_n is onto with $D \subset \ker \phi_n$

(2) for large n, $\psi_n \circ \phi_n$ is just the map $K_2(E) \to K_2(E)/D$.

It then follows that for n sufficiently large ψ_n is an isomorphism and it will be clear from the construction of the ψ_n that they are compatible with the inverse system $(*)$. Hence

$$\pi_1^{\text{top}} SL(\mathfrak{l}+1, E) \simeq K_2(E)/D \simeq \mu(E).$$

The maps ϕ_n will essentially be those defined in §2 modulo the equivalence $K_2(E) \simeq \pi_1^{ab} SL(\mathfrak{l}+1, E)$ for $\mathfrak{l} \geqslant 3$ demonstrated in [10]. However to avoid invoking this isomorphism we will give some details of the construction of the ϕ_n which will be needed anyway to establish (1) and (2).

Step 1. Defining ϕ_n for $n \geqslant 1$ and $\mathfrak{l} \geqslant 2$.

Let $z \in U_C^n$ be represented as a product

$$z = z_1 \cdot z_2 \cdot \cdots \cdot z_s$$

where $z_i \in U_{F_i}$ for some facette F_i in the linear stratification. To this product we will associate a loop

$$z_0 * z_1 * \cdots * z_s \subset SL_n^{\mathrm{top}}(\mathfrak{l}+1, E) \tag{∇}$$

from the base point U_C^n to itself as follows:

(a) First choose for $i = 0, \ldots, s+1$ an affine face G_i such that $z_i \in U_{G_i}^n$ for $i = 1, \ldots, s$ and $G_0 = G_{s+1} = C$.

(b) Then choose a path γ_i from the vertex $U_{G_i}^n$ to the vertex $U_{G_{i+1}}^n$ in the standard apartment $A \subset SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)$.

Let $\beta_0 = 1$ and for $1 \leqslant k \leqslant s$ let $\beta_k = z_1 \cdot \cdots \cdot z_k$. Let $\bar{z}_k = \beta_k \cdot \gamma_k$ be the translate of γ_k by β_k under the action of $SL(\mathfrak{l}+1, E)$ on $SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)$. The endpoint of $\bar{z}_{k-1}$ is $\beta_{k-1} \cdot U_{G_k}^n$ and $\beta_{k-1} \cdot U_{G_k}^n = \beta_{k-1} \cdot z_k \cdot U_{G_k}^n = \beta_k \cdot U_{G_k}^n$ is the initial point of $\bar{z}_k$. Hence the paths $\bar{z}_k$ can be strung end to end to produce the loop (∇).

First we show (∇) is independent of the choices (a) and (b): It is clearly independent of the choice of path γ_k because the standard apartment is contractible. To show the choice in (a) doesn't matter let $U_{G'_i}^n$ be another group containing z_i. By Lemma 4 there is a piecewise linear path ϱ_i from $U_{G_i}^n$ to $U_{G'_i}^n$ in the standard apartment such that each vertex U_G^n of ϱ_i contains z_i. Then (∇) constructed using the $U_{G'_i}^n$ is represented, in virtue of the independence from the choice (b), by the concatenation

$$\beta_0 \cdot (\gamma_0 * \varrho_1) * \cdots * \beta_k \cdot (\varrho_k^{-1} * \gamma_k * \varrho_{k+1}) * \cdots * \beta_s \cdot (\varrho_s^{-1} * \gamma_s)$$

We write this as

$$\cdots * \beta_{k-1} \cdot (\varrho_{k-1}^{-1} * \gamma_{k-1} * \varrho_k) * \beta_k \cdot (\varrho_k^{-1} * \gamma_k * \varrho_{k+1}) * \cdots$$

which is the same as

$$\cdots * \beta_{k-1} \cdot \varrho_{k-1}^{-1} * \beta_{k-1} \cdot \gamma_{k-1} * \beta_{k-1} \cdot \varrho_k * \beta_k \cdot \varrho_k^{-1} * \beta_k \cdot \gamma_k * \beta_k \cdot \varrho_{k+1} * \cdots$$

Now $\beta_{k-1} \cdot \varrho_k = \beta_k \cdot \varrho_k$ because each vertex in the path ϱ_k contains z_k. Hence the terms $\beta_{k-1} \cdot \varrho_k * \beta_k \cdot \varrho_k^{-1}$ cancel out to give the loop

$$\cdots * \beta_{k-1} \cdot \gamma_{k-1} * \beta_k \cdot \gamma_k * \cdots$$

which is the path representing (∇) obtained from the original choice (a).

Actually, since the path $\bar{z}_0$ is determined by z_1, it will be convenient from now on to denote the loop (∇) by $\bar{z}_1 * \cdots * \bar{z}_s$.

The construction of (∇) shows that

(c) if $z_1 \in U_C^n$, then $\bar{z}_1 * \cdots * \bar{z}_s = \bar{z}_2 * \cdots * \bar{z}_s$ and if $z_s \in U_C^n$, then $\bar{z}_1 * \cdots * \bar{z}_s = \bar{z}_1 * \cdots * \bar{z}_{s-1}$

(d) if for some $1 \leqslant k \leqslant s-1$ there is a linear face F and an affine face G such that z_k and z_{k+1} both belong to U_F and U_G^n, then

$$\bar{z}_1 * \cdots * \bar{z}_k * \bar{z}_{k+1} * \cdots * \bar{z}_s = \bar{z}_1 * \cdots * \overline{z_k \cdot z_{k+1}} * \cdots * \bar{z}_s.$$

Now to get ϕ_n, let $y \in K_2(E)$ be represented as the word

$$y = \prod_{\alpha=1}^{s} x_{i_\alpha j_\alpha}(\lambda_\alpha).$$

Then

$$\mathrm{id} = \prod_{\alpha=1}^{s} e_{i_\alpha j_\alpha}(\lambda_\alpha)$$

and we let

$$\phi_n(y) = \bar{e}_{i_1 j_1}(\lambda_1) * \cdots * \bar{e}_{i_s j_s}(\lambda_s).$$

Any two presentations of y as a product differ by Steinberg relations and these don't change $\phi_n(y)$ in view of (d). For example

$$\cdots * \overline{e_{ij}(\lambda) \cdot e_{jk}(\mu)} * \cdots = \cdots * \overline{e_{ik}(\lambda\mu) \cdot e_{jk}(\mu) \cdot e_{ij}(\lambda)} * \cdots$$

because all the generators in the third Steinberg relation belong to U_F for any linear facette on which $e_i - e_j$ and $e_j - e_k$ (and therefore $e_i - e_k$) are positive, and they also belong to U_G^n where G is any affine facette contained in

$$e_i - e_j + r = 0, \qquad e_j - e_k + r' = 0, \qquad e_i - e_k + r + r' = 0$$

for r and r' sufficiently negative.

