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The Universal Smooth Surgery Class

I. MADSEN and R. J. MILGRAM

1. Introduction

Geometric topology divides into 2 worlds: the world of the odd primes and the world of the prime 2[1]. The odd world has been beautifully explored by Sullivan, but only partial results have hitherto been available at the prime 2. In this paper we set up the machinery and prove the basic structure theorems necessary to demonstrate results analogous to Sullivan's, but for the prime 2. In a sequel [26] we apply these theorems to study the *2-local* structure of the oriented topological and PL-bordism rings, obtaining the algebraic structure of all the groups as well as much information on the explicit generating manifolds. In previous work (with G. Brumfiel) [9] we initiated work in this area by calculating the *mod. 2* cohomology structure of the classifying spaces BTOP and BPL. This gave us the unoriented PL-bordism ring and (except in dimension 4) the unoriented topological bordism ring as well.

To proceed from mod. 2 to 2-local cohomology which then allows one to proceed from unoriented to (2-local) oriented bordism requires much more technique than was available in [9]. On the other hand, with these new techniques we obtain much deeper insights into the precise differences between the theories of Differentiable, PL, Topological manifolds and Poincaré duality spaces, not to mention the K-theories KO, KPL, KTOP and KG (where KG is the theory of fibre homotopy sphere bundles).

All of these results follow from a study of the natural map

$$B(\pi): BG \to B(G/\mathrm{TOP})$$

whose fibre is the space BTOP; the injection of the fibre $j: B\mathrm{TOP} \to BG$ induces the forgetful functor for the associated cohomology theories KTOP to KG. In fact, we completely determine the 2-local homotopy type of $B(G/\mathrm{TOP})$ and the map $B(\pi)$, and obtain as an immediate corollary a precise determination of the 2-local obstruction to lifting a fibre homotopy sphere bundle to an honest sphere bundle.

We begin by determining $B(G/\mathrm{TOP})$.

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[1] A statement probably due to D. Sullivan.

THEOREM A. *At the prime* 2 *the space* $B^2(G/\mathrm{TOP})$ *is a product of Eilenberg-MacLane spaces,*

$$B^2(G/\mathrm{TOP})_{(2)} \simeq \prod_{n\geqslant 1} K(\mathbf{Z}_{(2)}, 4n+2)\times K(\mathbf{Z}/2, 4n).$$

This result is actually best possible since an analysis of the Dyer-Lashof operations in $H_*(G/\mathrm{TOP}; \mathbf{Z}/2)$ shows such a splitting to be impossible for $B^3(G/\mathrm{TOP})_{(2)}$, [24].

COROLLARY. *The* 2*-local part of the obstruction to reducing a stable spherical fibre space over a finite complex* X *to a topological sphere bundle is a graded cohomology class in*

$$\bigoplus_{i=1}^{\infty} H^{4i+1}(X; \mathbf{Z}_{(2)}) \oplus H^{4i-1}(X; \mathbf{Z}/2).$$

The corollary was also obtained by Brumfiel and Morgan [10], Jones [15], and Quinn [36], and was originally proved under the assumption that X is 4-connected by Levitt and Morgan [20]. The methods of these papers all use certain refinements of the "transversality obstruction" of Levitt to construct a fibration.

$$BS\mathrm{TOP} \xrightarrow{j} BSG \xrightarrow{T} \prod K(\mathbf{Z}_{(2)}, 4n+1)\times K(\mathbf{Z}/2, 4n-1),$$

where j is the forgetful map, but T is not, a priori, the natural map $B(\pi)$. Moreover, in these papers the authors are not able to do more than to study the $\mathbf{Z}/2$ and $\mathbf{Z}/4$ homotopy type of the map T, so precise information on T is lacking in their approaches.

Now we turn to the precise determination of the map $B(\pi)^*$. In view of Theorem A this involves defining suitable fundamental classes y_i in $H^*(B(G/\mathrm{TOP}))$ and calculating their images in $H^*(BSG)$. The map $B(\pi): BSG \to B(G/\mathrm{TOP})$ is an H-map (in fact an infinite loop map²) [4], and, again from Theorem A, we can assume our fundamental classes in $H^*(B(G/\mathrm{TOP}))$ are *primitive*. Hence $B(\pi)^*(y_i)$ is primitive with respect to the coproduct induced from Whitney sum.

On the other hand $B(\pi)$ factors as the composite

$$BSG \xrightarrow{B\lambda} B(G/O) \xrightarrow{B\tau} B(G/\mathrm{TOP})$$

where $B\lambda$ and $B\tau$ are again the natural maps – in fact infinite loop maps. Here $B\lambda$, in view of the close connection between SG and SO, is not too hard to analyse, so our main efforts go into studying the map $B(\tau)$, which is, of course, the map of classifying spaces associated with the natural map

$$\tau: G/O \to G/\mathrm{TOP}.$$

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²) It is this fact which ultimately enables us to complete the calculation.

By inspection one is able to check that τ^* determines $B(\tau)^*$ in cohomology. Thus our problem reduces to determining suitable primitive fundamental classes in G/TOP and evaluating their images in $H^*(G/O)$.

Specific cohomology classes were constructed in $H^{4i}(G/\mathrm{TOP}; \mathbf{Z}_{(2)})$ and $H^{4i-2}(G/\mathrm{TOP}; \mathbf{Z}/2)$ in [32], [35], [38]. For our purpose the class $K_{4i} \in H^{4i}(G/\mathrm{TOP}; \mathbf{Z}_{(2)})$ constructed in [32] is the more convenient one. It is not primitive, in fact

$$\psi(K_{4i}) = K_{4i} \otimes 1 + 8 \sum_{j=1}^{i-1} K_{4j} \otimes K_{4i-4j} + 1 \otimes K_{4i}.$$

A primitive class, agreeing with K_{4i} modulo decomposables, is obtained as

$$k_{4i} = \frac{1}{8i} s_i(8K_4, \ldots, 8K_{4i})$$

if we let s_i be the i'th Newton polynomial. The classes k_{4i-2} and k_{4i} together define a homotopy equivalence of H-spaces

$$K: G/\mathrm{TOP} \to \prod K(\mathbf{Z}_{(2)}, 4i) \times K(\mathbf{Z}/2, 4i-2)$$

where the H-structure on the right is the usual one.

In [23] the higher torsion structure of BSG and $B(G/O)$ as well as their loop spaces was examined. It was shown that $PH^{4k+1}(B(G/O); \mathbf{Z}_{(2)}) = \mathbf{Z}_{(2)} \oplus T$ where T is a $\mathbf{Z}/2$ vector space and a specific generator $\hat{e}_{4n+1}$ for the free summand was constructed. Let $\sigma^*: H^*(B(G/O); \mathbf{Z}_{(2)}) \to H^*(G/O; \mathbf{Z}_{(2)})$ denote the cohomology suspension.

THEOREM B. *The composite* $G/O \xrightarrow{\tau} G/\mathrm{TOP} \xrightarrow{k_{4n}} K(\mathbf{Z}_{(2)}, 4n)$ *defines the cohomology class* $2^{\alpha(n)-1}\sigma^*(\hat{e}_{4n+1})$ *where* $\alpha(n)$ *denotes the number of non-zero terms in the dyadic expansion of* n.

To obtain our main result from B we need, first of all, information on the primitive elements in $H^*(BSG; \mathbf{Z}_{(2)})$. From [23], §5, we have the exact sequences

$$\begin{aligned} &0 \to PH^{2n+1}(B(G/O); \mathbf{Z}_{(2)}) \xrightarrow{\sigma^*} PH^{2n}(G/O; \mathbf{Z}_{(2)}) \\ &0 \to \mathbf{Z}/2^{\nu(n)+1} \to PH^{2n+1}(BSG; \mathbf{Z}_{(2)}) \xrightarrow{\sigma^*} PH^{2n}(SG; \mathbf{Z}/2) \end{aligned} \tag{C}$$

where $\nu(n)$ is the 2-adic valuation on n. The natural map $B\lambda: BSG \to B(G/O)$ maps the element $\hat{e}_{4n+1} \in H^{4n+1}(B(G/O); \mathbf{Z}_{(2)})$ to an element $\hat{e}_{4n+1}$ of order $2^{\nu(n)+3}$ and $4 \cdot \hat{e}_{4n+1}$ is the generator in the kernel of $\sigma^*: PH^{4n+1}(BSG; \mathbf{Z}_{(2)}) \to PH^{4n}(SG; \mathbf{Z}_{(2)})$.

