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Maps Without Certain Singularities

by ANDREW DU PLESSIS

§0. Introduction

In this paper we study the problem of finding, for any smooth map $f: N \to P$ between manifolds, the maps $g: N \to P$ in the homotopy class of f which are as non-singular as possible. We shall discuss this only with regard to the Whitney-Thom singularities; although these singularities do not provide a complete description of the structure of f (even locally), they do go a long way towards it, while their geometric interpretation makes them natural objects to study.

The geometric situation is as follows:

Define $\Sigma^i(f) = \{x \in N \mid \text{kernel rank } Tf_x = i\}$. If, for a particular map f, $\Sigma^i(f)$ is submanifold of N, then $f \mid \Sigma^i(f)$: $\Sigma^i(f) \to P$ is a smooth map between manifolds, and we define $\Sigma^{ij}(f) = \Sigma^j(f \mid \Sigma^i(f))$. If this also is a manifold, we may define $\Sigma^{ijk}(f)$, and so on.

From this point of view it is not clear for which (if any!) maps f the sets $\Sigma^{ij}(f)$, $\Sigma^{ijk}(f)$, etc. are defined. However, a more subtle approach yields the following:

THEOREM 1 (Boardman [1]). *For each r-sequence (sequence of r integers) $I = (i_1, \ldots, i_r)$ there is a submanifold Σ^I of the infinite jet-space $J(N, P)$ of codimension v^I which is the inverse image by the natural projection $J(N, P) \to J^r(N, P)$ of a submanifold of $J^r(N, P)$.*

As I runs over all r-sequences, these submanifolds form a partition of $J(N, P)$ which is effectively finite, for Σ^I is empty unless

(a) $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_r \geqslant 0$

(b) $\dim N \geqslant i_1 \geqslant \dim N - \dim P$

(c) *if $i_1 = \dim N - \dim P$, then $i_1 = i_2 = \cdots = i_r$.*

If $f: N \to P$ is a smooth map, we define $\Sigma^I(f) = (Jf)^{-1} \Sigma^I$. These sets have the following properties:

(a) $\Sigma^j(f) = \{x \in N \mid \text{kr}\, Tf_x = j\}$.

(b) If Jf is transverse to Σ^I (so that $\Sigma^I(f)$ is a submanifold of N), then $\Sigma^{I,j}(f) = \Sigma^j(f \mid \Sigma^I(f))$. ($I, j$ is the $(r+1)$-sequence $(i, \ldots, i_r, j)$).

THEOREM 2 (Boardman [1]; a variant of the Thom transversality theorem).

The smooth maps $N \to P$ whose jet-sections are transverse to all the submanifolds Σ^I of $J(N, P)$ (the generic *maps) form a dense in $\mathscr{C}^\infty(N, P)$.*

Thus, for each r-sequence I and "most" maps f, $\Sigma^I(f)$ is a submanifold of N which may be constructed by the geometric method described above; and these submanifolds $\Sigma^I(f)$, as I runs over all r-sequences, give a finite partition of N.

Our aim is to construct a map g homotopic to f for which this partition has as few sets as possible, so that we regard the greatest r-sequence (*w.r.t.* lexicographic order) I s.t. $\Sigma^I(f) \neq \emptyset$ as the *singularity* of f and seek to reduce this by homotopy.

(Of course this definition applies equally to non-generic maps; and any non-generic map of singularity I may be fine-C^∞-approximated by generic maps of singularity $\leqslant I$, hence is homotopic to such a map.)

Let $\Omega^I = \bigcup \{\Sigma^K \mid r\text{-sequences } K \leqslant I\} \subset J(N, P)$. We shall say that a map $f: N \to P$ is Ω^I-regular if $Jf(N) \subset \Omega^I$; thus f is Ω^I-regular $\Leftrightarrow$ its singularity is $\leqslant I$.

Ω^I-regularity is a condition on r-jets (since for any r-sequence K, Σ^K is the inverse image of a submanifold of $J^r(N, P)$ by the natural projection $\pi^r: J(N, P) \to J^r(N, P)$) and it is in fact, in the terminology of [6], a stable, natural regularity condition. (This means that $\pi^r \Omega^I$ is an open sub-bundle of $J^r(N, P) \to N$ invariant under the natural action by local diffeomorphisms of N on $J^r(N, P)$; we will prove this later, in (1.4).) Hence the theorem of Gromov [4] (Theorem A of [6]) applies to Ω^I-regularity, and we have

THEOREM A. *Let $\mathscr{C}_{\Omega^I}(N, P)$ be the space of Ω^I-regular maps $N \to P$, with the C^r-topology, and let $\Gamma(\Omega^I(N))$ be the space of smooth sections of the bundle $\pi^r \Omega^I(N) \to N$ (with the compact-open topology).*

Then, if N is an open manifold,

$$j^r: \mathscr{C}_{\Omega^I}(N, P) \to \Gamma(\Omega^I(N))$$

is a weak homotopy equivalence.

In particular, we have the following

COROLLARY A. *Suppose N is an open manifold. Then a (smooth) map $f: N \to P$ is homotopic to a generic map with singularity $\leqslant I \Leftrightarrow$ there is a section of Ω^I covering f.*

Hence calculation of the minimal singularity in the homotopy class of f is reduced, when N is open, to a question of algebraic topology – the existence or otherwise of sections of the bundles Ω^I covering f.

However, the non-singular maps constructed need not be proper maps, and so while non-singular according to the letter of our definition may have rather pathological behaviour. Many non-singular maps may be constructed on an open manifold by "pushing singularities to infinity" – there is a Morse function without critical points on any open manifold, for example – and we should properly regard the constructions of Corollary A as an extension (albeit non-trivial) of this rather deceitful procedure for "hiding" singularities rather than actually getting rid of them.

Our theory should therefore be couched in terms of proper maps, or, more naturally, closed manifolds.

From [6], we have

THEOREM B. *If Ω^I is extensible, then*

$$j^r:\mathscr{C}_{\Omega^I}(N, P)\to\Gamma(\Omega(N))$$

is a weak homotopy equivalence, whether N is open or closed.

This result, together with the Approximation Theorem in the Appendix to [6] implies the following

COROLLARY B. *Let Ω^I be an extensible regularity condition. Then, whether N is open or closed, a proper smooth map $f: N\to P$ is homotopic to a proper map of singularity $\leqslant I \Leftrightarrow$ there is a section of Ω^I covering f.*

The notion of extensibility is fully explained in [6]; for Ω^I to be extensible it is sufficient to show that there exists a natural stable regularity condition $\Omega'\subset J^r(N\times\mathbf{R}, P)$ s.t. the natural projection $i^*J^r(N\times\mathbf{R}, P)\to J^r(N, P)$ (where $i: N=N\times 0\subset N\times\mathbf{R}$) carries $i^*\Omega'$ *onto* Ω^I.

The work of the paper will be concerned with discovering conditions under which Ω^I is extensible. It will turn out that Corollary B then gives considerable information on reducing singularity by homotopy, for the condition that Ω^I be extensible is not too restrictive: we will show that Ω^I $(I=(i_1,\dots, i_r))$ is extensible if

$$i_r > \dim N - \dim P - d^I.$$

Here $d^I=\Sigma_{s=1}^{r-1}\alpha_s$, where

$$\alpha_s=\begin{cases}1 & \text{if } i_s - i_{s+1} > 1.\\ 0 & \text{otherwise}\end{cases}$$

To obtain this result we shall require many results (and some slight extensions of results) from Boardman's paper [1]; these we introduce in §1. In §2 we prove the extensibility condition given above. §3 contains algebraic results required in this proof.

Notes

1. The result of Corollary B may hold even when Theorem B fails; for example, Eliashberg [2] has shown that Corollary B holds for $\Omega^{1,0}$ even though Theorem B fails (so of course $\Omega^{1,0}$ is not extensible).

In a later paper we will describe how Corollary B may be proved for certain other non-extensible Ω^I by application of transversality techniques.

Note however that the result of Corollary B is not always true; $\Omega^0 \subset J(S^1, \mathbf{R}^1)$ provides a counter-example.

2. From the point of view of the manifold N, at least, an equally natural measure of the singularity of a smooth map f might be

$$s(f) = \max\{s^I \mid \bar{\Sigma}^I(f) \neq \emptyset\}.$$

This does not give the same ordering of "singularity" as we have previously adopted (for example, when $\dim N = \dim P$, $\nu^{3,0} = 9$ and $\nu^{2,2} = 10$).

However, the partition $\{\Sigma^I\}$ of $J(N, P)$ by r-sequences I is not a stratification (for example, when $\dim N = \dim P$, $\Sigma^{2,0,0,0,0}$ contains points of $\bar{\Sigma}^{1,1,1,1,1}$ although $\nu^{2,0,0,0,0} = 4$ and $\nu^{1,1,1,1,1} = 5$). Thus there is no easy geometric interpretation of non-singularity according to this scheme. We shall nevertheless consider this kind of non-singularity in the Appendix.