PROPOSITION 5. *ϕ_n is onto for $\mathfrak{l} \geqslant 2$.*

Proof. The construction of (∇) gives a procedure for constructing a path $\bar{z}$ from any coset $\alpha \cdot U_F^n$ to any coset $\alpha \cdot \omega \cdot U_G^n$ given any word $z \in St(\mathfrak{l}+1, E)$ with $\varrho(z) = \omega \in SL(\mathfrak{l}+1, E)$. Here $\varrho : St \to SL$ is the natural homomorphism. The proof that ϕ_n is onto reduces to the special case of showing that a one-simplex of the form $(\alpha \cdot U_F^n < \beta \cdot U_G^n)$ is homotopic with endpoints fixed to the path $\bar{z}$ where $z \in St(\mathfrak{l}+1, E)$ is chosen so that $\varrho(z) = \alpha^{-1} \cdot \beta \in U_G^n$.

Write $\alpha^{-1}\cdot\beta=v_+\cdot g\cdot v_-$ as in Lemma 2. The upper and lower triangular matrices v_+ and v_- lift to well defined elements of $St\,(\mathrm{I}+1, E)$ which we continue to denote by v_+ and v_-. Let z_g be a lifting of g to a product of words of the form $h_{ij}(u)=$ $=w_{ij}(u)\cdot w_{ij}(-1)$ where $u\in 1+\mathfrak{p}^n$. See [6, §9]. Let $z=v_+\cdot z_g\cdot v_-$. Then $\bar{z}=\bar{v}_+ * \bar{z}_g * \bar{v}_-$ is a path from $\alpha\cdot U_F^n$ to $\beta\cdot U_G^n$ and we will show it is homotopic with endpoints fixed to the one-simplex $(\alpha\cdot U_F^n<\beta\cdot U_G^n)$. Since $v_+\in U_G^n$ the path $\bar{v}_+$ from $\alpha\cdot U_F^n$ to $\alpha\cdot U_G^n$ is $(\alpha\cdot U_F^n<\alpha\cdot v_+\cdot U_G^n)=(\alpha\cdot U_F^n<\alpha\cdot U_G^n)$. By Lemma 6 below the path $\bar{z}_g$ from $\alpha\cdot U_G^n$ $=\alpha\cdot v_+\cdot U_G^n$ to $\alpha\cdot U_G^n=\alpha\cdot v_+\cdot g\cdot U_G^n$ (obtained by translating via $\alpha\cdot v_+$ the path $\bar{z}_g$ considered as a loop from U_G^n to U_G^n) is homotopic to the constant path with endpoints fixed. The path $\bar{v}_-$ from $\alpha\cdot U_G^n=\alpha\cdot v_+\cdot g\cdot U_G^n$ to $\beta\cdot U_G^n=\alpha\cdot v_+\cdot g\cdot v_-\cdot U_G^n$ is homotopic to the constant path $(\alpha\cdot U_G^n<\beta\cdot U_G^n)$ because $v_-\in U_G^n$ and $\alpha\cdot U_G^n=\beta\cdot U_G^n$. Hence

$$\begin{aligned}\bar{v}_+ * \bar{z}_g * \bar{v}_- &\sim(\alpha\cdot U_F^n<\alpha\cdot U_G^n)*(\alpha\cdot U_G^n<\alpha\cdot U_G^n)*(\alpha\cdot U_G^n<\beta\cdot U_G^n)\\ &\sim(\alpha\cdot U_F^n<\beta\cdot U_G^n)\end{aligned}$$

q.e.d.

LEMMA 6. *The path $\bar{z}_g$ from U_G^n to U_G^n is homotopic to a constant keeping endpoints fixed.*

Proof. The general case reduces to the special case where $z_g=w_{ij}(u)\cdot w_{ij}(-1)$ and either $i<j$ or $i>j$. Suppose $i<j$. Then $\bar{z}_g$ from U_G^n to U_G^n is of the form $\eta*\bar{z}_g*\eta^{-1}$ where η is a path in the standard apartment from U_G^n to U_C^n and $\bar{z}_g$ is considered as a path from U_C^n to itself. We show that $\bar{z}_g$ is homotopic to a constant when viewed as a loop based at U_C^n. Recall that

$$C=\{1+x_{\mathrm{I}}>x_0>\cdots>x_{\mathrm{I}}\}.$$

Let

$$D=\{1+x_{\mathrm{I}}>x_0>\cdots>x_i=\cdots=x_j>x_{j+1}>\cdots>x_{\mathrm{I}}\}$$

and

$$\begin{aligned}C'&=\{1+x_{\mathrm{I}}>x_0>\cdots>x_j>\cdots>x_i>\cdots>x_{\mathrm{I}}\}, &&\text{if } j\neq \mathrm{I}\\ &=\{1+x_i>x_0>\cdots>x_{i-1}>x_{\mathrm{I}}>x_{i+1}>\cdots>x_i\}, &&\text{if } j=\mathrm{I}.\end{aligned}$$

Then $C>D<C'$ and each elementary matrix $e_{ij}(\lambda)$ or $e_{ji}(\lambda)$ appearing in the word $w_{ij}(u)\cdot w_{ij}(-1)$ lies in one of the subgroups U_C^n or $U_{C'}^n$. This means we can construct $\bar{z}_g$ as in (∇) by choosing as in (b) just the path $\gamma=U_C^n>U_D^n<U_{C'}^n$ or its reverse γ^{-1}. Write $u=1+\sigma$ and $u^{-1}=1+\tau$ for $\sigma,\tau\in\mathfrak{p}^n$ and note that $e_{ij}(\pm\sigma)$ and $e_{ji}(\pm\tau)$ lie in U_D. Then using properties (c) and (d)

$$\begin{aligned}\bar{z}_g&=\bar{e}_{ij}(1+\sigma)*\bar{e}_{ji}(-1-\tau)*\bar{e}_{ij}(1+\sigma)*\bar{e}_{ij}(-1)*\bar{e}_{ji}(1)*\bar{e}_{ij}(-1)\\ &=\bar{e}_{ji}(-1-\tau)*\bar{e}_{ij}(1+\sigma)*\bar{e}_{ij}(-1)*\bar{e}_{ji}(1)\\ &=\bar{e}_{ji}(-\tau)*\bar{e}_{ji}(-1)*\bar{e}_{ij}(\sigma)*\bar{e}_{ij}(1)*\bar{e}_{ij}(-1)*\bar{e}_{ji}(1)\\ &=\bar{e}_{ji}(-1)*\bar{e}_{ij}(\sigma)*\bar{e}_{ji}(1).\end{aligned}$$

This last path is $\gamma * \varepsilon \cdot \gamma^{-1}$ where $\varepsilon = e_{ji}(-1) \cdot e_{ij}(\sigma) \cdot e_{ji}(1)$ which lies in each of U_C^n, U_D^n, and $U_{C'}^n$. Hence $\varepsilon \cdot \gamma^{-1} = \gamma^{-1}$ and $\gamma * \varepsilon \cdot \gamma^{-1} = \gamma * \gamma^{-1}$ which is certainly contractible keeping the base point U_C^n fixed.

The case when $i > j$ is similar. q.e.d.

PROPOSITION 7. *$D \subset \ker \phi_n$ whenever n is large enough to satisfy* (ii).