As the next step we "deloop" the primitive element k_{4i} in $H^{4i}(G/\mathrm{TOP}; \mathbf{Z}_{(2)})$. This is not necessarily possible "on the nose" since not all 2-local primitives in the cohomology of a $K(\mathbf{Z}_{(2)}, n)$ are in the image of the suspension map, but we can show

THEOREM D. *There is a primitive graded class* $\hat{k}_{4*+1}\in PH^{4*+1}(B(G/\mathrm{TOP});\mathbf{Z}_{(2)})$ *satisfying*

(i) $\sigma^*(\hat{k}_{4n+1})-k_{4n}$ *has order* 2

(ii) $\tau^*(\sigma^*(\hat{k}_{4n+1})-k_{4n})=0$ in $H^{4n}(G/O;\mathbf{Z}_{(2)})$.

From these results it follows that $(B\pi)^*(\hat{k}_{4n+1})=2^{\alpha(n)-1}\hat{e}_{4n+1}$ and we get ((b) below is immediate from [9]).

COROLLARY E. *The* 2*-local part of the obstruction to reducing a stable spherical fibration* ξ *over* X *to a topological bundle is a graded cohomology class*

$$\sigma_{4*-1}(\xi)+\sigma_{4*+1}(\xi)\in H^{4*-1}(X;\mathbf{Z}/2)\oplus H^{4*+1}(X;\mathbf{Z}_{(2)}).$$

Furthermore,

(a) $\sigma_{4n+1}(\xi)$ *has order at most* $2^{\nu(n)-\alpha(n)+4}$

(b) $\sigma_{4n-1}(\xi)=0$ *unless* n *is a power of* 2.

The class $\sigma_{4n+1}(\xi)$ is almost explicit. We know [23] that if w_n is the n'th Stiefel-Whitney class in $H^*(BSG;\mathbf{Z}/2)$ then w_{2n}^2 is the restriction of a universally defined $\mathbf{Z}/8$ class, p_n. If it were known that the coproduct for p_n had the form

$$\psi(p_n)=\sum p_i\otimes p_{n-i}+\sum w_{2i+1}^2\otimes w_{2(n-i)-1}^2 \qquad (*)$$

as a class of $H^*(BSG\times BSG;\mathbf{Z}/8)$ then the class $\hat{e}_{4n+1}$ in $H^{4n+1}(BSG;\mathbf{Z}_{(2)})$ could be written explicitly as a Bockstein of the "primitive" in the p_{4i}. Unfortunately, we have not been able to prove $(*)$ so we leave it as a conjecture.

The class $\sigma_{2^i-1}(\xi)$ is connected with the secondary cohomology operation $\psi_{i,i}$ based on the Adém relation

$$Sq^{2^i-1}Sq^{2^i-1}+\sum_{0<j<i-1}Sq^{2^i-2^j}Sq^{2^j}=0.$$

Indeed, if all Stiefel-Whitney classes of ξ vanish then $\sigma_{2^i-1}(\xi)$ is defined by setting

$$\psi_{i,i}(U)=\sigma_{2^i-1}(\xi)\cup U,$$

where U is the Thom class of ξ in the Thom complex (Mahowald, unpublished). Here, we note, that $\psi_{i,i}$ has zero indeterminacy, so $\psi_{i,i}(U)$ is well defined.

Recently Ravenel [37] has introduced certain twisted secondary Stiefel-Whitney classes $\lambda_i(\xi)$ defined without any preconditions on the Stiefel-Whitney classes of ξ and has proved that

$$\lambda_{2^i-1}(\xi)=\sigma_{2^i-1}(\xi)$$

at least modulo decomposables. It would be very useful if we knew the exact difference between these two classes. For example, they would have to be equal if $\lambda_{2^i-1}(\xi)$ were universally primitive. What seems to be needed is a Cartan formula for the λ_i.

In the special case where X is a Poincaré duality space of dimension n all of whose Stiefel-Whitney classes vanish and ξ is its Spivak normal fibration we have the following partial characterisation of the class $\sigma_{2^i-1}(\xi)$:

PROPOSITION F. *If* $x \in H^{n-2^i+1}(X; \mathbf{Z}/2)$ *and* $\psi_{i,i}$ *is defined on* x, *then*

$$\langle \sigma_{2^i-1}(\xi) \cup x, [X] \rangle = \langle \psi_{i,i}(x), [X] \rangle$$

and $\psi_{i,i}$ *is defined with zero indeterminacy.*

Remark. It should be possible to give a similar Wu formula for the Ravenel operations.

In particular, if the $\psi_{i,i}$ are defined on the entirety of $H^{n-2^i+1}(X; \mathbf{Z}/2)$ then the $\sigma_{2^i-1}(\xi)$ are uniquely determined by Proposition F. This will be the case if and only if all the Sq^i vanish identically in $H^{n-2^i+1}(X; \mathbf{Z}/2)$.

We conclude by pointing out

PROPOSITION G. *Let* X *be a simply connected Poincaré duality space of dimension at least* 5 *and* ξ *its Spivak normal fibration. Suppose*

(i) $\sigma_{2^i-1}(\xi) = 0$ *for all* i

(ii) $2^{\alpha(k)-1} H^{4k+1}(X; \mathbf{Z}_{(2)})$ *is torsion free for all* k.

(iii) X *is orientable with respect to* $KO(\ )\otimes \mathbf{Z}[\frac{1}{2}]$.

Then there is a PL-*manifold* M *and a map* $f: M \to X$ *which is a homotopy equivalence.*

We have organized the paper in five sections,

§1 Introduction

§2 The 2-local structure of $B^2(G/\mathrm{TOP})$

§3 Delooping the universal surgery class

§4 The universal smooth surgery class

§5 Topological reduction of spherical fibrations.

In §2 we prove Theorem A and in §3 Theorem D. The evaluation of the natural maps $B\tau: B(G/O) \to B(G/\mathrm{TOP})$ is done in §4. In §5 we prove the rather obvious geometric corollaries listed above.

2. The 2-local Structure of $B(G/\mathrm{TOP})$

In this section all spaces and maps are to be taken in the 2-local category (see e.g. D. Sullivan [41] for the definition and the simple properties of the 2-local category).

The spaces G/TOP and G/PL are the fibres of the natural maps $B S\mathrm{TOP} \xrightarrow{i} BSG$ and $BS\mathrm{PL} \xrightarrow{i'} BSG$, respectively. In [4], Boardman and Vogt proved that $BS\mathrm{TOP}$, $BS\mathrm{PL}$ and BSG have natural structures as infinite loop spaces (the underlying H-space structure in each case is the one associated to Whitney sum). They also proved that i and i' are infinite loop maps. This gives G/TOP and G/PL an infinite loop space structure. We prove that $B(G/\mathrm{TOP})$ is a product of Eilenberg-MacLane spaces and that $B(G/\mathrm{PL})$ has a single non-zero K-invariant in dimension 6. We also show that $B^2(G/\mathrm{TOP})$ is a product of Eilenberg-MacLane spaces whereas $B^2(G/\mathrm{PL})$ has one non-zero K-invariant in dimension 7.

The proofs are very formal, based on the known structure of $H_*(G/\mathrm{TOP}; \mathbf{Z}/2)$ as a module over the Dyer-Lashof algebra of homology operations and on standard results about primitives in a differential Hopf algebra. There is one slightly unusual argument though (Theorem 2.15) in which we find it necessary to bring in higher Massey products and their connection with Eilenberg-Moore spectral sequences as well as with Dyer-Lashof operations. The techniques here may have wider implications for H-spaces, so they could well have a certain independent interest.

The main line of argument is to first restrict, for dimensional reasons the types of integral primitives in the cohomology of the r'th stage in a Postnikov resolution of $B(G/\mathrm{TOP})$. Next we show (using the Massey products) that $E_2 = E_\infty$ in the Eilenberg-Moore spectral sequence converging to $H^*(B(G/\mathrm{TOP}); \mathbf{Z}/2)$. Combining this fact with our previous study of the possible primitives quickly gives the main results.

In our original exposition of these results [25], we outlined a somewhat different proof. Using the notion of a Mahowald orientation we gave geometric reasons why most of the differentials in the Eilenberg-Moore spectral sequence converging to $H^*(B(G/\mathrm{TOP}); \mathbf{Z}/2)$ had to vanish. But we needed the algebraic techniques used here to handle some special cases. It then turned out that the algebraic techniques actually applied to all the differentials and there was no need anymore to use the geometric arguments. One might wonder, though, if our geometric arguments could not themselves be strengthened to prove the entire theorem.

From [40] we know that G/PL is almost a product of Eilenberg-MacLane spaces. In fact,

$$G/\mathrm{PL} \simeq \Omega E_3 \times \prod_{n>1} K(\mathbf{Z}_{(2)}, 4n) \times \prod_{n>1} K(\mathbf{Z}/2, 4n-2)$$

where E_3 is the 2-stage Postnikov system obtained as the fibre in the fibration

$$E_3 \to K(\mathbf{Z}/2, 3) \to K(\mathbf{Z}_{(2)}, 6)$$

with K-invariant $\beta_1(\mathrm{Sq}^2 \iota_3)$.