3. The result of Theorem B for Ω^0 was obtained by Hirsch [5] for $\dim N < \dim P$; and the result was obtained for Ω^{i_1} by Feit [3] for $i_1 > \dim N - \dim P$.

§1

In this chapter we introduce some results due to Boardman [1]; and we make some extensions to these results.

(1.1) *The Total Tangent Bundle*

Let N, P be smooth manifolds.

We recall that the topology of the infinite-jet space $J(N, P)$ has as base the sets $(\pi^r)^{-1}U$, where $r < \infty$ and U is open in $J^r(N, P)$ ($\pi^r: J(N, P) \to J^r(N, P)$ is the natural projection). $J(N, P)$ also has a "limit differential structure", defined as follows: if U is an open set in $J(N, P)$, we say a function $\Phi: U \to \mathbf{R}$ is smooth if it is locally of the form $\psi \cdot \pi^r$, where ψ is a smooth function on some open subset of $J^r(N, P)$.

It follows that $J(N, P)$ has a tangent bundle.

Now let $f: N \to P$ be a germ of smooth map at $x \in N$. Then the germ of infinite-jet section $Jf: N \to J(N, P)$ is also smooth, and so there is a tangent map $T(Jf)_x: TN_x \to TJ(N, P)$ (which is clearly an injection). It is easy to check (in local co-ordinates) that if two germs f, g have the same infinite jet at $x \in N$, then $T(Jf)_x = T(Jg)_x$. Thus the images of such tangent maps define a sub-bundle of $TJ(N, P)$, the *total tangent bundle D*.

Let $\pi_N: J(N, P) \to N$ be the natural projection; it follows at once that $T\pi_N \mid D: D \to TN$ is an isomorphism of fibres covering π_N.

We shall need the following result in §2:

LEMMA (1.1.1). *Let $F: J(N, P) \to J(N', P')$ be a smooth map of jet spaces covering a smooth map $f: N \to N'$. Suppose also that F is induced by a continuous transformation of smooth germs.*

Then there is a commutative diagram

$$\begin{array}{ccc} D_x & \xrightarrow{TF \mid D} & D'_{F(x)} \\ \| & & \| \\ TN_{\pi_N x} & \xrightarrow{Tf} & TN'_{\pi_{N'} F(x)} \end{array} \qquad (x \in J(N, P))$$

(where D′ is the total tangent bundle of $J(N', P')$).

Proof. Since F is induced by a continuous transformation of germs, say F', we have $F(Jg) = J(F'(g))$. So $TF \cdot T(Jg) = T(J(F'(g)))$, and thus $TF(D) \subset D'$.

Now $\pi_{N'} \cdot F = f \cdot \pi_N$, so $T\pi_N \cdot TF = Tf \cdot T\pi_N$.

Restricted to D_x, this gives the required result.

(1.2) *Intrinsic Derivatives*

The concept of intrinsic derivative is due to Porteous [7]; our treatment, however, is a reworking and extension of the more general results of Boardman [1].

Let $E \xrightarrow{p} N$ be a smooth vector bundle. Then the projection p induces a short exact sequence

$$0 \to E_n \to TE_e \xrightarrow{Tp_e} TN_n \to 0 \qquad (n = p(e)).$$

If e is in the image of the zero section z, then the tangent map $Tz_n: TN_n \to TE_e$ induces a canonical splitting $TE_e = TN_n \oplus E_n$.

Now suppose that $\chi: N \to E$ is any section vanishing at n. Then we define a map $i_\chi: TN_n \to E_n$ as the composite $TN_n \xrightarrow{T\chi_n} TE_e = TN_n \oplus E_n \xrightarrow[q]{\text{projection}} E_n$.

LEMMA (1.2.1). *Let $\chi: N \to E$ be a section.*

(i) *If $a: E \to F$ is a smooth vector bundle homomorphism over N and $\chi(n) = 0$, then $i_{a\chi} = a i_\chi$.*

(ii) *If $f: M \to N$ is a smooth map and $\chi(f(m)) = 0$, then $i_{f^*\chi} = i_\chi \cdot TF$.*

Proof. If $\Phi: \xi \to \xi'$ is a smooth vector bundle homomorphism covering a map $\phi: B \to B'$, then it is easy to see that, for any point e in the zero section of ξ, $(T\Phi)_e = T\phi_{p(e)} \oplus \Phi_{p(e)}$ w.r.t. the canonical splitting $(T\xi)_e = TB_{p(e)} \oplus \xi_{p(e)}$.

We now apply this fact to the situations of (i) and (ii). For (i), it gives us $(Ta)_{\chi(n)} = (T1_N)_n \oplus a_n$, and thus $qT(a\chi)_n = q(Ta_{\chi(n)} \cdot T\chi_n) = a_n \cdot qT\chi_n$, i.e. $i_{a\chi} = ai_\chi$. For (ii), if $1_f : f^*E \to E$ is the canonical identification covering f, the fact gives us $(T1_f)_{\chi(f(m))} = Tf_m \oplus 1_{E_m}$. Now $\chi f = 1_f(f^*\chi)$, so $qT\chi \cdot Tf = qT(1_f).\ T(f^*\chi) = qT(f^*\chi)$ i.e. $i_\chi Tf = i_{f^*\chi}$.

Now let $a: E \to F$ be a smooth vector bundle homomorphism over N. Then the *intrinsic derivative* of a at n, $d(a)_n : TN_n \to \mathrm{Hom}(\mathrm{Ker}\, a_n, \mathrm{Coker}\, a_n)$, is defined as follows: $d(a)_n(v)\ (\chi(n)) = [i_{a\chi}(v)]\ (v \in TN_n)$, where $\chi: N \to E$ is any smooth section s.t $a\chi(n) = 0$.
(1.2.1.) (i) shows this to be well-defined, since if $\chi(n) = 0$, $[i_{a\chi}(v)] = [ai_\chi(v)] = 0$.
These intrinsic derivatives have the following naturality properties:

LEMMA (1.2.2.). (i) *If $E \xrightarrow{a} F \xrightarrow{b} G$ are smooth vector bundle homomorphisms over N, then $d(ba)_n(v) \mid \mathrm{Ker}\, a_n = \bar{b}_n \cdot d(a)_n(v)$ (where $\bar{b}_n : \mathrm{Coker}\, a_n \to \mathrm{Coker}\, ba_n$ is the obvious map induced by b_n), and $d(b)_n(v) \cdot a_n = \bar{j}_n \cdot d(ba)_n(v)$* (where $\bar{j}_n : \mathrm{Coker}\, ba_n \to \mathrm{Coker}\, b_n$ *is the obvious map induced by the inclusion $j_n : \mathrm{Im}\, ba_n \subset \mathrm{Im}\, b_n$) for each $n \in N$ and $v \in TN_n$.*
(ii) *If $f: M \to N$ is a smooth map, then $d(a)_{f(m)} Tf_m = d(f^*a)_m$ for each $m \in M$.*
Proofs. These follow directly from (1.2.1) (i) and (ii).
In particular, we note that if $\mathrm{Ker}\, a_n = \mathrm{Ker}\, ba_n$, then $d(ba)_n = \mathrm{Hom}(1, \bar{b}_n) \cdot d(a)_n$; and if $a_n \mid \mathrm{Ker}\, ba_n : \mathrm{Ker}\, ba_n \to \mathrm{Ker}\, b_n$ is an isomorphism, then $d(b)_n = \mathrm{Hom}(a_n^{-1}, \bar{j}_n) \cdot d(ba)_n$.

(1.3) *Inductive Definition of the Boardman Variety Σ^I*
The definition we give here is due to Boardman ([1], §7); proofs that the induction we describe actually works may be found there. We record it here because the notation developed will be used extensively in §2.
Let $I = (i_1, \ldots, i_r)$ be an r-sequence. We will proceed by defining successively the varieties Σ^{i_1}, $\Sigma^{i_1 i_2}$.... For convenience, we shall write Σ_s for $\Sigma^{i_1 \cdots i_s}$ $(s \leqslant r)$ and T_s for $T\Sigma_s$, the tangent bundle of Σ_s.
First, let $E = \pi_P^* TP/J$ (where $\pi_P : J(N, P) \to P$ is the natural projection).
Now define
1) $\Sigma_0 = J(N, P)$
2) $K_0 = D$ (the total tangent bundle)
3) $E_0 = E$, and $c_0 = 1_E : E \to E_0$
4) $T_{-1} = T_0$, and $d_1 = T\pi_P : T_{-1} \to E$
and suppose inductively that we have defined
1) a submanifold Σ_{s-1} of $J(N, P)$
2) sub-bundles $K_{s-1} \subset \cdots \subset K_0 = D$ over Σ_{s-1} s.t. $K_{s-1} \subset T_{s-2}$
3) a bundle surjection $c_{s-1} : \mathrm{Hom}(K_{s-1} \circ \cdots \circ K_1, E) \xrightarrow{\text{onto}} E_{s-1}$ defined over Σ_{s-1} (no confusion attaches to this symmetric tensor product; see [1], (4.3)).
4) a bundle map $d_s : T_{s-2} \to E_{s-1}$ defined over Σ_{s-1}.