Proof. Let $\{u, v\} \in D$ where $u \in 1 + \mathfrak{p}^n$ and $v \in \mathfrak{O}^*$. Then $\phi_n(\{u, v\}) = \bar{h}_{01}(u) * \bar{h}_{02}(v) * \bar{h}_{01}(u)^{-1} * \bar{h}_{02}(v)^{-1}$ which is homotopic to $\bar{h}_{13}(v) * \bar{h}_{13}(v)^{-1}$ by Lemma 6. This last loop is clearly contractible. q.e.d.

Step 2. Construction of ψ_n.

Throughout this part we will assume $n \geqslant n_0 + 1$ where n_0 is as in (ii) and we assume $\mathfrak{l} \geqslant 2$. Let $m =$ order of $\mu(E)$ and let $(\ ,\ )_m : E^* \times E^* \to \mu(K)$ be the mth power norm residue symbol. Let

$$1 \to \mu(E) \to X \to SL(\mathfrak{l}+1, E) \to 1$$

be the continuous central extension associated to $(\ ,\ )_m$. By [6, A.14 and p. 95] this is algebraically equivalent to

$$1 \to K_2(E)/D \to St\,(\mathfrak{l}+1, E)/D \xrightarrow[\varrho]{} SL(\mathfrak{l}+1, E) \to 1\,.$$

LEMMA 8. *For each affine face F there is a homomorphism $s_F : U_F^n \to St\,(\mathfrak{l}+1, E)/D$ such that if $F < F'$, then $s_{F'} \mid U_F^n = s_F$ and such that $\varrho \circ s_F = \mathrm{id}$.*

Proof. Given an affine facette F let X_F^+ (resp. X_F^-) be the subgroup of $St\,(\mathfrak{l}+1, E)/D$ generated by $x_{ij}(\lambda)$ with $i<j$ (resp. $i>j$) and $v(\lambda) \geqslant k(F, e_i - e_j)_n$, and let Y be the subgroup of $St\,(\mathfrak{l}+1, E)/D$ generated by the words $h_{ij}(u)$ with $u \in 1 + \mathfrak{p}^n$. Let G_F be the subgroup of $St\,(\mathfrak{l}+1, E)/D$ generated by X_F^+, X_F^-, and Y. Then

$$G_F = X_F^+ \cdot Y \cdot X_F^- = X_F^- \cdot Y \cdot X_F^+\,.$$

The proof of this is similar to (B) in Lemma 2 and the main point is that for $i<j$ we have

$$x_{ij}(\lambda) \cdot x_{ji}(\mu) = h_{ij}(z) \cdot x_{ji}(\mu z) \cdot x_{ij}(z^{-1}\lambda) \tag{α}$$

modulo D where $\lambda, \mu \in E$ and $z = 1 + \lambda\mu \in 1 + \mathfrak{p}^n$. Here we continue to use the notation of (B) of Lemma 2. Not both k and k' are zero because $k' = r \cdot n - k$ for $r > 0$. Thus the argument breaks down into two steps.

Case 1. Both k and k' are non-negative and so at least one of them, say k, is strictly positive. Then $k \geqslant n$ because $k \in n \cdot z$. In particular $v(\lambda) \geqslant n$. Thus when we consider the element

$$x_{ij}(\lambda) \cdot x_{ji}(\mu) \cdot x_{ij}(z^{-1}\lambda)^{-1} \cdot x_{ji}(\mu z)^{-1} \cdot h_{ij}(z)^{-1}$$

of $K_2(\mathfrak{O})$ as an element of $K_2(\mathfrak{O}/\mathfrak{p}^n)$ it becomes zero. By (ii), i.e. by [2, §4], it lies in D so (α) holds modulo D.

Case 2. One of k or k' is negative. Say $k=-s$ where $s>0$. Then $k'\geqslant n$ because $k+k'\geqslant n$. Let $p\neq i,j$. Since the subgroup D is in the center of $St\,(\mathfrak{l}+1, E)$ the equation (α) holds modulo D iff its conjugate by $h_{ip}(\pi^s)$ holds modulo D:

$$\begin{aligned}x_{ij}(\pi^s\lambda)\cdot x_{ji}(\pi^{-s}\mu)&=h_{ij}(\pi^s z)\cdot h_{ij}(\pi^s)^{-1}\cdot x_{ji}(\pi^{-s}\mu z)\cdot x_{ij}(\pi^s z^{-1}\lambda)\\&=\{\pi, z\}^s\cdot h_{ij}(z)\cdot x_{ji}(\pi^{-s}\mu z)\cdot x_{ij}(\pi^s z^{-1}\lambda).\end{aligned}$$

Note that $v(\pi^s\lambda)\geqslant 0$, $v(\pi^{-s}\mu)\geqslant n$, $v(\pi^s z^{-1}\lambda)\geqslant 0$ and $v(\pi^{-s}\mu z)\geqslant n$. Let $z=1+v\pi^n$. Then by [2, §2]

$$\{\pi, z\}=\{\pi, 1+v\pi^n\}=\left\{-\frac{1+v\pi^{n-1}}{1-\pi}, \frac{1+v\pi^n}{1-\pi}\right\}^{-1}$$

and so $\{\pi, z\}\in K_2(\mathfrak{O})$ has the same image in $K_2(\mathfrak{O}/\mathfrak{p}^{n-1})$ as

$$\left\{-\frac{1}{1-\pi}, \frac{1}{1-\pi}\right\}^{-1}$$

which is zero by [6, Lemma 9.8]. The word

$$x_{ij}(\pi^s\lambda)\cdot x_{ji}(\pi^{-s}\mu)\cdot x_{ij}(\pi^s z^{-1}\lambda)^{-1}\cdot x_{ji}(\pi^{-s}\mu z)^{-1}\cdot h_{ij}(z)^{-1}\cdot\{\pi, z\}^{-s}$$

in $K_2(\mathfrak{O})$ therefore goes to zero in $K_2(\mathfrak{O}/\mathfrak{p}^{n-1})$; so again by [2, §4] it lies in D and (α) holds modulo D. Now to complete the proof of Lemma 8: By [6, Lemma 9.14] the maps $\varrho: X_F^{\pm}\to {}^{\pm}U_F^n$ are isomorphisms. The map $\varrho: Y\to H^n$ is a surjection. Any element $\omega\in Y$ with $\varrho(\omega)=1$ lies in the kernel of the map $K_2(\mathfrak{O})\to K_2(\mathfrak{O}/\mathfrak{p}^n)$ and therefore in D by [2, §4]. Hence $\varrho: Y\to H^n$ is an isomorphism also and consequently $\varrho: G_F\to U_F^n$ is an isomorphism. We define $s_F: U_F^n\to St\,(\mathfrak{l}+1, E)/D$ to be the inverse of ϱ restricted to G_F. If $F<F'$ then $G_F\subset G_{F'}$, so $s_{F'}\mid U_F^n=s_F$. q.e.d.