From Kirby and Siebenmann [17] it follows that G/TOP has the homotopy type of a product of Eilenberg-MacLane spaces, namely

$$G/\mathrm{TOP} \simeq \prod_{n \geqslant 1} K(\mathbf{Z}_{(2)}, 4n) \times \prod_{n \geqslant 1} K(\mathbf{Z}/2, 4n-2).$$

(In Section 3 we review the construction of a specific identification of G/TOP with the given product of Eilenberg-MacLane spaces. This refined statement is not needed however for the conclusions of this section).

The natural map $G/\mathrm{PL} \to G/\mathrm{TOP}$ has fibre $K(\mathbf{Z}/2, 3)$. From [32], [35] and [38] we know that $H^*(G/\mathrm{TOP}; \mathbf{Z}/2)$ is a primitively generated Hopf algebra while in $H^*(G/\mathrm{PL}; \mathbf{Z}/2)$ we have

$$\Psi(k_4) = k_4 \otimes 1 + k_2 \otimes k_2 + 1 \otimes k_4,$$

where k_2 is the non-zero class in $H^2(G/\mathrm{PL}; \mathbf{Z}/2)$ (compare [9], 9.16). Apart from the unusual behaviour of k_4, the fundamental classes $k_{2i} \in H^{2i}(G/\mathrm{PL}; \mathbf{Z}/2)$ are all primitive. The classes k_{4i} are $\mathbf{Z}/2$-reductions of integral primitive fundamental classes (cf. §3).

For a space X, let $(E_r(X), d_r)$ denote its mod. 2 Bockstein spectral sequence in cohomology [5],

$$E_1(X) = H^*(X; \mathbf{Z}/2)$$
$$E_\infty(X) = H^*(X; \mathbf{Z}_{(2)})/\mathrm{Tor}.$$

When X is an H-space then $(E_r(X), d_r)$ is a spectral sequence of Hopf algebras. Let $j_r: H^*(X; \mathbf{Z}/2^r) \to E_r(X)$ denote the reduction homomorphism. It is a surjection with kernel $2^* H^*(X; \mathbf{Z}/2^{r-1}) + \varrho_r \beta_{r-1} H^*(X; \mathbf{Z}/2^{r-1})$, where 2^* is induced from the inclusion $\mathbf{Z}/2^{r-1} \subset \mathbf{Z}/2^r$, β_{r-1} is the integral Bockstein homomorphism associated with the coefficient sequence $0 \to \mathbf{Z}_{(2)} \xrightarrow{2^{r-1}} \mathbf{Z}_{(2)} \to \mathbf{Z}/2^{r-1} \to 0$ and ϱ_r is the reduction to $\mathbf{Z}/2^r$ coefficients. If $j_r(x) \neq 0$ then x has order 2^r in $H^*(X; \mathbf{Z}/2^r)$.

We recall that an element $x \in H^*(X; \mathbf{Z}_{(2)})$ is called *primitive* if $\Delta(x) = \mu(x \otimes 1 + 1 \otimes x)$, where

$$\Delta: H^*(X; \mathbf{Z}_{(2)}) \to H^*(X \times X; \mathbf{Z}_{(2)})$$

is induced from the multiplication in X and

$$\mu: H^*(X; \mathbf{Z}_{(2)}) \otimes H^*(X; \mathbf{Z}_{(2)}) \to H^*(X \times X; \mathbf{Z}_{(2)})$$

is the exterior product. The subgroup of primitive elements is denoted $PH^*(X; \mathbf{Z}_{(2)})$. We observe

LEMMA 2.1. *Let $x \in H^*(X; \mathbf{Z}/2^r)$, where X is any H-space. Then $2^{r-1}x$ is primitive if and only if $j_r(x)$ is primitive.*

We shall examine the structure of $PH^*(X; \mathbf{Z}_{(2)})$ in the case where the underlying space has the homotopy type of a product of Eilenberg-MacLane spaces $K(\Lambda, n)$ with $\Lambda = \mathbf{Z}_{(2)}$ or $\mathbf{Z}/2$. We begin by reviewing the Bockstein structure of a single

$K(\Lambda, n)$. Let $B\{x\}$ be the following DG-Hopf algebra over $\mathbf{Z}_{(2)}$,

$$\begin{aligned}&B\{x\}=P\{x\}\otimes E\{y\}, \qquad \delta x=4y\\ &\deg x=4n, \qquad \deg y=4n+1\\ &\psi(x)=1\otimes x+x\otimes 1, \qquad \psi(y)=1\otimes y+y\otimes 1.\end{aligned}$$

The associated Bockstein spectral sequence is

$$\begin{aligned}&E_{r+2}B\{x\}=P\{x^{2^r}\}\otimes E\{yx^{2^r-1}\}\\ &d_{r+2}(x^{2^r})=yx^{2^r-1}.\end{aligned}$$

The structure of $E_r(K(\Lambda, n))$ is for $r\geqslant 2$ expressable in terms of these model spectral sequences (see e.g. Browder [5])

$$\begin{aligned}&\text{(i) } E_r(K(\mathbf{Z}/2, n))=\otimes\, E_rB\{x_i\}\\ &\text{(ii) } E_r(K(\mathbf{Z}_{(2)}, 2n))=P\{\iota_{2n}\}\otimes\otimes\, E_rB\{x_i\},\\ &\text{(iii) } E_r(K(\mathbf{Z}_{(2)}, 2n-1))=E\{\iota_{2n-1}\}\otimes\otimes\, E_rB\{x_i\},\end{aligned} \tag{2.2}$$

where ι_{2n} and ι_{2n-1} are reductions of integral primitive elements. The number of factors in each of the cases above as well as the naming of the elements x_i in $E_1(K(\Lambda, n))=H^*(K(\Lambda, n); \mathbf{Z}/2)$ is available but irrelevant for our purpose. We shall however use that each $x_i\in H^{4*}(K(\Lambda, n); \mathbf{Z}/2)$ is a square of a primitive (indecomposable) element.

Let $P: H^*(X; \mathbf{Z}/2^i)\to H^*(X; \mathbf{Z}/2^{i+1})$ be the Pontrjagin squaring operation (Thomas [42]) and let $P^{(r-1)}: H^*(X; \mathbf{Z}/2)\to H^*(X; \mathbf{Z}/2^r)$ be the $(r-1)$ st iterate. The Pontrjagin square is a refinement of the cup product square; in particular, $i_rP^{r-1}(x)=x^{2^r-1}$. From the remarks following 2.2 we know that $j_rP^{(r-1)}(x_i)=z_i^{2^r}$ for a certain indecomposable and primitive element $z_i\in E_1(K(\Lambda, n))$.

LEMMA 2.3. *The subgroup of primitive torsion elements in* $H^*(K(\Lambda, n); \mathbf{Z}_{(2)})$ *form a vector space over* $\mathbf{Z}/2$. *In fact* $\mathrm{Tor}PH^*(K(\Lambda, n); \mathbf{Z}_{(2)})$ *is spanned by the elements*

$$\begin{aligned}&\text{(i) } 2^{r-1}\beta_rP^{(r-1)}(z), \qquad z\in \mathrm{Tor}PH^{2i}(K(\Lambda, n); \mathbf{Z}/2)\\ &\text{(ii) } (\beta_1(z))^{2^a} \qquad\qquad z\in \mathrm{Tor}PH^{i}(K(\Lambda, n); \mathbf{Z}/2).\end{aligned}$$

Proof. It is a consequence of 2.1 that the elements $2^{r-1}\beta_rP^{(r-1)}(z)$ are primitive. It suffices to prove that a primitive torsion element p is a linear combination of the elements listed in (i) and (ii). Suppose inductively that

$$q_r=p+\sum_{i=1}^{r-1}2^{i-1}\beta_iP^{(i-1)}(z_i)$$

is divisible by 2^{r-1} in $H^*(K(\Lambda, n); \mathbf{Z}_{(2)})$. From 2.1 it follows that $j_r((1/2^{r-1})\,q_r)$ is primitive and from 2.2 that there is an element $z_r\in PH^{ev}(K(\Lambda, n); \mathbf{Z}/2)$ with

$i_r((1/2^{r-1})\, q_r) = j_r(\beta_r P^{(r-1)}(z_r))$. But then $q_r + 2^{r-1}\beta_r P^{(r-1)}(z_r)$ reduces to zero in $H^*(K(A, n); \mathbf{Z}/2^r)$ and is therefore divisible by 2^r. This process stops since $K(A, n)$ is of finite type. We finally note that if $j_1(p) = (\beta_1(z))^{2^a}$ for $a > 0$ then $p = \beta_1(z)^{2^a}$ since the elements $\beta_r P^{(r-1)}(z)$ for $r > 1$ all have dimension congruent to $1 \pmod 4$. This completes the proof.