Then define

1) $\Sigma_s = \{x \in \Sigma_{s-1} \mid d_s \mid K_{s-1}$ has kernel rank i_s at $x\}$; this is a submanifold of Σ_{s-1}.

2) $K_s \subset K_{s-1}$ is the kernel bundle of $d_s \mid K_{s-1}$ over Σ_s; $K_s \subset T_{s-1}$ over Σ_s.

3) Let $e_s: E_{s-1} \to Q_s$ be the cokernel bundle (and projection) for $d_s \mid K_{s-1}$ over Σ_s. Define $u_s: \mathrm{Hom}(K_s \circ \cdots \circ K_1, E) \to \mathrm{Hom}(K_s, Q_s)$ over Σ_s to be the composite $\mathrm{Hom}(K_s \circ \cdots \circ K_1, E) \subset \mathrm{Hom}(K_s \otimes (K_{s-1} \circ \cdots \circ K_1), E) \cong \mathrm{Hom}(K_s, \mathrm{Hom}(K_{s-1} \circ \cdots \circ K_1, E)) \xrightarrow{\mathrm{Hom}(1, c_{s-1})} \mathrm{Hom}(K_s, E_{s-1}) \xrightarrow{\mathrm{Hom}(1, e_s)} \mathrm{Hom}(K_s, Q_s)$. This has constant rank, so we may define its coimage bundle map $c_s: \mathrm{Hom}(K_s \circ \cdots \circ K_1, E) \xrightarrow{\text{onto}} E_s = \mathrm{Im}\, u_s$ over Σ_s.

4) The intrinsic derivative bundle map $d(d_s \mid K_{s-1}): T_{s-1} \to \mathrm{Hom}(K_s, Q_s)$ over Σ_s factors through E_s, and thus we obtain $d_{s+1}: T_{s-1} \to E_s$, and the induction step is complete.

(1.4) Ω^I *as a Natural Stable Regularity Condition*

Let $f: V \to V'$ be a diffeomorphism of open sets in N. Then we have a diffeomorphism $\bar{f}: J(V, P) \to J(V', P)$ by $\bar{f}(Jg) = J(gf)$. Σ^I is invariant under the action of such diffeomorphisms (this is true for Σ_0, and if it is true for Σ_{s-1} it is true for Σ_s by (1.2.2) (ii)). Hence Ω^I is an invariant sub-bundle of $J(N, P)$.

The closure of Σ_s in Σ_{s-1} is $\{x \in \Sigma_{s-1} \mid d_s \mid K_{s-1}$ has $\mathrm{kr} \geqslant i_s\}$. Hence $\bigcup\{\Sigma^K \mid K > I\}$ is closed in $J(N, P)$ and so Ω^I, the complement of this set, is open.

So Ω^I is an open invariant sub-bundle of $J(N, P) \xrightarrow{\pi_N} N$; as we have previously observed, it is the inverse image of a sub-bundle of $J^r(N, P)$ which is therefore also open and invariant; i.e. Ω^I defines a stable, natural regularity condition on smooth maps $N \to P$.

§2

In this chapter we shall find conditions under which $\Omega^I \subset J(N, P)$ defines an extensible regularity condition on maps $N \to P$. From the definition of extensibility (in [6], §0), it is enough to show that $\bar{\imath}(\Omega') = \Omega^I$ for some open, invariant subbundle $\Omega' \subset J(N \times \mathbf{R}, P)$ which is the inverse image of a subbundle in $J^r(N \times \mathbf{R}, P)$. ($\bar{\imath}: J(N \times \mathbf{R}, P) \to J(N, P)$ is defined by $\bar{\imath}(Jf) = J(fi_r)$, where $\pi_N Jf = (x, r)$ and $i_r: N \to N \times \mathbf{R}$ is the identification $N = N \times r$).

We shall in fact show that, under the conditions stated in §0, $\bar{\imath}(\Omega^I) = \Omega^I$. To do so, we shall study the relation of the Boardman varieties in $J(N, P)$ with those in other jet-spaces; during this investigation we shall take the notation of (1.3) as standard in $J(N, P)$, and distinguish similar structures in other jet-spaces by $'$.

(2.1) We begin by showing that $\bar{\imath}(\Omega^I) \subset \Omega^I$ for any I. This is an immediate consequence of the following

LEMMA

$$x \in \Sigma^I \subset J(N \times \mathbf{R}, P) \Rightarrow \bar{\imath}(x) \in \Omega^I \subset J(N, P).$$

Proof. There is a commutative diagram

$$\begin{array}{ccc} D' & \xrightarrow{d'_1 \mid D'} & E' \\ {\scriptstyle T\bar{\imath}}\downarrow & & \| \\ \bar{\imath}^*D & \xrightarrow{\bar{\imath}^*(d_1 \mid D)} & \bar{\imath}^*E \end{array} \qquad \text{over } J(N \times \mathbf{R}, P).$$

(Recall from (1.3) that $d_1 = T\pi_P$.)

Suppose $x \in \Sigma^I \subset J(N \times \mathbf{R}, P)$. Then $x \in \Sigma^{i_1}$ and $\operatorname{kr} d'_1 \mid D' = i_1$.

Hence $\bar{\imath}(x) \in \Sigma^{i_1} \Leftrightarrow \operatorname{Ker} d'_1 \mid D' \cap \operatorname{Ker} T\bar{\imath} \mid D = \{0\}$, and otherwise (since $T\bar{\imath} \mid D'$ is the obvious surjection induced by $TN \oplus \mathbf{R}^1 \xrightarrow{1 \times 0} TN$, by (1.1.1)) $\bar{\imath}(x) \in \Sigma^{i_1 - 1} \subset \Omega^I$.

Thus if $\bar{\imath}(x) \in \Sigma^{i_1}$, $T\bar{\imath}$ provides the natural identification of $K'_1 = \operatorname{Ker} d'_1 \mid D'$ with $\bar{\imath}^*K_1 = \bar{\imath}^*(\operatorname{Ker} d_1 \mid D)$ near x in Σ^{i_1}.

Now take intrinsic derivatives of the diagram above; by their naturality properties (see (1.2.2)), we have

$$\operatorname{Hom}((T\bar{\imath} \mid K'_1)^{-1}, \bar{\jmath})\, d(d'_1 \mid D') = d(d_1 \mid D)\, T\bar{\imath}.$$

Restricted to K'_1, $T\bar{\imath}$ is the identity, so we have a commutative diagram

$$\begin{array}{ccc} K'_1 & \xrightarrow{d'_2 \mid K_1} & E'_1 \\ \| & & \uparrow {\scriptstyle y_1} \\ \bar{\imath}^*K_1 & \xrightarrow{\bar{\imath}^*(d_2 \mid K_1)} & \bar{\imath}^*E_1 \end{array} \qquad \text{near } x \text{ in } \Sigma^{i_1}$$

where $y_1 = \operatorname{Hom}((T\bar{\imath} \mid K'_1)^{-1}, \bar{\jmath})$ and the identification on the left is that induced by $T\bar{\imath}$.

(Recall E_1 is a sub-bundle of $\operatorname{Hom}(\operatorname{Ker} d_1 \mid D, \operatorname{coker} d_1 \mid D)$, and that d_2 is $d(d_1 \mid D)$, which factors through E_1.)

Now suppose inductively that $\bar{\imath}(x) \in \Omega^{i_1 \cdots i_{s-1}}$, and that if $\bar{\imath}(x) \in \Sigma^{i_1 \cdots i_{s-1}}$, then there is a commutative diagram

$$\begin{array}{ccc} K'_{s-1} & \xrightarrow{d'_s \mid K'_{s-1}} & E'_{s-1} \\ \| & & \uparrow {\scriptstyle y_{s-1}} \\ \bar{\imath}^*K_{s-1} & \xrightarrow{\bar{\imath}^*(d_s \mid K_{s-1})} & \bar{\imath}^*E_{s-1} \end{array} \qquad \text{near } x \text{ in } \Sigma^{i_1 \ldots i_{s-1}}$$

where the identification on the left is provided by $T\bar{\imath}$.

Thus $\operatorname{kr} d_s \mid K_{s-1} \leqslant \operatorname{kr} d'_s \mid K'_{s-1}$. If $\operatorname{kr} d_s \mid K_{s-1} < \operatorname{kr} d'_s \mid K'_{s-1}$, then $\bar{\imath}(x) \in \Omega^{i_1 \cdots i_s} -$

$\Sigma^{i_1 \cdots i_s}$. Otherwise, taking intrinsic derivatives in the diagram above, and using the naturality properties (1.2.2) we have

$$\operatorname{Hom}(1, \overline{y_{s-1}})\, d(d_s \mid K_{s-1})\, T\bar{\imath} = d(d_s' \mid K_{s-1}') \qquad \text{restricted to } T_{s-1}.$$

Restricted to K_s', $T\bar{\imath}$ is the identity $K_s' = \bar{\imath}^* K_s$ (where $K_s = \operatorname{Ker} d_s \mid K_{s-1}$, $K_s' = \operatorname{Ker} d_s' \mid K_{s-1}'$), so that we have a commutative diagram

$$\begin{array}{ccc} K_s' & \xrightarrow{d'_{s+1} \mid K'_s} & E_s' \\ \| & & \uparrow y_s = \operatorname{Hom}(1, \overline{y_{s-1}}). \\ \bar{\imath}^* K_s & \xrightarrow{\bar{\imath}^*(d_s \mid K_s)} & \bar{\imath}^* E_s \end{array}$$

This completes the induction step and hence the proof.