To define the homomorphism ψ_n we use essentially the same method as in [10, Prop. 2]. If $\mathfrak{l}$ denotes the directed one-simplex $(\alpha\cdot U_F^n<\beta\cdot U_G^n)$, we let $\mathfrak{l}^{-1}$ denote the same one-simplex directed from $\beta\cdot U_G^n$ back to $\alpha\cdot U_F^n$. Any piecewise linear loop γ from U_G^n to itself is a concatenation

$$\gamma=\mathfrak{l}_1^{\varepsilon_1} * \mathfrak{l}_2^{\varepsilon_2} * \cdots * \mathfrak{l}_s^{\varepsilon_s}$$

where $\varepsilon_i=\pm 1$. Choose an element in each vertex $\alpha\cdot U_F^n$ of $SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)$. For each $\mathfrak{l}_i$ we have the element $g_i^{-1}\cdot h_i$ where g_i is the chosen element in the initial vertex of $\mathfrak{l}_i$ and h_i is in the final vertex of $\mathfrak{l}_i$. Since $g_{i+1}=h_i$ the element

$$u=\prod_{i=1}^{s}(g_i^{-1}\cdot h_i)^{\varepsilon_i}$$

lies in U_C^n. If $\mathfrak{l}_i = (\alpha \cdot U_F^n < \beta \cdot U_G^n)$ we let $L_i = s_G(g_i^{-1} \cdot h_i)$. Let w be a lifting of u to $St(\mathfrak{l}+1, E)$ obtained as in the proof of Lemma 8 and define

$$\psi_n(\gamma) = L_1^{\varepsilon_1} \cdot L_2^{\varepsilon_2} \cdot \dots \cdot L_s^{\varepsilon_s} \cdot w^{-1} \in K_2(E)/D.$$

Any two such liftings of u as in Lemma 8 are congruent modulo D so $\psi_n(\gamma)$ doesn't depend on which lifting we choose. Using Lemma 8 it is not hard to see that $\psi_n(\gamma)$ is independent of the choice of elements in the $\alpha \cdot U_F^n$ and also is independent of any simplicial homotopy. For example suppose we have a segment in γ which looks like

$$\alpha_1 \cdot U_{F_1}^n < \alpha_2 \cdot U_{F_2}^n < \alpha_3 \cdot U_{F_3}^n$$

and that $g_i \in \alpha_i \cdot U_{F_i}^n$ are the chosen elements. Let g_2 be changed to h and write $g_2 = h \cdot v$ where $v \in U_{F_2}^n$. The corresponding subwords of $\psi_n(\gamma)$ are $s_{F_2}(g_1^{-1} \cdot g_2) \cdot s_{F_3}(g_2^{-1} \cdot g_3)$ and $s_{F_2}(g_1^{-1} \cdot h) \cdot s_{F_3}(h^{-1} \cdot g_3)$. But we have

$$\begin{aligned} s_{F_2}(g_1^{-1} \cdot g_2) \cdot s_{F_3}(g_2^{-1} \cdot g_3) &= s_{F_2}(g_1^{-1} \cdot h \cdot v) \cdot s_{F_3}(v^{-1} \cdot h^{-1} \cdot g_3) \\ &= s_{F_2}(g_1^{-1} \cdot h)\, s_{F_2}(v) \cdot s_{F_3}(v^{-1})\, s_{F_3}(h^{-1} \cdot g_3) \\ &= s_{F_2}(g_1^{-1} \cdot h) \cdot s_{F_3}(h^{-1} \cdot g_3). \end{aligned}$$

The above procedure gives the desired homomorphism from the edge path presentation of $\pi_1 SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)$ into $K_2(E)/D$. We have verified (1) in Propositions 5 and 7. Property (2) is an immediate consequence of the construction of ϕ_n and ψ_n.

§4. π_i^{top} for Discrete Valuation Rings

Let $\mathfrak{O}$ be a discrete valuation ring with unique maximal ideal $\mathfrak{p}$. Let $v: E \to \mathbf{Z}$ be the associated discrete valuation on the quotient field E of $\mathfrak{O}$. This section outlines the modifications necessary to define

$$\pi_i^{\mathrm{top}} SL(\mathfrak{l}+1, \mathfrak{O})$$

so that we have

THEOREM B. *If $\mathfrak{O}$ is complete with finite residue field of characteristic p and $\mathfrak{l} \geqslant 2$, then*

$$\pi_1^{\mathrm{top}} SL(\mathfrak{l}+1, \mathfrak{O}) = \mu(E)_p$$

where $\mu(E)_p$ is the p-primary component of the group of roots of unity in E.

Let F be a facette in the affine stratification of $\mathbf{R}^{\mathfrak{l}+1}$ and let $n \geqslant 1$. Define V_F^n to be the subgroup of $SL(\mathfrak{l}+1, \mathfrak{O})$ generated by the $e_{ij}(\lambda)$ where $\lambda \in \mathfrak{O}$ and $v(\lambda) \geqslant$

$k(F, e_i - e_j)_n$. The analogue of Lemma 1 remains valid and wel et $SL_n^{\mathrm{top}}(\mathfrak{l}+1, \mathfrak{O})$ be the realization of the simplicial set whose k-simplices are $(k+1)$-tuples

$$(\alpha_0 \cdot V_{F_0}^n < \cdots < \alpha_k \cdot V_{F_k}^n)$$

When m divides n there is a map

$$SL_m^{\mathrm{top}}(\mathfrak{l}+1, \mathfrak{O}) \leftarrow SL_n^{\mathrm{top}}(\mathfrak{l}+1, \mathfrak{O})$$

and we let

$$\pi_i^{\mathrm{top}} SL(\mathfrak{l}+1, \mathfrak{O}) = \lim_{\substack{\leftarrow \\ m \mid n}} \pi_i SL_n^{\mathrm{top}}(\mathfrak{l}+1, \mathfrak{O})$$

and

$$K_i^{\mathrm{top}}(\mathfrak{O}) = \lim_{\substack{\rightarrow \\ \mathfrak{l}}} \pi_{i-1}^{\mathrm{top}} \mathrm{SL}(\mathfrak{l}+1, \mathfrak{O})$$

The analogues of Lemma 2 and Lemma 3 hold although (B) of Lemma 3 is valid for the V_F^n only for $w \in W_0$. This is sufficient however to prove Lemma 4 for the V_F^n because in the proof the conjugation is by an element of W_0. The proof of Theorem B is similar to the argument in Theorem A but uses the following result of Dennis-Stein [2, §4]: For $r \geqslant 3$

$$K_2(r, \mathfrak{O}) = K_2(\mathfrak{O}) = \mu(E)_p \oplus D$$

where D is the same subgroup as in section 3. All the steps in the proof can be done by replacing E by $\mathfrak{O}$. One point to mention is that in Case 2 of Lemma 8 it is not necessary to conjugate the equation (α) to get an equivalent equation in $St\,(\mathfrak{l}+1, \mathfrak{O})/D$ because the terms in (α) already lie in $St\,(\mathfrak{l}+1, \mathfrak{O})$.