A product of Eilenberg-MacLane spaces can have several H-space structures. Let $E_{4,k}$ be the fibre in the fibration

$$E_{4,k} \longrightarrow K(\mathbf{Z}/2, k+3) \xrightarrow{\mathrm{Sq}^4} K(\mathbf{Z}/2, k+7).$$

Then $\Omega E_{4,k} = E_{4,k-1}$. In particular, $E_{4,0}$ has the homotopy type of $K(\mathbf{Z}/2, 3) \times \times K(\mathbf{Z}/2, 6)$. The H-space structure on $E_{4,0}$ however is distinct from the ordinary structure on the product, since in $H^*(E_{4,0}; \mathbf{Z}/2)$,

$$\psi(\iota_6) = \iota_6 \otimes 1 + \iota_3 \otimes \iota_3 + 1 \otimes \iota_6.$$

(Compare [1]).

More generally, if X is an H-space which is homotopy equivalent to a product of $K(\mathbf{Z}/2, i)$'s and $K(\mathbf{Z}_{(2)}, j)$'s and if

$$K(\mathbf{Z}/2, 4n+1) \xrightarrow{j} E \xrightarrow{i} X \xrightarrow{\pi} K(\mathbf{Z}/2, 4n+2)$$

is a fibration sequence with $\pi^*(\iota_{4n+2}) = \mathrm{Sq}^{2n+1}(x)$ for some primitive element $x \in H^{2n+1}(X; \mathbf{Z}/2)$, then in $H^*(\Omega E; \mathbf{Z}/2)$ there is a class ι_{4n} with $j^*(\iota_{4n})$ the generator of $H^{4n}(K(\mathbf{Z}/2, 4n); \mathbf{Z}/2)$ and such that

$$\bar{\psi}(\iota_{4n}) = \sigma^*(i^*(x)) \otimes \sigma^*(i^*(x)),$$

where $\bar{\psi}$ is the reduced diagonal. This follows easily using the methods of [18] or [33].

By an *abelian* Hopf algebra we shall mean a commutative and cocommutative Hopf algebra. Let A be an algebra over $\mathbf{Z}/2$ equipped with two coalgebra structures ψ_1 and ψ_2 and such that (A, ψ_i) are abelian Hopf algebras. Further, suppose that (A, ψ_2) is primitively generated. (A, ψ_1) is a tensor product of monogenic Hopf algebras by a theorem of Milnor and Moore [34]. Moreover, the primitive elements of (A, ψ_1) are contained among the indecomposables and elements of the form x^{2^i} with x primitive. We conclude that the primitive elements of (A, ψ_1) occur in a subset of the same dimensions as the primitive elements of (A, ψ_2). As a corollary of the proof of 2.3 we then get

LEMMA 2.4. *Let X be a homotopy commutative H-space and suppose the underlying space has the homotopy type of a product of Eilenberg-MacLane spaces, $X \simeq \prod K(\mathbf{Z}_{(2)}, j)$. Then a primitive torsion element of $H^*(X; \mathbf{Z}_{(2)})$ either occurs in dimension $4t+1$ or it has a non-zero $\mathbf{Z}/2$ reduction.*

We shall now consider the Eilenberg-Moore spectral sequences of a fibration of infinite loop spaces

$$X \to EX \to BX \qquad (EX \simeq *)$$

converging either to $H_*(BX; \mathbf{Z}/2)$ or $H^*(BX; \mathbf{Z}/2)$. The latter is a first quadrant spectral sequence of cohomology type with

$$\begin{aligned} &E_2^{s,t} = \mathrm{Ext}^{s,t}_{H^*(X;\,\mathbf{Z}/2)}(\mathbf{Z}/2, \mathbf{Z}/2) \\ &E_\infty = E_0 H^*(BX; \mathbf{Z}/2). \end{aligned} \tag{2.5}$$

The spectral sequence is associated to the usual geometric filtration $B_1X \subset B_2X \subset \cdots \subset \subset B_nX \subset \cdots$ of BX by the "number of joins" [29]. In particular, the spectral sequence admits an action of the Steenrod algebra. A result of A. Clark [12] asserts that $\{E_r, d_r\}$ is a spectral sequence of differential abelian Hopf algebras.

There is a natural identification $\Sigma X = B_1X$ and the resulting inclusion $\sigma: \Sigma X \to BX$ may be identified with the usual suspension map $\Sigma\Omega BX \to BX$ ([29], [39]). Thus $E_\infty^{1,*} \subset E_2^{1,*} = PH^*(X; \mathbf{Z}/2)$ determines exactly the image of the cohomology suspension σ^*.

Dually we have a first quadrant homology type spectral sequence with

$$\begin{aligned} &E^2 = \mathrm{Tor}_{H_*(X,\,\mathbf{Z}/2)}(\mathbf{Z}/2, \mathbf{Z}/2) \\ &E^\infty = E^0 H_*(BX; \mathbf{Z}/2). \end{aligned} \tag{2.6}$$

Again, $\{E^r, d^r\}$ is a spectral sequence of differential abelian Hopf algebras, and the elements of $E^\infty_{1,*}$ give the image of $\sigma_*: QH_*(X; \mathbf{Z}/2) \to PH_*(BX; \mathbf{Z}/2)$.

The following two lemmas are often useful when dealing with the Eilenberg-Moore spectral sequences. We recall that a Hopf algebra A is called primitive if the natural map $P(A) \xrightarrow{j} Q(A)$ is surjective and is called biprimitive if j is an isomorphism.

LEMMA 2.7. *Suppose A is a primitive abelian differential Hopf algebra. Then $H(A, d)$ is again primitive.*

Proof. The Hopf algebra A is primitive if and only if $P(A^*) \to Q(A^*)$ is injective. The lemma now follows from the exact sequence

$$0 \to P(H(A^*) \xrightarrow{\xi} P(H(A^*)) \to Q(H(A^*)),$$

since $\xi \equiv 0$ on $P(A^*)$ implies that $\xi \equiv 0$ on $P(H(A^*))$.

LEMMA 2.8. *Let $A = \{A^{r,s}\}$ be a primitive abelian differential bigaded Hopf algebra with differential of bidegree $(n, -n+1)$. Suppose A has the property that every primitive element $p \in A^{r,s}$ with $r \geqslant 3$ occurs in odd total degree or in total degree congruent to $0 \pmod 4$. Then $H(A)$ has the same property.*

Proof. First, if A is biprimitive, then

$$A = \bigotimes_i E\{x_i\} \otimes E\{y_i\} \otimes \bigotimes_j E\{z_j\}$$

with differential $dx_i = y_i$, $dz_j = 0$, and the lemma follows easily by direct computation,

$$H(A) = \bigotimes E\{x_i y_i\} \otimes \bigotimes E\{z_j\}.$$

When A is not biprimitive, we use the spectral sequence of Browder ([6], 3.3 and 3.4). It is a spectral sequence of biprimitive Hopf algebras with $E_1(A)$ the biprimitive form of A and $E_\infty(A)$ the biprimitive form of $H(A)$. Since a primitive Hopf algebra and its biprimitive form have the same primitive elements, the lemma follows.

As a final preparation for our main theorems we review the connection between matric Massey products and the Eilenberg-Moore spectral sequence as well as the connection of matric Massey products with the Dyer-Lashof operations. The references for this are [14], [21] and [30].

Let (A, d) be a DG-algebra. Massey products are higher order operations in $H(A, d)$ which arise whenever $H(A, d)$ has more multiplicative relations than A. The simplest case is the triple product $\langle \hat{a}, \hat{b}, \hat{c} \rangle$ defined for elements $\hat{a}$, $\hat{b}$ and $\hat{c}$ of $H(A, d)$ with $\hat{a}\hat{b} = 0$ and $\hat{b}\hat{c} = 0$. Choose a, b and c in A representing the respective classes. Then $ab = du$ and $bc = dv$ for some u and v in A and $uc + av$ is a cycle (we are working over $\mathbf{Z}/2$). The set of all the associated homology classes $\{uc + av\}$ is denoted $\langle a, b, c \rangle$. It is easy to see that this set determines a unique element in the quotient group $H(A)/\hat{a}H(A) + H(A)\,\hat{c}$.

DEFINITION 2.9. Let A be as above and suppose M and N are matrices with entries in A of type $n \times m$ and $m \times k$, respectively. We say that M and N are multipliable if $\deg(m_{ij}) + \deg(n_{jk})$ depends only on i and k.