(2.2) Next, we recollect a "suspension" for Boardman varieties.

LEMMA

$\mathscr{I}(\Sigma^I) \subset \Sigma^I$ *for all* I.

(*Define* $\mathscr{I}: J(N, P) \to J(N \times \mathbf{R}, P \times \mathbf{R})$ by $\mathscr{I}(Jf) = J(f \times 1_{\mathbf{R}})$).

Proof. We have a commutative diagram

$$\begin{array}{ccc} D \oplus \mathbf{R} & \xrightarrow{(d_1 \mid D) \times 1_{\mathbf{R}}} & E \oplus \mathbf{R} \\ \| & & \| \\ \mathscr{I}^* D' & \xrightarrow{\mathscr{I}^*(d'_1 \mid D')} & \mathscr{I}^* E' \end{array} \qquad \text{over } J(N, P)$$

where the isomorphism on the left is induced by $T\mathscr{I}$. (This is easily checked in local co-ordinates.)

Thus $\operatorname{Ker} d_1' \mid D' = \operatorname{Ker} d_1 \mid D$, so that $\mathscr{I}(\Sigma^{i_1}) \subset \Sigma^{i_1}$. Taking intrinsic derivatives, we have

$$d(d_1' \mid D')\, T\mathscr{I} = d((d_1 \mid D) \times 1_{\mathbf{R}}) = d(d_1 \mid D).$$

Restricting to $K_1 = \operatorname{Ker} d_1 \mid D$, we have the commutative diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{d_2 \mid K_1} & E_1 \\ \| & & \| \\ \mathscr{I}^* K_1' & \xrightarrow{\mathscr{I}^*(d'_2 \mid K'_1)} & \mathscr{I}^* E_1' \end{array}$$

where the isomorphism on the left is induced by $T\mathscr{I}$.

Now suppose inductively that $\mathscr{I}(\Sigma^{i_1 \cdots i_{s-1}}) \subset \Sigma^{i_1 \cdots i_{s-1}}$, and that we have the commutative diagram

$$\begin{array}{ccc} K_{s-1} & \xrightarrow{d_s \mid K_{s-1}} & E_{s-1} \\ \| & & \| \\ \mathscr{I}^* K'_{s-1} & \xrightarrow{\mathscr{I}^*(d'_s \mid K'_{s-1})} & \mathscr{I}^* E'_{s-1} \end{array} \quad \text{over } \Sigma_{s-1}$$

where the isomorphism on the left is induced by $T\mathscr{I}$.

Clearly $K_s = \operatorname{Ker} d_s \mid K_{s-1} = \operatorname{Ker} \mathscr{I}_*(d'_s \mid K'_{s-1}) = \mathscr{I}^* K'_s$, so $\mathscr{I}(\Sigma^{i_1 \cdots i_s}) \subset \Sigma^{i_1 \cdots i_s}$.

Taking intrinsic derivatives, we have

$$d(d_s \mid K_{s-1}) = d(d'_s \mid K'_{s-1})\, T\mathscr{I} \mid T_{s-1}.$$

Restricted to $K_s = \mathscr{I}^* K'_s$, we have the commutative diagram

$$\begin{array}{ccc} K_s & \xrightarrow{d_{s+1} \mid K_s} & E_s \\ \| & & \| \\ \mathscr{I}^* K'_s & \xrightarrow{\mathscr{I}^*(d'_{s+1} \mid K'_s)} & \mathscr{I}^* E'_s \end{array}$$

where the isomorphism on the left is induced by $T\mathscr{I}$.

This completes the induction step and hence the proof.

(2.3) Let $f: (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a smooth map.

We identify $\mathbf{R}^p = \mathbf{R}^q \times \mathbf{R}^{p-q}$. Define a map $F: \mathbf{R}^n \times \mathbf{R}^{p-q} \to \mathbf{R}^p$ by $F(x, y) = f(x) + y$. If $k: \mathbf{R}^p \to \mathbf{R}^q$, $l: \mathbf{R}^p \to \mathbf{R}^{p-q}$ are the natural projections, we may write

$$F(x, y) = (kf(x), lf(x) + y).$$

Let h be the diffeomorphism of $\mathbf{R}^n \times \mathbf{R}^{p-q}$ defined by $h(x, y) = (x, y - lf(x))$. Then $Fh = kf \times 1$. Since $h(0,0) = (0,0)$, it follows from the lemma above that $J_{(0,0)}F \in \Sigma^I \Leftrightarrow \bar{k}(J_0 f) = J_0(kf) \in \Sigma^I$. (For any smooth map $k: P \to Q$, we define $\bar{k}: J(N, P) \to J(N, Q)$ by $\bar{k}(Jf) = J(kf)$).

Now suppose that $x \in \Omega^I \subset J(N, P)$; in local co-ordinates it may be represented as a smooth map as above. Hence if there is a submersion k of a neighbourhood of $\pi_P x$ s.t. $\bar{k}x \in \Omega^I$ also, we may construct a jet $x' \in \Omega^I \subset J(N \times \mathbf{R}, P)$ s.t. $\bar{\imath}x' = x$. Thus, to show $\bar{\imath}(i^*\Omega^I) = \Omega^I$ it is enough to show that for each $x \in \Omega^I \subset J(N, P)$ there is a local submersion $k: U \to \mathbf{R}^q$ (where $q < \dim P$) of a neighbourhood U of $\pi_P x$ in P s.t. $\bar{k}x \in \Omega^I$.

We investigate this in the following lemmas.

(2.4) LEMMA. *Let $x \in \Sigma^I \subset J(N, P)$, and let $k: U \to Q$ be a submersion of a neighbourhood U of $\pi_P x$ in P onto a neighbourhood Q of* 0 *in $\mathbf{R}^q$; let $G = \pi_P^* \operatorname{Ker} Tk \subset E$.*

Then

(a) *if* u_s Hom$(K_s \circ \cdots \circ K_1, G) \cap d_{s+1}(K_s) = \{0\}$ *at* $x \forall s < r$, *then* $\bar{k}(x) \in \Sigma^I \subset J(U, Q)$.

(b) *if* s *is the smallest integer s.t.* u_s Hom$(K_s \circ \cdots \circ K_1, G) \cap d_{s+1}(K_s) \neq \{0\}$ at x, *then* $\bar{k}(x) \in \Sigma^{i_1 \cdots i_s j_{s+1}}$, *where* $j_{s+1} = i_{s+1} + \dim(u_s$ Hom$(K_s \circ \cdots \circ K_1, G) \cap d_{s+1}(K_s))$.

Proof. (a) We have a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{d_1 \mid D} & E \\ \| & & \downarrow z_0 \\ \bar{k}^* D' & \xrightarrow{\bar{k}^*(d'_1 \mid D')} & \bar{k}^* E' \end{array} \quad \text{over } J(N, P) \mid N \times U$$

where z_0 is the obvious map induced by $Tk: TU \to TQ$. Clearly Ker $z_0 = G$.

Now suppose inductively that

(i) $\bar{k}(x) \in \Sigma_{s-1}$

(ii) $K_t = \bar{k}^* K'_t \ \forall t \leqslant s-1$ near x.

(iii) There is a commutative diagram

$$\begin{array}{ccc} K_{s-1} & \xrightarrow{d_s \mid K_{s-1}} & E_{s-1} \\ \| & & \downarrow z_{s-1} \\ \bar{k}^* K'_{s-1} & \xrightarrow{\bar{k}^*(d'_s \mid K'_{s-1})} & \bar{k}^* E_{s-1} \end{array} \quad \text{near } x \text{ in } \Sigma_{s-1}$$

s.t. Ker $z_{s-1} = u_{s-1}$ Hom$(K_{s-1} \circ \cdots \circ K_1, G)$.

Since Ker $z_{s-1} \cap d_s(K_{s-1}) = \{0\}$, Ker $\bar{k}^*(d'_s \mid K'_{s-1}) =$ Ker $d_s \mid K_{s-1}$. Hence $\bar{k}(x) \in \Sigma_s$.

Now take intrinsic derivatives in the diagram above:

$$d(d'_s \mid K'_{s-1})\, T\bar{k} = \mathrm{Hom}(1, \bar{z}_{s-1})\, d(d_{s-1} \mid K_{s-1}) \qquad \text{restricted to } T_{s-1}.$$

Since $T\bar{k}$ induces the identification $D = \bar{k}^* D'$ (by (1.1.1)), and $K_s \subset D$, we have a commutative diagram

$$\begin{array}{ccc} K_s & \xrightarrow{d_{s+1} \mid K_s} & E_s \\ \| & & \downarrow z_s \\ \bar{k}^* K'_s & \xrightarrow{\bar{k}^*(d'_{s+1} \mid K'_s)} & \bar{k}^* E'_s \end{array} \quad \text{near } x \text{ in } \Sigma_s$$

where $z_s = \mathrm{Hom}(1, \overline{z_{s-1}}) \mid E_s$.