The natural inclusions $V_F^n \subset U_F^n$ induce a map

$$SL_n^{\mathrm{top}}(\mathfrak{l}+1, \mathfrak{O}) \rightarrow SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)$$

of inverse systems and we get a homomorphism

$$K_i^{\mathrm{top}}(\mathfrak{O}) \rightarrow K_i^{\mathrm{top}}(E).$$

Following the method of §2 one constructs a natural homomorphism

$$K_i^{ab}(\mathfrak{O}) \rightarrow K_i^{\mathrm{top}}(\mathfrak{O})$$

compatible with the above homomorphism. It seems plausible to conjecture that there is a short exact sequence

$$0 \rightarrow K_i^{\mathrm{top}}(\mathfrak{O}) \rightarrow K_i^{\mathrm{top}}(E) \rightarrow K_{i-1}(\mathfrak{O}/\mathfrak{p}) \rightarrow 0$$

similar to the one in algebraic K-Theory. See [3, Th. 1.3] and [8, §5]. For $i=2$ Theorems A and B show this is true; namely, we get

$$\begin{array}{ccccccccc} 0 \to & K_2^{top}(\mathfrak{O}) & \to & K_2^{top}(E) & \to & K_1(\mathfrak{O}/\mathfrak{p}) & \to 0 \\ & \| & & \| & & \| & \\ 0 \to & \mu(E)_p & \to & \mu(E) & \to & (\mathfrak{O}/\mathfrak{p})^* & \to 0. \end{array}$$

In a future paper we will show that

$$K_i^{top}(\mathfrak{O}) = \varprojlim_n K_i(\mathfrak{O}/\mathfrak{p}^n).$$

Since the kernel of the homomorphism $GL(\mathfrak{O}/\mathfrak{p}^{n+1}) \to GL(\mathfrak{O}/\mathfrak{p}^n)$ is a p-group, the maps $K_i(\mathfrak{O}/\mathfrak{p}^{n+1}) \to K_i(\mathfrak{O}/\mathfrak{p}^n)$ are isomorphisms on the $\mathfrak{l}$-primary part whenever $(\mathfrak{l}, p)=1$. Hence by Quillen's computation of the K-theory for a finite field [12] we have

$$\begin{aligned} &K_{2i}^{top}(\mathfrak{O})_{(\mathfrak{l})} = 0, && i \geqslant 1 \\ &K_{2i-1}^{top}(\mathfrak{O})_{(\mathfrak{l})} = [\mathbf{Z}/(q^i_{-1}) \cdot \mathbf{Z}]_{(\mathfrak{l})}, && i \geqslant 2 \end{aligned}$$

where $(\mathfrak{l}, p)=1$ and $q = |\mathfrak{O}/\mathfrak{p}|$.

§5. Relation to Buildings

In this section we discuss the relationship of the spaces $SL^{ab}(\mathfrak{l}+1, E)$ and $SL_n^{top}(\mathfrak{l}+1, E)$ to the buildings corresponding to the linear and affine BN-pair structures on $SL(\mathfrak{l}+1, E)$. The significance of this is not clear, but one possible motivation concerns unitary representations of the p-adic group $SL(\mathfrak{l}+1, E)$. Borel and Serre have shown that the Hilbert space of square summable harmonic forms in dimension $\mathfrak{l}$ on the $\mathfrak{p}$-adic building associated to $SL(\mathfrak{l}+1, E)$ is the special representation and that the cohomology with compact support in dimension $\mathfrak{l}$ is contained in it as the set of admissible vectors. Perhaps the L^2 harmonic forms on some of the $SL(\mathfrak{l}+1, E)$-spaces below decompose to give useful realizations for other irreducible representations of $SL(\mathfrak{l}+1, E)$.

Recall that the linear building I^{ab} (resp. the $\mathfrak{p}$-adic or affine building I^{aff}) is the realization of the simplicial set whose k-simplices are the $(k+1)$-tuples

$$(\alpha_0 \cdot P_{S_0} \supset \cdots \supset \alpha_K \cdot P_{S_K})$$

where $\alpha_i \in SL(\mathfrak{l}+1, E)$, S_i is a linear (resp. affine) facette contained in the closure of the fundamental chamber C_0 (resp. C), and P_{S_i} is the parabolic (resp. parahoric) subgroup associated to S_i. In the linear case we require S_i to be a facette of $(R^{\mathfrak{l}+1}$-diagonal$)\cap \bar{C}_0$. Actually, I^{ab} and I^{aff} as defined here are the first barycentric subdivisions of the buildings as usually defined. In both cases the action of $SL(l+1, E)$ is given by

$$\alpha\cdot(\alpha_0\cdot P_{S_0}\supset\cdots\supset\alpha_k\cdot P_{S_k})=(\alpha\alpha_0\cdot P_{S_0}\supset\cdots\supset\alpha\alpha_k\cdot P_{S_k}).$$

See [4, chap. II].

The following two lemmas will be useful. Let $S=\{u, v, w\ldots\}$ be a set partially ordered by a relation "$\leqslant$" such that if $u\leqslant v$ and $v\leqslant u$ then $u=v$. Suppose G is a group acting on the right of S and preserving the ordering in such a way that

(β) if $u, v\leqslant w$ and $v=u\cdot g$, then $u=v$.

Let S/G denote the space of orbits partially ordered by the condition that $u\cdot G\leqslant v\cdot G$ iff there is a $g\in G$ with $u\cdot g\leqslant v$. Note that $u\cdot G\leqslant v\cdot G$ and $v\cdot G\leqslant u\cdot G$ implies $u\cdot G=v\cdot G$.

Let $|S|$ and $|S/G|$ denote the geometric realizations of the nerves of these partially ordered sets.

LEMMA 9. (A) $|S|/G=|S/G|$

(B) *if G acts freely on S, then G acts freely and properly on $|S|$ as a discrete group. In particular the quotient space $|S|/G$ is not just triangulable but has a natural triangulation.*

Proof. (A). There is a natural simplicial map $\varrho:|S|\to|S/G|$ given by the correspondence

$$(v_0\leqslant\cdots\leqslant v_k)\to(v_0\cdot G\leqslant\cdots\leqslant v_k\cdot G)$$

which takes non-degenerate simplices to non-degenerate simplices. We must show that if $\sigma=(v_0\leqslant\cdots\leqslant v_k)$ and $\tau=(w_0\leqslant\cdots\leqslant w_k)$ are two non-degenerate simplices with $\varrho(\tau)=\varrho(\sigma)$, then there is a $g\in G$ with $\tau\cdot g=\sigma$. By hypotheses we know that for each w_i there is a $g_i\in G$ with $w_i\cdot g_i=v_i$. Consider the simplex $\tau\cdot g_k=(w_0\cdot g_k\leqslant\cdots\leqslant w_k\cdot g_k)$. Then $w\cdot g_k=v_k$ and for each $0\leqslant i<k$ we have $v_i=w_i\cdot g_k\cdot(g_k^{-1}g_i)$. Hence, by (β), $v_i=w_i\cdot g_k$ and so $\tau\cdot g_k=\sigma$.

(B) Let $\sigma=(v_0\leqslant\cdots\leqslant v_k)$ be a non-degenerate simplex. Let $st\ (\sigma)$ be the open star of σ consisting of all simplices τ having σ as a face. We shall show that if $st\ (\sigma)\cap st\ (\sigma)\cdot g$ is not empty then $g=1$. Let τ be a simplex such that $\sigma\leqslant\tau$ and $\sigma\leqslant\tau\cdot g$, and let v be a vertex of σ. Then $v\cdot g$ is a vertex of $\tau\cdot g$ and v is a vertex of $\tau\cdot g$. Thus either $v\cdot g\leqslant v$ in which case $v\cdot g=v$ by (β) and $g=1$ since G acts freely; or $v\leqslant v\cdot g$ so that $v\cdot g^{-1}\leqslant v$ and $g^{-1}=1$. q.e.d.