When M and N are multipliable matrices, then $M \cdot N$ is again a matrix with entries in A.

DEFINITION 2.10. Let $M_1, \ldots, M_n$ be a system of matrices in $H(A, d)$ such that M_1 is a row and M_n a column and such that M_i and M_{i+1} are multipliable for all i. The n-fold matric Massey product $\langle M_1, \ldots, M_n \rangle$ is said to be defined if there exist matrices N_{ij} $(1 \leqslant i \leqslant j \leqslant n+1$ and $1 \leqslant j-i \leqslant n-1)$ with entries in A satisfying

$$d_{ij}N = \Sigma_k N_{ik} N_{kj}, \qquad dN_{i,i+1} = 0$$

and with the class of $N_{i,i+1}$ in $H(A, d)$ equal to M_i. The value of $\langle M_1, \ldots, M_n \rangle$ is the set of all classes in $H(A, d)$ represented by cycles of the form $\Sigma N_{1,k} N_{k,n+1}$.

It should be noted that any two values of $\langle M_1, \ldots, M_n \rangle$ differ by elements in certain $(n-1)$-fold matric Massey products (the reader might consult [30] pp. 41 and 42 for examples of these products).

The next theorem which is due to J. P. May [27] connects matric Massey products with the Eilenberg-Moore spectral sequence of a fibration $X \to EX \to BX$ (see 2.6).

THEOREM 2.11. (May). *Let X be a connected strictly associative H-space with a strict unit. Then the suspension map*

$$\sigma_r : H_j(X; \mathbf{Z}/2) \to E^r_{1,j}(BX; \mathbf{Z}/2)$$

has kernel the set of all k-fold matric Massey products with $2 \leqslant k \leqslant r$.

Suppose now that X is an infinite loop space. Passing to the Moore loop space we can assume that X is strictly associative with a strict unit. Then the singular chain complex $C_*(X; \mathbf{Z}/2)$ is a DG-algebra and matric Massey products make sense. The infinite loop space structure gives among other things a map (Dyer-Lashof [13])

$$\Theta : W \otimes_{\mathbf{Z}/2[\Sigma_2]} C_*(X; \mathbf{Z}/2) \otimes C_*(X; \mathbf{Z}/2) \to C_*(X; \mathbf{Z}/2),$$

where W is the standard $\mathbf{Z}/2[\Sigma_2]$-free resolution of $\mathbf{Z}/2$ with a single generator e_i in each dimension i. Let $x \cup_i y = \Theta(e_i \otimes x \otimes y)$ and define chain level operations

$$q_i(x) = x \cup_i x + \partial x \cup_{i+1} x.$$

There are induced operations in homology

$$Q_i : H_n(X; \mathbf{Z}/2) \to H_{2n+i}(X; \mathbf{Z}/2).$$

(The Dyer-Lashof operations Q^i are defined as $Q^i(x) = Q_{i-n}(x)$ for $x \in H_n(X; \mathbf{Z}/2)$).

Matric Massey products on differential graded algebras with additional structure were considered in [30]. It is not hard to see that the singular chains of an infinite loop space have the required extra structure to assure that Theorem 0 of [30] is valid (compare [21]). Thus we have

PROPOSITION 2.12. *Let X be an infinite loop space and let $x \in H_*(X; \mathbf{Z}/2)$ be an element of the matric Massey product $\langle M_1, \ldots, M_n \rangle$. Then $Q_2(x)$ is contained in the n-fold Massey product*

$$\left\langle (Q_2M_1, Q_1M_1, Q_0M_1), \begin{pmatrix} Q_0M_2 & 0 & 0 \\ Q_1M_2 & Q_0M_2 & 0 \\ Q_2M_2 & Q_1M_2 & Q_0M_2 \end{pmatrix}, \ldots, \begin{pmatrix} Q_0M_n \\ Q_1M_n \\ Q_2M_n \end{pmatrix} \right\rangle.$$

In [24] the action of homology operations in $H_*(G/\mathrm{TOP}; \mathbf{Z}/2)$ was determined. Let $k_{2n} \in H^{2n}(G/\mathrm{TOP}; \mathbf{Z}/2)$ be any fundamental class, that is, a class which projects non-trivialy to the quotient group $\mathbf{Z}\otimes_A QH^*(G/\mathrm{TOP};\mathbf{Z}/2)$ of indecomposable elements over the Steenrod algebra. From [24] we have

PROPOSITION 2.13. *For every class* x *in* $H_*(G/\mathrm{TOP};\mathbf{Z}/2)$, $Q_0(x)=0$ *and* $Q_1(x)=0$. *However, if* $\langle x, k_{4i+2}\rangle\neq 0$ *then* $\langle Q_2(x), k_{8i+6}\rangle\neq 0$ *as well.*

Let (E^r, d^r) denote the Eilenberg-Moore spectral sequence of the fibration $G/\mathrm{TOP}\to E(G/\mathrm{TOP})\to B(G/\mathrm{TOP})$ (compare 2.6). In view of 2.11, 2.12 and 2.13 we get

PROPOSITION 2.14. *Let* $x\in H_j(G/\mathrm{TOP};\mathbf{Z}/2)$ *and suppose that the suspension* $\sigma_r(x)\in E^r_{1,j}$ *is a boundary,* $\sigma_r(x)=d^r(y)$ *for some* $y\in E^r_{r+1,j-r-1}$. *Then* $\sigma_r(Q_2(x))=0$ *in* $E^r_{1,2j+2}$.

In the beginning of this section we remarked that $H^*(G/\mathrm{TOP};\mathbf{Z}/2)$ was a primitive Hopf algebra. Therefore $H_*(G/\mathrm{TOP};\mathbf{Z}/2)$ is an exterior algebra and the E_2-term of the Eilenberg-Moore spectral sequence converging to $H^*(B(G/\mathrm{TOP});\mathbf{Z}/2)$ has the form

$$E_2=P\{[p]\mid p\in PH^*(G/\mathrm{TOP};\mathbf{Z}/2)\}.$$

Moreover, since all the generators have filtration degree 1, they are primitive and E_2 is consequently a primitive abelian Hopf algebra.

THEOREM 2.15. *The Eilenberg-Moore spectral sequence converging to* $H^*(B(G/\mathrm{TOP});\mathbf{Z}/2)$ *collapses, i.e.* $E_2=E_\infty$. *In particular* $\sigma: QH^*(B(G/\mathrm{TOP});\mathbf{Z}/2)\to$ $\to PH^*(G/\mathrm{TOP};\mathbf{Z}/2)$ *is an isomorphism.*

Proof. Since the spectral sequence is a module over the mod. 2 Steenrod algebra A (and in particular d_r is an A-homomorphism) and since G/TOP is a product of Eilenberg-MacLane spaces, it suffices to prove that $[k_{4n+2}]$ and $[k_{4n}]$ in E_2 are infinite cycles. First consider the $[k_{4n}]$. They are primitive and therefore if $d_r([k_{4n}])\neq 0$, it must be a primitive element of total degree $4n+2$ and with filtration degree $r+1\geqslant 3$. But in E_2 the primitives of filtration degree at least 3 all have total degrees congruent to $0 \pmod 4$. According to 2.7 and 2.8, each stage E_r in the spectral sequence has no primitive elements in filtration degree $\geqslant 3$ and total degree congruent to $2 \pmod 4$. Thus $[k_{4n}]$ is an infinite cycle.

We next consider the elements $[k_{4n+2}]$. To prove that these elements are infinite cycles we first note that the Eilenberg-Moore spectral sequences, E_r and E^r converging to $H^*(B(G/\mathrm{TOP});\mathbf{Z}/2)$ and $H_*(B(G/\mathrm{TOP});\mathbf{Z}/2)$, respectively, are dual to each other. Suppose that $d_r([k_{4n+2}])\neq 0$, then there exist $y\in E^r$ and $x\in H_*(G/\mathrm{TOP};\mathbf{Z}/2)$ such that $d^r(y)=\sigma_r(x)$ and $\langle [k_{4n+2}], \sigma_r(x)\rangle=1$. But then 2.13 implies that $\langle [k_{8n+6}], \sigma_r(Q_2(x))\rangle=1$ and in particular $\sigma_r(Q_2(x))\neq 0$. This contradicts 2.14 and finishes the proof.