Then

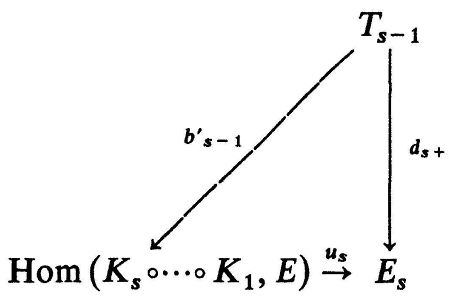
$$\begin{aligned} \mathrm{Ker}\, z_s &= \mathrm{Im}\, u_s \cap \{\alpha \in \mathrm{Hom}(K_s, Q_s) \mid \overline{z_{s-1}} \cdot \alpha = 0\} \\ &= \mathrm{Im}\, u_s \cap \mathrm{Hom}(K_s, \mathrm{Ker}\, \overline{z_{s-1}}) \\ &= \mathrm{Im}\, u_s \cap \mathrm{Hom}(K_s, e_s u_{s-1}\, \mathrm{Hom}(K_{s-1} \circ \cdots \circ K_1, G)) \\ &= u_s\, \mathrm{Hom}(K_s \circ \cdots \circ K_1, G). \end{aligned}$$

Thus we have completed the induction step, and hence the proof.

(b) If s is the smallest integer s.t. $u_s\,\mathrm{Hom}(K_s\circ\cdots\circ K_1, G)\cap d_{s+1}(K_s)\neq\{0\}$, then by (a) $\bar{k}(x)\in\Sigma_s$, and we may construct $z_s: E_s\to \bar{k}^*E_s'$ as above s.t. $zd_{s+1}\mid K_s=\bar{k}^*(d_{s+1}'\mid K_s')$ and $\mathrm{Ker}\,z_s=u_s\,\mathrm{Hom}(K_s\circ\cdots\circ K_1, G)$. Then $j_{s+1}=\mathrm{kr}\,d_{s+1}'\mid K_s'=\mathrm{kr}(z_s d_{s+1}\mid K_s)=\mathrm{kr}(d_{s+1}\mid K_s)+\dim(\mathrm{Ker}\,z_s\cap d_{s+1}(K_s))$. So $\bar{k}(x)\in\Sigma^{i_1\cdots i_s j_{s+1}}$, where $j_{s+1}=i_{s+1}+\dim(u_s\,\mathrm{Hom}(K_s\circ\cdots\circ K_1, G)\cap d_{s+1}(K_s))$.

Any subspace $G\subset E_x$ is of the form $\pi_p^*\,\mathrm{Ker}\,Tk$ for some submersion k; so the problem of finding submersions k for $x\in\Sigma^I$ s.t. $\bar{k}(x)\in\Sigma^I$ is equivalent to the algebraic problem of finding subspaces $G\subset E_x$ with the properties (2.4) (a). In attacking this problem, we shall use the following additional property of the intrinsic derivatives d_{s+1}:

LEMMA (2.5). (i) *There is a bundle map* $b_{s-1}': T_{s-1}\to\mathrm{Hom}(K_s\circ\cdots\circ K_1, E)$ *over* Σ_s *s.t. the following diagram commutes over* Σ_s:



(The notation is, again, that of (1.3).)

(ii) b_{s-1}' *is symmetric.*

(By this we mean that b_{s-1}' may be regarded as an element of $\mathrm{Hom}(T_{s-1}\circ K_s\circ\cdots\circ K_1, E)\subset\mathrm{Hom}(T_{s-1}\otimes(K_s\circ\cdots\circ K_1), E)$ via the natural isomorphism $\mathrm{Hom}(T_{s-1}\otimes(K_s\circ\cdots\circ K_1), E)\cong\mathrm{Hom}(T_{s-1}, \mathrm{Hom}(K_s\circ\cdots\circ K_1, E))$).

Proof. See Boardman [1], (7.11) and (7.7)

We have the following result:

LEMMA (2.6). *Let* $x\in\Sigma^I$, $I=(i_1,\ldots,i_r)$. *Define* $d^I=\sum_{s=1}^{r-1}\alpha_s$, *where* $\alpha_s=1$ *if* $i_s-i_{s+1}>1$ *and* 0 *otherwise. Then*

(a) *If* $p-n+i_r+d^I\geqslant g$, *then there is a subspace* $G\subset E_x$ *of dimension* g *s.t.*

$$u_s\,\mathrm{Hom}\,(K_s\circ\cdots\circ K_1, G)\cap d_{s+1}(K_s)=\{0\}\quad \text{at}\quad x\ \forall s<r.$$

(b) *If* $p-n+i_{r-1}+d^{i_1\cdots i_{r-1}}=g_{r-1}>0$ *and* $p-n+i_r+d^I\leqslant 0$, *then there is a subspace* $G\subset E_x$ *of dimension* 1 *s.t.*

$$u_s\,\mathrm{Hom}\,(K_s\circ\cdots\circ K_1, G)\cap d_{s+1}(K_s)=\{0\}\quad \text{at}\quad x\ \forall s<r-1$$

and

$$\dim(u_{r-1}\operatorname{Hom}(K_{r-1}\circ\cdots\circ K_1, G)\cap d_r(K_{r-1}))\leqslant\begin{cases} i_{r-1}-i_r-g_{r-1} & g_{r-1}>1\\ i_{r-1}-i_r & g_{r-1}=1\end{cases}$$

Proof. (a) Let G_1 be a complement to $\operatorname{Im} d_1 \mid D$ in E_x, so that $\dim G_1=g_1=p-n+i_1$. Clearly $u_0 G\cap d_1(D)=\{0\}$. We shall construct inductively $G_t\subset G_1$ s.t. $u_s \operatorname{Hom}(K_s\circ\cdots\circ K_1, G_t)\cap d_{s+1}(K_s)=\{0\}$ $\forall s<t$, where $\dim G_t\geqslant g_t=p-n+i_t+d^{i_1\cdots i_t}$.

Suppose, then, that we have constructed G_t as above. By the definition of u_s, $u_s \mid \operatorname{Hom}(K_s\circ\cdots\circ K_1, G_t)$ is injective$\Leftrightarrow u_{s-1} \mid \operatorname{Hom}(K_{s-1}\circ\cdots\circ K_1, G_t)$ is and $u_{s-1}\operatorname{Hom}(K_{s-1}\circ\cdots\circ K_1, G_t)\cap d_s(K_{s-1})=\{0\}$. Hence since $u_0 \mid G_t=1_{G_t}$, we see by induction that $u_t \mid \operatorname{Hom}(K_t\circ\cdots\circ K_1, G_t)$ is injective.

Now let $L=d_{t+1}^{-1}(u_t \operatorname{Hom}(K_t\circ\cdots\circ K_1, G_t))\cap K_t$. Then $b'_{t-1}(L)\subset\operatorname{Hom}(K_t\circ\cdots\circ K_1, G_t)$, so that we have a map $b:L\to\operatorname{Hom}(K_t\circ\cdots\circ K_1, G_t)$ with the following property; for any subspace $G'\subset G_t$,

$$b(L)\cap\operatorname{Hom}(K_t\circ\cdots\circ K_1, G')\cong d_{t+1}(K_t)\cap u_t \operatorname{Hom}(K_t\circ\cdots\circ K_1, G').$$

(i) Suppose $L\subsetneq K_t$; then b has $\operatorname{rank}\leqslant\operatorname{rk}(d_{t+1} \mid K_t)-1=i_t-i_{t+1}-1$. Now apply (3.1) (a); and we find that there is a subspace $G_{t+1}\subset G_t$ of dimension $\geqslant p-n+i_t+d^{i_1\cdots i_t}-(i_t-i_{t+1}-1)\geqslant p-n+i_{t+1}+d^{i_1\cdots i_{t+1}}=g_{t+1}$ s.t.

$$\operatorname{Im} b\cap\operatorname{Hom}(K_t\circ\cdots\circ K_1, G_{t+1})=\{0\}.$$

(ii) Suppose $L=K_t$; then $b:K_t\to\operatorname{Hom}(K_t\otimes(K_{t-1}\circ\cdots\circ K_1), G_t)$ is a symmetric map of $\operatorname{rank}=\operatorname{rk}(d_{t+1} \mid K_t)=i_t-i_{t+1}$. Thus, by (3.2) (a), there is a subspace $G_{t+1}\subset G_t$ of dimension$\geqslant p-n+i_t+d^{i_1\cdots i_t}-(i_t-i_{t+1})+\alpha_{t+1}=p-n+i_{t+1}+d^{i_1\cdots i_{t+1}}=g_{t+1}$ s.t.

$$\operatorname{Im} b\cap\operatorname{Hom}(K_t\circ\cdots\circ K_1, G_{t+1})=\{0\}.$$

Thus, in either case, we have constructed G_{t+1} s.t.

$$d_{t+1}(K_t)\cap u_t \operatorname{Hom}(K_t\circ\cdots\circ K_1, G_{t+1})=\{0\}.$$

Thus the induction step is complete. The proof is finished by defining $G=G_r$.

b) By (a), there is a subspace G_{r-1} of dimension $\geqslant g_{r-1}$ s.t.

$$u_s \operatorname{Hom}(K_s\circ\cdots\circ K_1, G_{r-1})\cap d_{s+1}(K_s)=\{0\}\qquad \forall s<r-1.$$

As in the proof of (a), $u_{r-1} \mid \operatorname{Hom}(K_{r-1}\circ\cdots\circ K_1, G_{r-1})$ is therefore injective. Define

$L=d_r^{-1}(u_{r-1}\,\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G_{r-1}))\cap K_{r-1}$. Then $b'_{r-2}(L)\subset\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G_{r-1})$, so we have a map $b:L\to\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G_{r-1})$ s.t., for any subspace $G'\subset G_{r-1}$, $b(L)\cap\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G')\cong d_r(K_{r-1})\cap u_{r-1}\,\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G')$.