Now let $S'=\{u, v, w,\ldots\}$ be partially ordered by a relation "$\geqslant$" and let G act on the right of S' preserving the ordering and satisfying

(β′) if $w\geqslant u, v$ and $u\cdot g=v$, then $w\cdot g=w$.

Partially order the orbit space S'/G by setting $u\cdot G\geqslant v\cdot G$ iff there is a $g\in G$ with $u\cdot g\geqslant v$.

LEMMA 10. (A) $|S'|/G=|S'/G|$

(B) *if G acts freely on the set of vertices, then G acts freely and properly on* $|S'|$.

Proof. (A). The natural map $\varrho:|S'|\to|S'|/G$ takes non-degenerate simplices to non-degenerate simplices and we must show that if $\sigma=(v_0\geqslant\cdots\geqslant v_k)$ and $\tau=(w_0\geqslant\cdots\geqslant w_k)$ have the same image under ϱ then there is a $g\in G$ with $\sigma=\tau\cdot g$. By hypothesis we know there are elements $g_i\in G$ with $w_i\cdot g_i=v_i$ for $i=0,\ldots,k$. Since $w_0\cdot g_0=v_0$ and $w_1\cdot g_1=w_1\cdot g_0\cdot(g_0^{-1}g_1)=v_1$, we conclude from (β′) that $w_0\cdot g_1=v_0$. By induction one has $w_i\cdot g_k=v_i$ for $i=0,\ldots,k$. Hence $\sigma=\tau\cdot g_k$. The proof of (B) is similar to (B) of Lemma 9. q.e.d.

The Linear Case. Compare [10, §2]. For any associative ring E there is an equivariant map $\theta:SL^{ab}(\mathrm{l}+1,E)\to I^{ab}$ defined by the correspondence which takes the simplex $(\alpha_0\cdot U_{F_0}<\cdots<\alpha_k\cdot U_{F_k})$ to the simplex $(\alpha_0\omega_0\cdot P_{S_0}\supset\cdots\supset\alpha_k\omega_k\cdot P_{S_k})$ where S_i is the unique linear facette in $R^{\mathrm{l}+1}$-diag and in the closure of C_0 such that $\omega_i\cdot S_i=F_i$ for some $\omega_i\in W_0$. The cosets $\alpha_i\omega_i\cdot P_{S_i}$ are independent of the choice of the ω_i.

We shall construct a space K on which $SL(\mathrm{l}+1,E)$ acts on the left such that the map θ factors as the composition of $SL(\mathrm{l}+1,E)$-equivariant simplicial maps

$$SL^{ab}(\mathrm{l}+1,E)\to K\xrightarrow{\Phi}I^{ab}$$

where the first map is a covering map with group N of the linear BN-pair structure on $SL(\mathrm{l}+1,E)$.

The group N acts on the right of $SL^{ab}(\mathrm{l}+1,E)$ as follows: Let $\alpha\cdot U_F$ be a vertex of $SL^{ab}(\mathrm{l}+1,E)$ and let $\eta\in N$. Define

$$(\alpha\cdot U_F)\cdot\eta=\alpha\eta\cdot U_{\eta^{-1}\cdot F}.$$

This is well defined; for if $\alpha\cdot U_F=\beta\cdot U_F$, then $\beta=\alpha\cdot u$ with $u\in U_F$ and

$$\beta\eta\cdot U_{\eta^{-1}\cdot F}=\alpha u\eta\cdot U_{\eta^{-1}\cdot F}=\alpha\eta(\eta^{-1}u\eta)\cdot U_{\eta^{-1}\cdot F}=\alpha\eta\cdot U_{\eta^{-1}\cdot F}$$

because $\eta^{-1}u\eta\in U_{\eta^{-1}\cdot F}$ by (A) of Lemma 3. This action preserves the relation "$<$" on the cosets $\alpha\cdot U_F$ and induces an action on $SL^{ab}(\mathrm{l}+1,E)$.

LEMMA 11. *The action of N on* $SL^{ab}(\mathrm{l}+1,E)$ *satisfies condition* (β) *and is free and proper.*

Proof. Suppose $\alpha\cdot U_F<\gamma\cdot U_H$, $\beta\cdot U_G<\gamma\cdot U_H$, and $\alpha\cdot U_F=\beta\eta\cdot U_{\eta^{-1}\cdot G}$. The definition of the partial ordering of the $\alpha\cdot U_F$ in §1 implies that $F=\eta^{-1}\cdot G$. Now $\gamma^{-1}\alpha\cdot U_F\subset U_H$ and $\gamma^{-1}\beta\cdot U_G\subset U_H$, and so $\beta^{-1}\alpha\in U_H$. Also $\alpha\cdot U_F=\beta\eta\cdot U_{\eta^{-1}\cdot G}=\beta\eta\cdot U_F$ so that $\eta^{-1}\beta^{-1}\alpha\in U_F$. Hence $\eta\in U_H$ and since $N\cap U_H=1$ we have $\eta=1$. Thus $\alpha\cdot U_F=\beta\cdot U_G$. It is also easy to see that N acts freely on the vertices $\alpha\cdot U_F$. Thus (β) is satisfied and N acts freely and properly by (B) of Lemma 9.

Let $K=SL^{ab}(\mathfrak{l}+1, E)/N$. Since θ is constant on the orbits of N we obtain the desired factorization.

Recall from [10, §2] that θ, and therefore Φ, is onto in homology in dimension $\mathfrak{l}-1$. For E a finite field both $H_{\mathfrak{l}-1}(SL^{ab}(\mathfrak{l}+1, E); \mathbf{Z})$ and $H_{\mathfrak{l}-1}(K; \mathbf{Z})$ are free abelian groups of finite rank mapping equivariantly onto the Steinberg representation $H_{\mathfrak{l}-1}(I^{ab}; \mathbf{Z})$. What can be said about them as $SL(\mathfrak{l}+1, E)$ modules?

Here is an alternate description of K. For any linear facette F let N_F be the subgroup of elements η of N such that $\eta\cdot F=F$. Then N_F normalizes U_F by (A) of Lemma 3 and so $Q_F=N_F\cdot U_F$ is a subgroup of $SL(\mathfrak{l}+1, E)$. Note that $F<G$ implies $N_F\supset N_G$ so that Q_F is not a subgroup of Q_G unless $F=G$. Define a partial ordering on the cosets $\alpha\cdot Q_F$ by the condition that $\alpha\cdot Q_F\prec\beta\cdot Q_G$ iff $F<G$ and there is an element $\eta\in N_F$ such that $\alpha\eta\cdot U_F\subset\beta\cdot U_G$.