Let

$$\begin{tikzcd}
\cdots \arrow[r] & BE_5 \arrow[r] & BE_4 \arrow[r] \arrow[d, "\pi_4"] & BE_3 \arrow[r] \arrow[d, "\pi_3"] & BE_2 \arrow[r] \arrow[d, "\pi_2"] & K(\mathbf{Z}/2, 3) \arrow[d, "\pi_1"] \\
 & & K(\mathbf{Z}/2, 12) & K(\mathbf{Z}_{(2)}, 10) & K(\mathbf{Z}/2, 8) & K(\mathbf{Z}_{(2)}, 6)
\end{tikzcd}$$

be a Postnikov decomposition of $B(G/\mathrm{TOP})$. It is completely determined by specifying the K-invariants $K_r=\pi_r^*(\iota)$ in $H^*(BE_r;\, \pi_*(G/\mathrm{TOP}))$. Since G/TOP is an infinite loop space, the same is true of each stage BE_r. In particular K_r must be in the image of the suspension map and hence primitive. This fact sharply limits the possibilities for the K-invariants.

THEOREM 2.16. *There is a* (2-*local*) *homotopy equivalence*

$$B(G/\mathrm{TOP})\simeq \prod_{n=1}^{\infty} K(\mathbf{Z}/2, 4n-1)\times K(\mathbf{Z}_{(2)}, 4n+1).$$

Proof. The proof is by induction over the Postnikov decomposition of $B(G/\mathrm{TOP})$. Suppose that the r'th stage BE_r has the homotopy type of a product of Eilenberg-MacLane spaces. We must show that the K-invariant in the next stage is zero. Consider the projection $\pi: B(G/\mathrm{TOP})\to BE_r$. The K-invariant is determined by the first dimension in which π is not a homotopy equivalence and is non-zero only if

$$\pi^*: H^{s+1}(BE_r;\, \pi_s(B(G/\mathrm{TOP})))\to H^{s+1}(B(G/\mathrm{TOP});\, \pi_s(B(G/\mathrm{TOP})))$$

is not injective. In our case the kernel must be cyclic with a primitive generator.

If $s=4i+1$, we require a primitive element K_r of $H^{4i+2}(BE_r;\, \mathbf{Z}_{(2)})$ and from 2.4 either $K_r=0$ or $\varrho_1(K_r)\neq 0$ in $H^{4i+2}(BE_r;\, \mathbf{Z}/2)$. In the latter case, consider $\sigma^*(\varrho_1(K_r))$. It is surely zero since G/TOP is a product of Eilenberg-MacLane spaces. Hence $\varrho_1(K_r)=y^2$ for some primitive element y. This follows from the exact sequence (Milnor-Moore [34])

$$0\to PH^*(BE_r;\, \mathbf{Z}/2)\xrightarrow{\xi} PH^*(BE_r;\, \mathbf{Z}/2)\to QH^*(BE_r;\, \mathbf{Z}/2)$$

together with 2.15. Since y is odd dimensional, it is indecomposable and thus $\sigma^*(y)\neq 0$ in $H^*(G/\mathrm{TOP};\, \mathbf{Z}/2)$. In this case we would have in $H^*(G/\mathrm{TOP};\, \mathbf{Z}/2)$

$$\bar{\psi}(\varrho_1(k_{4i}))=\sigma^*(y)\otimes\sigma^*(y)$$

(compare the paragraphs preceding 2.4). This contradicts the fact that $H^*(G/\mathrm{TOP};\, \mathbf{Z}/2)$ is a primitive Hopf algebra.

If $s=4i-1$, then the possible K-invariant K_r belongs to $H^{4i}(BE_r;\, \mathbf{Z}/2)$. That this

must be zero follows by a counting argument and uses the fact that the additive structure of $H^*(B(G/\mathrm{TOP}); \mathbf{Z}/2)$ is the same as the additive structure of $H^*(\prod K(\mathbf{Z}_{(2)}, 4n+1)\times K(\mathbf{Z}_{(2)}, 4n-1); \mathbf{Z}/2))$. This completes the proof.
It is now easy to prove the main result of this section

THEOREM 2.17. *There is a* (2*-local*) *homotopy equivalence*

$$B^2(G/\mathrm{TOP}) \simeq \prod_{n=1}^{\infty} K(\mathbf{Z}_{(2)}, 4n+2)\times K(\mathbf{Z}/2, 4n).$$

Proof. First, it is a simple dimensional argument to see that the Eilenberg-Moore spectral sequence converging to $H^*(B^2(G/\mathrm{TOP}; \mathbf{Z}/2)$ collapses. Therefore

$$\sigma: QH^*(B^2(G/\mathrm{TOP}); \mathbf{Z}/2) \to PH^*(B(G/\mathrm{TOP}); \mathbf{Z}/2)$$

is an isomorphism. For $B^2(G/\mathrm{TOP})$ the K-invariants occur in dimensions $4s+3$ and $4s+1$. Let $B^2(E_r)$ denote the r'th stage in the Postnikov decomposition for $B^2(G/\mathrm{TOP})$ and assume it is a product of Eilenberg-MacLane spaces. Then the r'th K-invariant is a primitive element in either $H^{4s+3}(B^2E_r; \mathbf{Z}_{(2)})$ or in $H^{4s+1}(B^2E_r; \mathbf{Z}/2)$. In the first case K_r is non-zero only if $\varrho_1(K_r)\neq 0$. But $\varrho_1(K_r)$ is an odd-dimensional primitive and hence indecomposable. Since $\sigma^*(\varrho_1(K_r))=0$ we conclude that $\varrho_1(K_r)$ is itself zero. In the second case a similar remark applies. This proves the theorem.

We shall finally determine the spaces $B(G/\mathrm{PL})$ and $B^2(G/\mathrm{PL})$. Let E_3 and $E_{3,1}$ be the fibres in the fibrations

$$E_3 \to K(\mathbf{Z}/2, 3) \xrightarrow{\beta_1 \mathrm{Sq}^2} K(\mathbf{Z}_{(2)}, 6),$$
$$E_{3,1} \to K(\mathbf{Z}/2, 4) \xrightarrow{\beta_1 \mathrm{Sq}^2} K(\mathbf{Z}_{(2)}, 7).$$

THEOREM 2.18. *There are* (2*-local*) *homotopy equivalences*

$$B(G/\mathrm{PL}) \simeq E_3 \times \prod_{n=2}^{\infty} K(\mathbf{Z}_{(2)}, 4n+1)\times K(\mathbf{Z}/2, 4n-1)$$
$$B^2(G/\mathrm{PL}) \simeq E_{3,1} \times \prod_{n=2}^{\infty} K(\mathbf{Z}_{(2)}, 4n+2)\times K(\mathbf{Z}/2, 4n).$$

Proof. Consider the fibration

$$K(\mathbf{Z}/2, 4) \to B(G/\mathrm{PL}) \to B(G/\mathrm{TOP}).$$

It is of course a fibering in the category of infinite loop spaces and thus classified by a stable mapping

$$B(G/\mathrm{TOP}) \xrightarrow{\lambda} K(\mathbf{Z}/2, 5).$$

In particular $B(G/\mathrm{PL})$ is the fibre of λ. But

$$PH^5(B(G/\mathrm{TOP}); \mathbf{Z}/2)=\mathbf{Z}/2\oplus\mathbf{Z}/2$$

with generators $\mathrm{Sq}^2(\iota_3)$ and $\varrho_1(\iota_5)$, respectively. Moreover, in view of the known structure of G/PL the only possibility for $\lambda^*(\iota)$ is $\lambda^*(\iota)=\mathrm{Sq}^2(\iota_3)+\varrho_1(\iota_5)$, and the result on $B(G/\mathrm{PL})$ easily follows. The result for $B^2(G/\mathrm{PL})$ is shown in a similar fashion.

3. Delooping the Universal Surgery Class

The space G/TOP is the classifying space for "normal maps". A homotopy class $M\xrightarrow{f}G/\mathrm{TOP}$ (M a manifold, $\dim M>4$) is equivalent to a normal cobordism class $M'\to M$. The simply connected surgery obstructions thus give invariants of the set of homotopy classes $[M, G/\mathrm{TOP}]$ – in fact of the smooth bordism of G/TOP. If $\dim M\leq 4$ one first cross with CP^2 and then take the simply connected surgery invariants. These invariants are expressable in terms of characteristic classes of the map $f: M\to G/\mathrm{TOP}$. Indeed, there is a class ([38])

$$k_{4*-2}=k_2+k_6+\cdots H^{4*-2}(G/\mathrm{TOP}; \mathbf{Z}/2)$$

such that the Kervaire invariant $s_K(M^{2n},f)$ of the normal cobordism class associated with f is given by the formula

$$s_K(M^{2n},f)=\langle f^*(k_{4*-2})\cdot V(M)^2, [M]\rangle, \tag{3.1}$$

where $V(M)$ is the total Wu class of M.