(i) Suppose $L\subsetneq K_{r-1}$; then b has rank $\leqslant\mathrm{rk}(d_r \mid K_{r-1})-1=i_{r-1}-i_r-1$. Thus, by (3.1)(b), there is a 1-dimensional subspace $G\subset G_{r-1}$ s.t.

$$\dim(\mathrm{Im}\,b\cap\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G))\leqslant i_{r-1}-i_r-g_{r-1}.$$

(ii) Suppose $L=K_{r-1}$; then $b:K_{r-1}\to\mathrm{Hom}(K_{r-1}\otimes(K_{r-2}\circ\cdots\circ K_1), G_{r-1})$ is a symmetric map of rank$=\mathrm{rk}(d_r \mid K_{r-1})=i_{r-1}-i_r$. Thus, by (3.2)(b), there is a 1-dimensional subspace $G\subset G_{r-1}$ s.t.

$$\dim(\mathrm{Im}\,b\cap\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G))\leqslant\begin{cases} i_{r-1}-i_r-g_{r-1} & g_{r-1}>1\\ i_{r-1}-i_r & g_{r-1}=1\,. \end{cases}$$

Thus, in either case,

$$\dim(u_{r-1}\,\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1, G)\cap d_r(K_{r-1}))\leqslant\begin{cases} i_{r-1}-i_r-g_{r-1} & g_{r-1}>1\\ i_{r-1}-i_r & g_{r-1}=1 \end{cases}$$

which is the required result.

We may now prove our main theorem.

THEOREM (2.7). *Let I be the r-sequence $(i_1,\dots,i_r)$, $i_1\geqslant\cdots\geqslant i_r$. If $i_r>n-p-d^I$, then $\Omega^I\subset J(N,P)$ is extensible. (As previously, $d^I=\sum_{s=1}^{r-1}\alpha_s$, where $\alpha_s=1$ if $i_s-i_{s+1}>1$ and 0 otherwise.)*

Proof. We note that $i_s-i_{s+1}\geqslant\alpha_s=d^{i_1\dots i_{s+1}}-d^{i_1\dots i_s}$, and hence that

$$i_s-(n-p-d^{i_1\dots i_s})\geqslant i_{s+1}-(n-p-d^{i_1\dots i_{s+1}})\qquad(\forall s<r).$$

Thus $g_1\geqslant\cdots\geqslant g_s\geqslant\cdots\geqslant g_r>0$, where $g_s=i_s-(n-p-d^{i_1\dots i_s})$.

By (2.1), (2.2) and (2.3) it is enough to show that for $x\in\Sigma^K\subset J(N,P)$ with $K\leqslant I$, there is a submersion k s.t. $\bar{k}(x)\in\Sigma^{K'}$, where $K'\leqslant I$.

If $x\in\Sigma^I$, this submersion exists by (2.4) (a) and (2.6) (a).

If $x\in\Sigma^K$, with $K<I$, let s be the first integer s.t. $k_t=i_t$ $\forall t<s$ and $k_s<i_s$.

If $k_s>n-p-d^{i_1\dots i_{s-1}k_s}$, there is, by (2.4) (a) and (2.6) (a), a submersion k s.t. $\bar{k}(x)\in\Sigma^{i_1\dots i_{s-1}k_s}\subset\Omega^I$.

If $k_s\leqslant n-p-d^{i_1\dots i_{s-1}k_s}$, then by (2.4) (b) and (2.6) (b) there is a submersion k s.t. $\bar{k}(x)\in\Sigma^{i_1\dots i_{s-1}j_s}$, where

$$j_s\leqslant\begin{cases} k_s+(i_{s-1}-k_s-g_{s-1}) & g_{s-1}>1\\ k_s+(i_{s-1}-k_s) & g_{s-1}=1 \end{cases}$$

Thus

$$i_s-j_s\geqslant\begin{cases} g_{s-1}-i_{s-1}+i_s & g_{s-1}>1 \\ i_s-i_{s-1} & g_{s-1}=1 \end{cases}$$

If $i_s>j_s$, $\bar{k}(x)\in\Omega^I$.

Suppose $i_s\leqslant j_s$:

(i) if $g_{s-1}>1$, then $i_s-j_s\geqslant g_{s-1}-i_{s-1}+i_s=g_s-\alpha_s\geqslant 0$ (since $g_s\geqslant 1$, $\alpha_s\leqslant 1$). So if $i_s\leqslant j_s$, $i_s=j_s$ and $g_s=1$, $\alpha_s=1$. But $g_s=1$ implies $g_s=g_{s+1}=\cdots=g_r=1$, and thus

$$\begin{aligned} &i_t-(n-p-d^{i_1\cdots i_t})=i_{t+1}-(n-p-d^{i_1\cdots i_{t+1}}) && \forall t\geqslant s \\ &\Leftrightarrow i_t-i_{t+1}=\alpha_t\Leftrightarrow i_t=i_{t+1} && \forall t\geqslant s. \end{aligned}$$

Hence $\Sigma^{i_1\cdots i_s}\subset\Omega^I$, so $\bar{k}(x)\in\Omega^I$.

(ii) If $g_{s-1}=1$, then $g_{s-1}=\cdots=g_r=1$ and $i_{s-1}=\cdots=i_r$. Thus $i_s=j_s$ and $\bar{k}(x)\in\Omega^I$ since $\Sigma^{i_1\cdots i_s}\subset\Omega^I$.

This completes the proof.

§3

In this chapter we prove the algebraic results used in §2.

We begin with

LEMMA (3.1). *Let U, V, G be vector spaces, and let $b\colon U\to \mathrm{Hom}(V, G)$ be a linear map of rank r.*

(a) *If $r<\dim G$, then there is a subspace $A\subset G$ of dimension $a=\dim G-r$ s.t.* $\mathrm{Im}\, b\cap \mathrm{Hom}(V, A)=\{0\}$.

(b) *If $r\geqslant\dim G$, then there is a* 1*-dimensional subspace $A\subset G$ s.t.* $\dim(\mathrm{Im}\, b\cap\mathrm{Hom}(V, A))\leqslant r-\dim G+1$.

Proof. (a) Suppose the contrary, so that for each a-dimensional subspace $A\subset G$, $\exists u\in U$, $v\in V$ s.t. $\mathrm{Im}\, b(u)\subset A$ and $b(u)\,(v)=a\in A-\{0\}$.

Let A_1 be any a-dimensional subspace. Choose $u_1\in U$, $v_1\in V$ s.t. $\mathrm{Im}\, b(u_1)\subset A_1$ and $b(u_1)\,(v_1)=a_1\in A_1-\{0\}$. Let A_2 be an a-dimensional subspace of G s.t. $a_1\notin A_2$. Choose $u_2\in U$, $v_2\in V$ s.t. $\mathrm{Im}\, b(u_2)\subset A_2$ and $b(u_2)\,(v_2)=a_2\in A_2-\{0\}$. Let A_3 be an a-dimensional subspace of G s.t. $\langle a_1, a_2\rangle\cap A_3=\{0\}$. Choose $u_3\in U$, $v_3\in V$ s.t. $\mathrm{Im}\, b(u_3)\subset A_3$ and $b(u_3)\,(v_3)=a_3\in A_3-\{0\}$.

Continue in this way, defining A_i s.t. $A_i\cap\langle a_1,\ldots, a_{i-1}\rangle=\{0\}$, so that eventually we have defined $u_1,\ldots, u_{r+1}\in U$ s.t. $b(u_1),\ldots, b(u_{r+1})$ are linearly independent.

This contradicts $\mathrm{rk}\, b=r$, so our supposition was false, and the result is proved.

(b) For some subspace $U'\subset U$ of dimension $\dim G-1$, $\mathrm{rk}\, b\mid U'=\dim G-1$.