LEMMA 12. *K is isomorphic as a simplicial complex with a left $SL(\mathfrak{l}+1, E)$ action to the space whose k-simplices are $(k+1)$-tuples*

$$(\alpha_0\cdot Q_{S_0}\prec\cdots\prec\alpha_k\cdot Q_{S_k})$$

where each S_i is a linear facette of $\mathbf{R}^{\mathfrak{l}+1}$-diag *contained in the closure of C_0.*

The proof is straight forward because there is a bijection between orbits $(\alpha\cdot U_F)\cdot N$ and the cosets $\alpha\cdot Q_S$.

Finally, we calculate $\Phi^{-1}(\tau)$ for any simplex τ of I^{ab}. Actually, since $\tau=\alpha\cdot\sigma$ where $\alpha\in SL(\mathfrak{l}+1, E)$ and $\sigma=(P_{S_0}\supset\cdots\supset P_{S_k})$, it suffices to describe $\Phi^{-1}(\sigma)$. Here is the formula:

$$\Phi^{-1}(\sigma)=\beta\cdot(Q_{S_0}\prec\cdots\prec Q_{S_k})$$

where $\beta\in P_{S_k} \bmod N_{S_k}\cdot U_{S_0}$. Thus we also have

$$\theta^{-1}(\sigma)=\beta\eta\cdot(U_{\eta^{-1}\cdot S_0}<\cdots<U_{\eta^{-1}\cdot S_k})$$

for $\beta\in P_{S_k} \bmod N_{S_k}\cdot U_{S_0}$ and $\eta\in N$.

The Affine Case. Let E be a local field, complete with finite residue field. For each $n\geqslant 1$ we shall construct a sequence of $SL(\mathfrak{l}+1, E)$-equivariant simplicial maps

$$SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)\to K_n \xrightarrow[\Psi_n]{} I_n^{\mathrm{aff}}$$

such that whenever m divides n there is a commutative diagram of $SL(\mathfrak{l}+1, E)$-equivariant simplicial maps

$$\begin{array}{ccccc} SL_n^{\mathrm{top}}(\mathfrak{l}+1, E) & \to & K_n & \xrightarrow{\Psi_n} & I_n^{\mathrm{aff}} \\ \downarrow & & \downarrow & & \downarrow \\ SL_m^{\mathrm{top}}(\mathfrak{l}+1, E) & \to & K_m & \xrightarrow{\Psi_m} & I_m^{\mathrm{aff}} \end{array} \qquad (\kappa)$$

satisfying the properties

(i) for $n=1$, I_n^{aff} is the affine building I^{aff},

(ii) each Ψ_n and each vertical map in the diagram is a proper map,

(iii) the map $SL_n^{\mathrm{top}}(\mathrm{l}+1, E)\to K_n$ is a covering space with fiber the group M^n defined as follows: let $T^n\subset T$ be the subgroup of translations in W of the form $\mathrm{diag}(\pi^{n_0},\ldots,\pi^{n_l})$ where $n_i\equiv 0 \bmod n$. Let W^n be the subgroup of W generated by T^n and W_0. Let $N^n\subset N$ be the subgroup of elements mapping to W^n under the arrow $N\to W$. Finally let $M^n=N^n \bmod H^n$.

First we define an action of N^n on the right of $SL_n^{\mathrm{top}}(\mathrm{l}+1, E)$ and prove

LEMMA 13. *The induced action of M^n is free, proper, and satisfies condition* (β). Let $\eta\in N^n$. Define

$$(\alpha\cdot U_F^n)\cdot\eta=\alpha\eta\cdot U^n_{\eta^{-1}\cdot F}.$$

This is well defined: for let $\alpha\cdot U_F^n=\beta\cdot U_F^n$ where $\beta=\alpha\cdot u$. Then

$$\beta\eta\cdot U^n_{\eta^{-1}\cdot F}=\alpha u\eta\cdot U^n_{\eta^{-1}\cdot F}=\alpha\eta(\eta^{-1}u\eta)\cdot U^n_{\eta^{-1}\cdot F}=\alpha\eta\cdot U^n_{\eta^{-1}\cdot F}$$

because $\eta^{-1}u\eta\in U^n_{\eta^{-1}\cdot F}$ according to (B) of Lemma 3. If $\eta\in H^n$, then $\eta\cdot F=F$ so $(\alpha\cdot U_F^n)\cdot\eta=\alpha\cdot U_F^n$. Hence M^n acts on the right of $SL^n_{\mathrm{top}}(\mathrm{l}+1, E)$. This action is free on vertices. For suppose $\alpha\eta\cdot U^n_{\eta^{-1}\cdot F}=\alpha\cdot U_F^n$ in the sense of the partial ordering. Then $\eta^{-1}\cdot F=F$ and $\eta\cdot U^n_{\eta^{-1}\cdot F}=U_F^n$ as cosets. Thus $\eta\in N\cap U_F^n=H^n$. This shows the isotropy groups of N^n are just H^n. Hence M^n acts freely. The argument showing (β) is satisfied is similar to the proof of Lemma 11. We apply Lemma 9 to see that M^n acts freely and properly on $SL_n^{\mathrm{top}}(\mathrm{l}+1, E)$.

In view of Lemma 13 we can define K_n as $SL_n^{\mathrm{top}}(\mathrm{l}+1, E)/M^n$ and get a covering space map $SL_n^{\mathrm{top}}(\mathrm{l}+1, E)\to K_n$ which is $SL(\mathrm{l}+1, E)$-equivariant.

Next we define the spaces I_n^{aff}. First let J^{aff} be the realization of the simplicial set whose k-simplices are the $(k+1)$-tuples $(\alpha_0\cdot P_{F_0}\supset\cdots\supset\alpha_k\cdot P_{F_k})$ where each P_{F_i} is the parahoric subgroup corresponding to the affine facette F_i. Here F_i runs over all affine facettes and not just those in the closure of the fundamental chamber C. Recall that each P_F is of the form $w\cdot P_S\cdot w^{-1}$ where $w\in W$ and S is a facette in $\bar{C}$ with $w\cdot S=F$. Furthermore, for $\eta\in N$ one has $\eta\cdot P_F\cdot\eta^{-1}=P_{\eta\cdot F}$. The group $SL(\mathrm{l}+1, E)$ acts on the left of J^{aff} by

$$\alpha\cdot(\alpha_0\cdot P_{F_0}\supset\cdots\supset\alpha_k\cdot P_{F_k})=(\alpha\alpha_0\cdot P_{F_0}\supset\cdots\supset\alpha\alpha_k\cdot P_{F_k}).$$

LEMMA 14. *There is an action of N on the right of J^{aff} which satisfies condition* (β′) *and induces an action of W satisfying* (β′).