Next, let M^{4n} be a smooth $\mathbf{Z}/2^r$-manifold, that is, a smooth "manifold" with $\mathbf{Z}/2^r$ cone singularities along a codimension one submanifold δM (see [32] or [35] for a precise definition). Let $\nu: M\to BSO$ denote the $\mathbf{Z}/2^r$-normal bundle. As in the non-singular case a homotopy class of maps $f: M\to G/\mathrm{TOP}$ gives rise to a normal cobordism class af $\mathbf{Z}/2^r$-manifolds $M'\to M$ and hence an index obstruction $s_I(M,f)\in\mathbf{Z}/2^r$. The invariant $s_I(M,f)$ only depends on the bordism class of (M,f) as an element of $\Omega_*(G/\mathrm{TOP}; \mathbf{Z}/2^r)$ and is consequently expressable in terms of characteristic classes. Precise formulas were given in [32] and [35]. Let $\mathscr{L}\in H^*(BSO; \mathbf{Z}_{(2)})$ be the modified (inverse) Hirzebruch class [35]; it is the unique class whose rational reduction is the inverse Hirzebruch polynomial and whose $\mathbf{Z}/2$-reduction is the square of the total Wu-class. Let $\nu: M\to BSO$ denote the $\mathbf{Z}/2^r$ normal bundle. There is a graded class

$$K_{4*}=K_4+K_8+\cdots\in H^{4*}(G/\mathrm{TOP}; \mathbf{Z}_{(2)})$$

such that the index invariant $s_I(M,f)\in\mathbf{Z}/2^r$ is given as

$$\begin{aligned} s_I(M,f) &= \langle f^*(K_{4*})\, v^*(\mathscr{L}),\, [M]\rangle + \\ &\quad + 2^{k-1}\langle f^*(k_{4*-2})\, v^*(\Sigma v_{2i}\,\mathrm{Sq}^1 v_{2i}),\, [\delta M]\rangle . \end{aligned} \tag{3.2}$$

Here v_{2i} denotes the $2i$'th Wu class, k_{4*-2} the class in 3.1 and 2^{k-1} the injection $\mathbf{Z}/2\subset\mathbf{Z}/2^k$.

The classes k and K are uniquely characterised by 3.1 and 3.2 since the bordism groups $\mathfrak{N}_*(G/\mathrm{TOP})$ and $\Omega_*(G/\mathrm{TOP};\mathbf{Z}/2^r)$ map onto $H_*(G/\mathrm{TOP};\mathbf{Z}/2)$ and $H_*(G/\mathrm{TOP};\mathbf{Z}/2^r)$, respectively.

Remark. The class K_{4i} above is the class constructed in [32]. In [35] a different class L_{4i} was constructed using the genus $V\,\mathrm{Sq}^1\,V$ rather than $\Sigma v_{2i}\,\mathrm{Sq}^1 v_{2i}$. The difference between K_{4i} and L_{4i} is easily seen to be a class of order 2 in the subgroup of $H^*(G/\mathrm{TOP})$ generated by the action of the Steenrod algebra on the classes k_{4i-2}. The precise formula is (compare [8])

$$L_{4*}-K_{4*}=\beta_1\,\mathrm{Sq}(2^*)\,\mathrm{Sq}^1\,k_{4*-2}$$

where $\mathrm{Sq}(2^*)=1+\sum_{i=0}^{\infty}\mathrm{Sq}^{2^i}$. The classes k_{4n-2} are primitive, whereas the coproduct on K_{4n} is

$$\psi(K_{4n})=1\otimes K_{4n}+8\left(\sum_{i=1}^{n-1}K_{4i}\otimes K_{4(n-i)}\right)+K_{4n}\otimes 1$$

so that $8K_{4*}$ is a multiplicative class.

We recall that when X_{4*} is a multiplicative class then the Newton polynomial $s_n(X_4,\ldots,X_{4n})$ is an additive (i.e. primitive) class. It is given by the formula

$$s_n(X_4,\ldots,X_{4n})=\sum a(i_1,\ldots,i_n)\,X_4^{i_1},\ldots,X_{4n}^{i_n}$$

where the summation is over all n-tuples with $\sum r i_r=n$ and where the coefficient $a(i_1,\ldots,i_n)$ is

$$(-1)^\varrho\, a(i_1,\ldots,i_n)=n(i_1+\cdots+i_n-1)!/i_1!\ldots i_n!,\qquad \varrho=\sum i_r .$$

From the well known formula for the 2-adic valuation on $k!$, $v(k!)=k-\alpha(k)$ it follows easily that $8^\varrho a(i_1,\ldots,i_n)$ is divisible by $8n$ and in fact divisible by $32n$ when $(i_1,\ldots,i_n)\neq(0,\ldots,0,1)$. Let $\tilde{s}_n$ be the polynomial

$$\tilde{s}_n(X_4,\ldots,X_{4n})=\frac{1}{8n}\,s_n(8X_4,\ldots,8X_{4n}).$$

It has coefficients in $\mathbf{Z}_{(2)}$ and

$$\tilde{s}_n(X_4,\ldots,X_{4n})\equiv X_{4n}\ (\mathrm{mod}\,4).$$

The element $k_{4n}=\tilde{s}_n(K_4,\dots,K_{4n})$ is a primitive class in $H^*(G/\mathrm{TOP};\mathbf{Z}_{(2)})$. It differs from K_{4n} only by decomposable terms, in fact, by $4\cdot$(decomposable terms). The classes k_{4n-2} and k_{4n} together define a specific 2-local homotopy equivalence of H-spaces

$$K:(G/\mathrm{TOP})_{(2)}\to\prod_{n=1}^{\infty}K(\mathbf{Z}_{(2)},4n)\times K(\mathbf{Z}/2,4n-2). \tag{3.3}$$

Next, we recollect some results on the homological structure of G/O. First of all ([22])

$$H_*(G/O;\mathbf{Z}/2)=P\{u_{a,b}\mid b\leqslant a\leqslant 2b\}\otimes P\{u_I\mid I\in\mathscr{J}\},$$

where $\mathscr{J}$ is the set of sequences $I=(i_0,i_1,\dots,i_n)$ of positive integers which satisfy

$$2\leqslant n,\qquad i_{j-1}\leqslant 2i_j,\qquad 1\leqslant i_0-i_1-\cdots-i_n.$$

The degree of $u_{a,b}$ is $a+b$ and the degree of u_I is $i_0+\cdots+i_n$.

Let $\zeta:H_{2n}(G/O;\mathbf{Z}/2)\to H_n(G/O;\mathbf{Z}/2)$ denote the halving map. It is the $\mathbf{Z}/2$-dual of the cup-squaring map in cohomology, $\zeta^*(x)=x^2$. The value of ζ on the basis above is

$$\zeta(u_{2a,2b})=u_{a,b},\qquad \zeta(u_{2I})=u_I.$$

In particular ζ is surjective. Hence ζ^* is injective and $H^*(G/O;\mathbf{Z}/2)$ is a polynomial algebra. The space G/O is an infinite loop space ([4]) and as such it admits homology operations

$$\hat{Q}^a:H_n(G/O;\mathbf{Z}/2)\to H_{n+a}(G/O;\mathbf{Z}/2)$$

as well as Pontrjagin squaring operations ([23])

$$\hat{P}:H_n(G/O;\mathbf{Z}/2^r)\to H_{2n}(G/O;\mathbf{Z}/2^{r+1}).$$

Let β_r be the r'th order integral (or rather 2-local) Bockstein operator and ϱ_r the reduction homomorphism to $\mathbf{Z}/2^r$ coefficients. Then

$$\varrho_1\beta_1(u_I)=(i_0-1)\,u_{I-\Delta_0},\qquad \varrho_1\beta_1(u_{a,b})=(a-1)\,u_{a-1,b},$$

where $I-\Delta_0=(i_0-1,i_1,\dots,i_n)$. We note that the sequence $I-\Delta_0$ is not necessarily in $\mathscr{J}$, indeed $I-\Delta_0\notin\mathscr{J}$ if and only if $i_0-i_1-\cdots-i_n=1$. In this case $u_I-\Delta_0$ is to be interpreted as u_J^2, $J=(i_1,\dots,i_n)$.

The higher torsion structure of G/O is a consequence of the following "universal" formulas

$$\varrho_r\beta_{r+1}(\hat{P}^{(r)}(u))=\hat{P}^{(r-1)}(u)\cdot\beta_r\hat{P}^{(r-1)}(u),\qquad r\geqslant 2$$
$$\varrho_1\beta_2(\hat{P}(u))=u\cdot\beta_1 u+\hat{Q}^{2n}(\varrho_1\beta_1(u)),$$

where $u \in H_{2n}(G/O; \mathbf{Z}/2)$ and $\hat{P}^{(r)}(u) \in H_*(G/O; \mathbf{Z}/2^{r+1})$ is the r'th iterated Pontrjagin square.