Hence, by (a), $\exists a$ 1-dimensional subspace $A \subset G$ s.t. $b(U') \cap \mathrm{Hom}(V, A) = \{0\}$. Thus

$$\dim(\mathrm{Im}\, b \cap \mathrm{Hom}(V, A)) \leqslant r - (\dim G - 1) = r - \dim G + 1.$$

These results may be improved somewhat if b is a symmetric map, when we have the following:

LEMMA (3.2). *Let* $b: K \to \mathrm{Hom}\ (K \otimes L, G)$ *be a linear map of rank* r *which is symmetric in* K.

(a) *If* $r \leqslant \dim G$, *there is a subspace* $A \subset G$ *of dimension*

$$\begin{cases} \dim G - r + 1 & r > 1 \\ \dim G - r & r \leqslant 1 \end{cases}$$

s.t.

$$\mathrm{Im}\, b \cap \mathrm{Hom}(K \otimes L, A) = \{0\}.$$

(b) *If* $r \geqslant \dim G$, *there is a* 1-*dimensional subspace* $A \subset G$ *s.t.*

$$\dim (\mathrm{Im}\, b \cap \mathrm{Hom}\, (K \otimes L, A)) \leqslant \begin{cases} r - \dim G & \dim G > 1 \\ r - \dim G + 1 & \dim G = 1 \end{cases}.$$

Proof. (a) By (3.1) (a), $\exists$ a subspace $A' \subset G$ of dimension $\dim G - r$ *s.t.*

$$\mathrm{Im}\, b \cap \mathrm{Hom}(K \otimes L, A') = \{0\}.$$

This gives the result if $r \leqslant 1$. Now suppose $r > 1$. We shall show that there is $a \notin A'$ s.t.

$$\mathrm{Im}\, b \cap \mathrm{Hom}(K \otimes L, \langle a, A' \rangle) = \{0\}.$$

This is equivalent to showing that there is $a \in G/A' - \{0\}$ s.t. $\mathrm{Im}\, b' \cap \mathrm{Hom}\, (K \otimes L, \langle a \rangle) = \{0\}$, where $b' = \mathrm{Hom}(1, q)\ b: K \to \mathrm{Hom}(K \otimes L, G/A')$ and $q: G \to G/A'$ is the projection. Note $\mathrm{rk}\, b' = \mathrm{rk}\, b = r$.

Suppose no such $a \in G/A' - \{0\}$ exists, so that for each $g \in G' = G/A'$, $\exists k \in K$ s.t. $\mathrm{Im}\, b'(k) = \langle g \rangle$. Let $\{g_1, \ldots, g_r\}$ be a basis for G', and let $\{k_1, \ldots, k_r\} \subset K$ be s.t. $\mathrm{Im}\, b'(k_i) = \langle g_i \rangle$.

Then $\{b'(k_1), \ldots, b'(k_r)\}$ is a linearly independent set.

For each pair $(k, k') \in K \times K$, we may regard $b'(k)\ (k')$ as a linear map $L \to G'$. Clearly $\mathrm{Im}\, b'(k_i)\ (k_j) \subset \langle g_i \rangle$ for each j. But $b'(k_i)\ (k_j) = b'(k_j)\ (k_i)$, since b is symmetric, and so

$$\mathrm{Im}\, b'(k_i)\ (k_j) \subset \langle g_i \rangle \cap \langle g_j \rangle = 0 \quad \text{if} \quad i \neq j.$$

Thus $b'(k_i)\ (k_j) = 0$ unless $i = j$.

Suppose $b'(k_i)(k_i)=0$; then $\exists k'\in K$ s.t. $\operatorname{Im} b'(k_i)(k')\subset\langle g_i\rangle-\{0\}$. Then $k'\notin\langle k_1,\dots,k_r\rangle$ (for if $k'=\sum_j \lambda_j k_j$, $b'(k_i)(k')=\lambda_i b'(k_i)(k_i)=0$, a contradiction), and $b'(k')$, $b'(k_1),\dots, b'(k_r)$ are linearly independent (for $b(k_1),\dots, b(k_r)$ are, and $b(k')(k_i)\neq 0$, $b(k_j)(k_i)=0$ for all $j=1,\dots,r$). This contradicts rank $b'=r$. Hence $b'(k_i)(k_i)\neq 0$ for each $i=1,\dots,r$.

By our initial assumption, $\exists k\in K$ s.t. $\operatorname{Im} b'(k)=\langle g_1+g_2\rangle$. Suppose $k=\sum_j \lambda_j k_j$; then $b'(k)(k_i)=\lambda_i b'(k_i)(k_i)$.

But $\operatorname{Im} b'(k_i)(k_i)\subset\langle g_i\rangle$, so $\lambda_i=0$ for each i. Thus $k=0$. Thus, by contradiction, $k\notin\langle k_1,\dots,k_r\rangle$. Similarly, $b'(k)$ is linearly independent of $b(k_1),\dots,b(k_r)$. This contradicts $\operatorname{rk} b'=r$.

Thus our assumption was false; so the result is proved.

(b) The result follows by (3.1)(b) if $\dim G=1$. Now suppose $\dim G>1$. If there is a 1-dimensional subspace $A\subset G$ s.t. $\operatorname{Im} b\cap \operatorname{Hom}(K\otimes L, A)=\{0\}$, we have the result. So suppose otherwise; then for each $g\in G$, $\exists k\in K$ s.t. $\operatorname{Im} b(k)=\langle g\rangle$. Let $\{g_i\}$ be a basis for G, and let $\{k_i\}$ be s.t. $\operatorname{Im} b(k_i)=\langle g_i\rangle$. Clearly $\{b(k_i)\}$ are linearly independent. b is symmetric, so $b(k_i)(k_j)=0$ if $i\neq j$. (Regarding $b(k)(k')$ as a linear map $L\to G$ for each pair $(k,k')\in K\times K$.)

Suppose two of $b(k_i)(k_i)$ are non-zero, w.l.o.g. $i=1, 2$. Then (by the same argument as we used in (a)),

$$b(\langle k_1,\dots,k_g\rangle)\cap\operatorname{Hom}(K\otimes L,\langle g_1+g_2\rangle)=\{0\}\qquad (g=\dim G)$$

Hence $\operatorname{Im} b\cap\operatorname{Hom}(K\otimes L,\langle g_1+g_2\rangle)$ has dimension at most $r-\dim G$.

Alternatively, suppose one of $b(k_i)(k_i)=0$, w.l.o.g. $i=1$. Let $k\in K$ be s.t. $b(k_1)(k)\neq 0$, so that $\operatorname{Im} b(k_1)(k)=\operatorname{Im} b(k)(k_1)=\langle g_1\rangle$. Then $k\notin\langle k_1,\dots,k_r\rangle$ (by the argument used in (a)), and $b(k)$ is linearly independent of $b(k_1),\dots,b(k_g)$. Then

$$b(\langle k, k_1, k_3,\dots,k_g\rangle)\cap\operatorname{Hom}(K\otimes L,\langle g_2\rangle)=\{0\}.$$

(For suppose $\xi=\mu b(k)+\sum_{i\neq 2}\alpha_i b(k_i)$ and $\operatorname{Im}\xi=\langle g_2\rangle$. Then $\xi(k_1)=\mu b(k)(k_1)$, and $\operatorname{Im} b(k)(k_1)\in\langle g_1\rangle-\{0\}$. Thus $\mu=0$. Then $\xi(k')=\sum_{i\neq 2}\alpha_i b(k_i)(k')$ so $\operatorname{Im}\xi(k')\subset\langle g_1, g_3,\dots,g_g\rangle$ since $\operatorname{Im} b(k_i)(k')\subset\langle g_i\rangle$ for all $k'\in K$. But $\operatorname{Im}\xi(k')\subset\langle g_2\rangle$, so $\xi(k')=0$ for all $k'\in K$. Hence $\xi=0$.)

Thus $\operatorname{Im} b\cap\operatorname{Hom}(K\otimes L,\langle g_2\rangle)$ has dimension at most $r-\dim G$.

APPENDIX. We consider very briefly some of the regularity conditions $\Omega^I=J(N,P)-\bar{\Sigma}^I$. These are open invariant sub-bundles of $J(N,P)$; we wish to investigate their extensibility. To do so, we require a description of $\bar{\Sigma}^I$.

First, we have $\bar{\Sigma}^i=U\{\Sigma^j\mid j\geqslant i\}$, so $\Omega^I=\Omega^{i-1}$.

The situation is rather more complicated for $\bar{\Sigma}^{ij}$.

Let $x\in\Sigma^{i+k}$ for some $k\geqslant 0$. Let $K=\operatorname{Ker} d_1 \mid D_x$, $I=\operatorname{Im} d_1 \mid D_x$. We have the second intrinsic derivative map at x $d_2 \mid K: K\to \operatorname{Hom}(K, E_x/I)$.