Proof. For $\eta\in N$ define

$$(\alpha\cdot P_F)\cdot\eta=\alpha\eta\cdot P_{\eta^{-1}\cdot F}$$

This is well defined: for let $\alpha\cdot P_F=\beta\cdot P_F$ where $\beta=\alpha\cdot p$. Then

$$\beta\eta\cdot P_{\eta^{-1}\cdot F}=\alpha p\eta\cdot P_{\eta^{-1}\cdot F}=\alpha\eta(\eta^{-1}p\eta)\cdot P_{\eta^{-1}\cdot F}=\alpha\eta\cdot P_{\eta^{-1}\cdot F}$$

because $\eta^{-1}p\eta\in P_{\eta^{-1}\cdot F}$. To see that ($\beta'$) holds suppose $\gamma\cdot P_H\supset\alpha\cdot P_F$, $\gamma\cdot P_H\supset\beta\cdot P_G$, and $\alpha\cdot P_F=\beta\eta\cdot P_{\eta^{-1}\cdot G}$ for some $\eta\in N$. Then $P_F=P_{\eta^{-1}\cdot G}$ and $\eta\cdot F=G$. Since $\bar{F}\supset H\subset\bar{G}$ we must have $\eta\cdot H=H$. Hence $\eta\in P_H$ so that $\gamma\cdot\eta\cdot P_{\eta^{-1}\cdot H}=\gamma\cdot P_H$, which is ($\beta'$). Since $N\cap B$ is contained in each P_F the action of N induces an action of $W=N/N\cap B$ satisfying (β'). q.e.d.

We define $I_n^{\mathrm{aff}}=J^{\mathrm{aff}}/W^n$. For each $n\geq 1$ there is a simplicial $SL(\mathfrak{l}+1, E)$-equivariant map

$$\psi_n: SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)\to J^{\mathrm{aff}}$$

defined on vertices by $\psi_n(\alpha\cdot U_F)=\alpha\cdot P_F$. It is compatible with the actions of M^n and W^n and therefore induces an $SL(\mathfrak{l}+1, E)$-equivariant map

$$\Psi_n: K_n=SL_n^{\mathrm{top}}(\mathfrak{l}+1, E)/M^n\to J^{\mathrm{aff}}/W^n=I_n^{\mathrm{aff}}.$$

When m divides n we clearly get the commutative diagram (κ).

To establish (ii) we use the fact that if $f: X\to Y$ is a simplicial map between locally finite, finite dimensional simplicial complexes such that the number of elements in $f^{-1}(\sigma)$ is bounded by a fixed constant as σ runs over the simplices of Y, then f is proper. Actually it suffices to show $f^{-1}(\sigma)$ bounded by a fixed constant for σ any vertex. Thus we must compute the inverse images of simplices under Ψ_n and under the vertical maps in the diagram (κ). To do this it will be convenient to give equivalent descriptions of the spaces K_n and I_n^{aff}.

For $n\geq 1$ we shall let $C_n=\{x_{\mathfrak{l}}+n>x_0>x_1>\cdots>x_{\mathfrak{l}}\}$. Then $\bar{C}_n$ is a fundamental domain for W^n acting on $R^{\mathfrak{l}+1}$ and is a union of affine facettes. For any affine facette F, let N_F^n be the stabilizer of F in N^n. N_F^n normalizes U_F^n by (B) of Lemma 3 and we let $Q_F^n=N_F^n\cdot U_F^n$. For any two cosets $\alpha\cdot Q_F^n$ and $\beta\cdot Q_G^n$ we let $\alpha\cdot Q_F^n\prec\beta\cdot Q_F^n$ iff $F<G$ and there is an $\eta\in N_F^n$ such that $\alpha\cdot\eta\cdot U_F^n\subset\beta\cdot U_G^n$.

LEMMA 15. (A) *K_n is the realization of the simplicial set whose k-simplices are $(k+1)$-tuples*

$$(\alpha_0\cdot Q_{S_0}^n\prec\cdots\prec\alpha_k\cdot Q_{S_k}^n)$$

where $S_i\subset\bar{C}_n$.

(B) *I_n^{aff} is the realization of the simplicial set whose k-simplices are $(k+1)$-tuples*

$$(\alpha_0\cdot P_{S_0}\supset\cdots\supset\alpha_k\cdot P_{S_k})$$

where $S_i\subset\bar{C}_n$.

The proof of (A) follows from the bijection between orbits $(\alpha\cdot U_F^n)\cdot N^n$ and cosets $\alpha\cdot Q_S^n$; similarly for the proof of (B). In particular $I_1^{\text{aff}}=I^{\text{aff}}$, which gives (i).

In view of Lemma 15 the map Ψ_n is induced by the correspondence $\alpha\cdot Q_S^n\to\alpha\cdot P_S$. The composite map

$$SL_n^{\text{top}}(\text{I}+1, E)\to K_n\to I_n^{\text{aff}}$$

is induced by the correspondence $\alpha\cdot U_F^n\to\alpha\omega\cdot P_S$ where $\omega\in W^n$ and $S\subset\bar{C}_n$ are such that $\omega\cdot S=F$.

Denote the vertical maps in the diagram (κ) by

$$p\colon SL_n^{\text{top}}(\text{I}+1, E)\to SL_m^{\text{top}}(\text{I}+1, E)$$
$$q\colon K_n\to K_m$$
$$r\colon I_n^{\text{aff}}\to I_m^{\text{aff}}$$

With the exception of the map q, we shall describe the inverse images of simplices lying in a region whose translates by $SL(\text{I}+1, E)$ fill up the space:

(a) $\psi_n^{-1}(P_{F_0}\supset\cdots\supset P_{F_k})=\alpha\cdot(U_{F_0}^n<\cdots<U_{F_k}^n)$
where $F_i\subset R^{\text{I}+1}$ where $\alpha\in P_{F_k}\bmod U_{F_0}^n$

(b) $\Psi_n^{-1}(P_{S_0}\supset\cdots\supset P_{S_k})=\alpha\cdot(Q_{S_0}^n\prec\cdots\prec Q_{S_k}^n)$
where $S_i\subset\bar{C}_n$ where $\alpha\in P_{S_k}\bmod N_{S_k}^n\cdot U_{S_0}^n$

(c) $p^{-1}(U_{F_0}^m<\cdots<U_{F_k}^m)=\alpha\cdot(U_{F_0}^n<\cdots<U_{F_k}^n)$
where $F_i\subset R^{\text{I}+1}$ where $\alpha\in U_{F_0}^m\bmod U_{F_0}^n$

(d) $r^{-1}(P_{S_0}\supset\cdots\supset P_{S_k})=(\eta^{-1}\cdot P_{\eta\cdot S_0}\supset\cdots\supset\eta^{-1}\cdot P_{\eta\cdot S_k})$
where $S_i\subset\bar{C}_m$ where $\eta\cdot S_i\subset\bar{C}_n$ for $\eta\in W^m$

We must check that in each case the cardinality of the inverse images of simplices is uniformly bounded. In (b) and (d) this is clear because on the left hand side there are only finitely many simplices and on the right hand side there are only finitely many α's and η's. Uniform boundedness in case (a) holds because $P_{F_k}\bmod U_{F_0}^n\simeq P_{S_k}\bmod U_{S_0}^n$ for some S_k, S_0 in $\bar{C}_n$; it holds for case (c) because $U_{F_0}^m\bmod U_{F_0}^n\simeq U_{S_0}^m\bmod U_{S_0}^n$ for some $S_0\subset\bar{C}_n$.

The inverse image of a simplex under q is tedious to describe; however, we still know that q is proper because r and Ψ_n are proper.

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