LEMMA. *$x\in\bar{\Sigma}^{ij} \Leftrightarrow$ there is a codim. -k subspace $L\subset K$, and a subspace $J\subset E_x$ of which I is a codim.-k subspace s.t.* $\operatorname{Hom}(i, j)\, d_2 \mid L: L\to \operatorname{Hom}(L, E_x/J)$ *has kernel rank $\geqslant j$ (where* $\operatorname{Hom}(i,j): \operatorname{Hom}(K, E_x/I)\to \operatorname{Hom}(L, E_x/J)$ *is the natural map induced by the inclusions $i: L\subset K$, $j: I\subset J$).*

Proof. Let $a: E\to F$ be a vector-bundle homomorphism over a smooth manifold X. W.r.t. local co-ordinates in a neighbourhood U of x in X, this may be represented as a smooth map $\tilde{a}: U\to \operatorname{Hom}(E_x, F_x)$. This map has derivative $d\tilde{a}_x: \mathbf{R}^{\dim X}\to \operatorname{Hom}(E_x, F_x)$ and it is easily seen that this, composed with the natural map $\operatorname{Hom}(E_x, F_x)\to \operatorname{Hom}(\operatorname{Ker}_x a, \operatorname{Coker}_x a)$ is the intrinsic derivative $d(a)_x$ of a at x.

Suppose a has kernel rank $i+k$ at x, and that x is the limit of a sequence $\{x_n\}$ s.t. a has kernel rank i at x_n for all n. Then $L=\lim \operatorname{Ker}_{x_n} a\subset \operatorname{Ker}_x a$, and $J=\lim \operatorname{Im}_{x_n} a\supset \operatorname{Im}_x a$, (these limits exist; think of $\operatorname{Ker} a$ as a section of the Grassman bundle $G_{\operatorname{kr} a}(E) \mid \{x \mid \operatorname{kr} a_x = i\}$, $\operatorname{Im} a$ as a section of $G_{\operatorname{rk} a}(F) \mid \{x \mid \operatorname{kr} a_x = i\})$, and $\lim d(a)_{x_n}$ is $d(a)_x$ composed with the natural map $\operatorname{Hom}(\operatorname{Ker}_x a, \operatorname{Coker}_x a)\to \operatorname{Hom}(L, F_x/J)$. (This follows because the derivative of $\tilde{a}$ is continuous). Moreover, if $d(a)_{x_n}$ has kernel rank $j\,\forall n$, this map has kernel rank $\geqslant j$.

Now apply these results to $d_1 \mid D$, $J(N, P)$ to prove $\Rightarrow$; $\Leftarrow$ is easy.

The complication of this result does not encourage an investigation of Σ^{ijk}....

We may show, by similar methods to those of §2, that the extensibility of $\bar{\Omega}^{ij}$ reduces to the existence of solutions to an algebraic problem analagous to (3.2)(a), which we formulate as follows:

let $\phi: K\to \operatorname{Hom}(K, Q)$ be a (symmetric) linear map with the following property $E(k, r)$ – for any codimension-k subspace $L\subset K$, and for any k-dimensional subspace $R\subset Q$, the map $\phi^1_{L,R}=\operatorname{Hom}(i, p_R)\,\phi \mid L: L\to \operatorname{Hom}(L, Q/R)$ has rank $\geqslant r$ (here $p_R: Q\to Q/R$ is the natural projection).

The problem we must solve is; does there exist a 1-dimensional subspace $A\subset Q$ s.t. $\bar{\phi}_A=\operatorname{Hom}(1, p_A)\,\phi: K\to \operatorname{Hom}(K, Q/A)$ has the property $E(k, r)$?

We have the following result:

LEMMA. *If* $\dim K<\dim Q$, *such a 1-dimensional subspace $A\subset Q$ exists for any ϕ with the property $E(k, r)$.*

Proof. Suppose the contrary, so that for each $q\in Q$, there is a codimension-k subspace $L\subset K$ and a dimension-k subspace $R\subset Q$, and a vector $l\in L$ s.t. $\phi(l)\,L=q \bmod R$. Thus $\operatorname{Im}\phi(l)=Z=\langle q, R, \phi(l)\,M\rangle$, where $M\oplus L=K$. (We have equality of Z with $\operatorname{Im}\phi(l)$ since if $\langle q, R\rangle=\langle q', R'\rangle$, then $\phi(l)\,L=q' \bmod R'$ for otherwise $\phi(l)\,L= 0 \bmod R'$, in contradiction to property $E(k, r)$ of ϕ.)

Since $\dim K < \dim Q$, $\dim Z < \dim Q$.

Choose $q_1 \in Q - \{0\}$, and obtain as above l_1, L_1, R_1 and hence Z_1.
Choose $q_2 \in Q - Z_1$, and hence obtain Z_2.
Choose $q_3 \in Q - (Z_1 \cup Z_2)$, and hence obtain Z_3.

Continue in this way, choosing $q_s \in Q - (Z_1 \cup \ldots \cup Z_{s-1})$, obtaining eventually a basis $\{q_i\}$ of Q for which the corresponding maps $\phi(l_i)$ are linearly independent. Hence $\{l_i\}$ are linearly independent.

Thus $\dim K \geqslant \dim Q$, in contradiction to the hypothesis of the lemma. Thus the lemma is proved.

It follows that $\Omega^{\overline{ij}}$ is extensible if $n < p$.

In some cases, this result may be improved (for example, $\Sigma^3 \subset \Sigma^{2,1}$, so $\Omega^{\overline{2,1}} = \Omega^{2,0}$, which is extensible if $n=p$) but in general this result is best possible, because the algebraic result above is. (For a counter-example when $\dim K = \dim Q$, take $K = Q$ of dimension 3, with basis $\{k_1, k_2, k_3\}$ and define ϕ by

$$\phi(k_i, k_j) = \phi(k_j, k_i) = 0 \quad \text{if} \quad i \neq j$$
$$\phi(k_i, k_i) = k_i.$$

It is not immediately obvious that this a counter-example, but the arguments of the next lemma may be used to show that.)

A more geometric reason why $\Omega^{\overline{ij}}$ is not always extensible if $n \geqslant p$ is the following: $\Sigma^3 \not\subset \bar{\Sigma}^{2,2}$ for $n = p$ but $\Sigma^3 \subset \bar{\Sigma}^{2,2}$ for $n = p+1$, so that $\Omega^{2,2}$ cannot be extensible for $n = p$. The first assertion $\Sigma^3 \not\subset \bar{\Sigma}^{2,2}$ is clear from the respective codimensions of Σ^3 and $\Sigma^{2,2}$ when $n = p$ (see §0, Note 2); or the example above gives the second intrinsic derivative of a jet in $\Sigma^3 - \bar{\Sigma}^{2,2}$. To see that $\Sigma^3 \subset \bar{\Sigma}^{2,2}$ for $n = p+1$, we study the second intrinsic derivative of $x \in \Sigma^3$, which may be expressed as a map $\phi: K \to \mathrm{Hom}(K, Q)$ symmetric in K, where $\dim K = 3$, $\dim Q = 2$. Our result will follow by showing that no such map has property $E(1, 1)$, which we do in the following lemma:

LEMMA. *Let* $\dim K = 3$, $\dim Q = 2$. *Then no homomorphism* $\phi: K \to \mathrm{Hom}(K, Q)$ *symmetric in K has property* $E(1, 1)$.

Proof. Let $\{q_1, q_2\}$ be any basis for Q. Then ϕ defines two quadratic forms ϕ_1, ϕ_2 given by its q_1- and q_2-co-ordinates, and hence a pencil $\Phi = \langle \phi_1, \phi_2 \rangle$ of quadratic forms.

Property $E(1, 1)$ would claim that the restriction of Φ to any 2-plane $L \subset K$ is non-trivial (i.e. $\phi_1 \mid L \circ L$ and $\phi_2 \mid L \circ L$ are linearly independent in $\mathrm{Hom}(L \circ L, \mathbf{R})$).

Let A_1, A_2 be matrices for ϕ_1, ϕ_2 w.r.t. some basis in K. Either A_2 is singular, or $\det(A_1 + \lambda A_2)$ is a cubic in λ, and so has a root μ, corresponding to a singular matrix $A_1 + \mu A_2$. So the pencil Φ contains a degenerate quadratic form. W.r.t. some basis

e_x, e_y, e_z in K this has one of the forms 0, x^2, x^2-y^2, x^2+y^2. In the first three cases counter-examples to property $E(1, 1)$ are provided respectively by any plane, $\{x=0\}$, and $\{x=y\}$. In the fourth case, any quadratic form of the pencil may be first put in the form $ax^2+2xy+y^2+vz^2$ by choice of e_z, and then diagonalised w.r.t. x^2+y^2; so that the pencil has the form

$$\langle x^2+y^2, \lambda x^2+\mu y^2+vz^2\rangle.$$

If $v=0$, the pencil is trivial restricted to $\{x=0\}$. If $\lambda=\mu$, the pencil is trivial restricted to $\{z=0\}$.

Otherwise, the pencil contains the degenerate quadratic forms $(\mu-\lambda)\,y^2+vz^2$, $(\lambda-\mu)\,x^2+vz^2$. One of these is indefinite; i.e. has the form $x'^2-y'^2$ w.r.t. some basis, so is zero on the plane $\{x'=y'\}$. Hence the pencil is trivial on that plane.

We have, therefore, shown that no ϕ has property $E(1, 1)$.

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