In [23] we found that G/O is *Henselian*. Roughly, this means that the higher torsion of $H_*(G/O; \mathbf{Z}_{(2)})$ is generated from $H_{ev}(G/O; \mathbf{Z}/2)$ under iterated use of the Pontrjagin square followed by a Bockstein. We list as an immediate consequence

LEMMA 3.4. *A primitive class in* $H^*(G/O; \mathbf{Z}_{(2)})$ *is determined by its* $\mathbf{Z}/2$ *and* $\mathbf{Q}$ *reductions together with its value on the classes* $\hat{P}^{(r)}(u)$, $u \in H_{ev}(G/O; \mathbf{Z}/2)$ *and* $r \geqslant 1$.

Let $\tau: G/O \to G/\mathrm{TOP}$ be the natural (infinite loop) map and consider the composite

$$\tau_1^*: \mathrm{Tor}\, PH^*(G/\mathrm{TOP}; \mathbf{Z}_{(2)}) \xrightarrow{\varrho_1} PH^*(G/\mathrm{TOP}; \mathbf{Z}/2) \xrightarrow{\tau^*} PH^*(G/O; \mathbf{Z}/2).$$

As a final preparation for the proof of Theorem D we shall need

LEMMA 3.5. $\mathrm{Im}\, \tau_1^* = \mathrm{Sq}^1 \mathrm{Im}(\tau^*)$.

Proof. One inclusion is obvious since Sq^1 is the reduction to $\mathbf{Z}/2$ coefficients of the integral Bockstein. The space G/TOP is a product of Eilenberg-MacLane spaces as far as $\mathbf{Z}_{(2)}$ homology goes. From 2.3 we see that it suffices to prove that any element $\tau^*(\mathrm{Sq}^1(l))^{2^r}$ with $l \in PH^*(G/\mathrm{TOP}; \mathbf{Z}/2)$ in fact belongs to $\tau^*(\mathrm{Sq}^1 PH^*(G/\mathrm{TOP}))$. To this end we shall use the main result of [9]: τ_* maps the elements $u_I \in H_*(G/O; \mathbf{Z}/2)$ to zero and defines a monomorphism from the vector space generated by the $u_{a,b}$ to the indecomposable elements of $H_*(G/\mathrm{TOP}; \mathbf{Z}/2)$.

Now, if $\mathrm{Sq}^1(l)^{2^r}$ evaluates non-zero on $\tau_*(u_{a,b})$ then a is even and $a > b$. If $l_1 \in PH^*(G/\mathrm{TOP}; \mathbf{Z}/2)$ is an element such that $\tau^*(l_1)$ is dual to $u_{a-1,b}$ then $\mathrm{Sq}^1(l)^{2^r} + \tau^*(\mathrm{Sq}^1 l_1)$ annihilates $u_{a,b}$ and evaluates as $\mathrm{Sq}^1(l)^{2^r}$ on the rest of the $u_{i,j}$. This proves the lemma.

In §2 we saw that the double delooping $B^2(G/\mathrm{TOP})$ is 2-locally a product of Eilenberg-MacLane spaces. In 3.3 we reviewed a specific identification K (as H-spaces) of $(G/\mathrm{TOP})_{(2)}$ with a product of Eilenberg-MacLane spaces. The natural question arises if $K \in H^*(G/\mathrm{TOP})$ is in the image of the double suspension

$$\sigma^2: H^*(B^2(G/\mathrm{TOP})) \to H^*(G/\mathrm{TOP}).$$

The $4n-2$ dimensional components of K are primitive classes with $\mathbf{Z}/2$ coefficients and they deloop. The $4n$-dimensional components of K, however, are classes k_{4n} with $\mathbf{Z}_{(2)}$ coefficients and they are not, a priori, in the image of σ^2. We have not been able to decide if K itself is in the image of σ^2, so we leave this as a conjecture.

A cohomology class $\hat{k} \in H^{2n}(B^2(G/\mathrm{TOP}); \Lambda)$ ($\Lambda = \mathbf{Z}/2$ or $\mathbf{Z}_{(2)}$) is called a *fundamental class* provided its value on the spherical $2n$-dimensional homology class is a unit in Λ.

THEOREM 3.6. *There are graded classes*

$$\hat{k}_{4*+2}=\hat{k}_6+\hat{k}_{12}+\cdots\in H^{4*+2}(B^2(G/\mathrm{TOP});\mathbf{Z}_{(2)})$$
$$\hat{k}_{4*}=\hat{k}_4+\hat{k}_8+\cdots\in H^{4*}(B^2(G/\mathrm{TOP});\mathbf{Z}/2)$$

which satisfy

(a) $\hat{k}_{2n}$ *is a fundamental class*

(b) $\sigma^2(\hat{k}_{4n})=k_{4n-2}$

(c) $\sigma^2(\hat{k}_{4n+2})-k_{4n}$ *has order* 2 *and is annihilated by* $\tau^*:H^*(G/\mathrm{TOP};\mathbf{Z}_{(2)})\to$ $\to H^*(G/O;\mathbf{Z}_{(2)})$.

Proof. The double cohomology suspension

$$\sigma^2:QH^*(B^2(G/\mathrm{TOP}))\to PH^*(G/\mathrm{TOP})$$

is an isomorphism with both $\mathbf{Z}/2$ and $\mathbf{Q}$ coefficients. From the previous lemma it follows that there is a fundamental class $\hat{k}_{4n+2}\in H^{4n+2}(B^2(G/\mathrm{TOP});\mathbf{Z}_{(2)})$ such that $\sigma^2(\hat{k}_{4n+2})-k_{4n}$ is a primitive torsion class whose reduction to $\mathbf{Z}/2$ coefficients maps to zero in $H^*(G/O;\mathbf{Z}/2)$. Moreover (2.3)

$$\sigma^2(\hat{k}_{4n+2})-k_{4n}=(\beta_1 y)^{2^a}$$

for some $y\in PH^*(G/\mathrm{TOP};\mathbf{Z}/2)$. We must argue that $\tau^*(\beta_1(y))^{2^a}=0$ in $H^*(G/O;\mathbf{Z}_{(2)})$. The $\mathbf{Z}/2$-reduction of $\tau^*(\beta_1(y))^{2^a}$ is zero (by construction) and since $H^*(G/O;\mathbf{Z}/2)$ is a polynomial algebra $\varrho_1\tau^*\beta_1(y)=0$.

To see that $\tau^*\beta_1(y)$ is itself zero it suffices to check that $\langle\tau^*\beta_1(y),\hat{P}^{(r)}(u)\rangle=0$ for all $u\in H_{ev}(G/O;\mathbf{Z}/2)$ and all $r\geqslant 1$, (3.4). But

$$\varrho_1\beta_{r+1}\hat{P}^{(r)}(u)=\varrho_1\hat{P}^{(r-1)}(u)\cdot\varrho_1\beta_r\hat{P}^{(r-1)}(u)\quad\text{for}\quad r\geqslant 2$$
$$\varrho_1\beta_2\hat{P}(u)=\hat{Q}^{2k}(\varrho_1\beta_1(u))+u\cdot\beta_1(u),$$

where $u\in H_{2k}(G/O;\mathbf{Z}/2)$. Furthermore,

$$\langle\beta_1\tau^*(y),\hat{P}^{(r)}(u)\rangle=\langle\tau^*(y),\varrho_1\beta_{r+1}\hat{P}^{(r)}(u)\rangle\in\mathbf{Z}/2\subset\mathbf{Z}/2^{r+1}.$$

Since $\tau^*(y)$ is primitive $\beta_1\tau^*(y)$ annihilates $\hat{P}^{(r)}(u)$ for $r\geqslant 2$. For $r=1$ we use ([22], §4) that $\hat{Q}^{2k}(u_{a,b})=u_{(2k,a,b)}+$ decomposable terms if $a+b=2k-1$. Now, $\tau_*(u_{(2k,a,b)})=0$ and the result follows. Finally the existence of the classes $\hat{k}_{4n}$ is immediate.

We note that Theorem D of the introduction is an obvious consequence of 3.6 since the image under the suspension map of a fundamental class in $H^*(B^2(G/\mathrm{TOP}))$ is a primitive fundamental class of $H^*(B(G/\mathrm{TOP}))$.

4. The Smooth Surgery Class

In this section we determine the composite

$$G/O \xrightarrow{\tau} G/\mathrm{TOP} \xrightarrow{K} \prod_{n=1}^{\infty} K(\mathbf{Z}_{(2)}, 4n) \times K(\mathbf{Z}/2, 4n-2)$$

where τ is the natural infinite loop map and K is the H-map equivalence of 3.3. At the same time we evaluate the 2-local part of the infinite loop maps

$$B\pi\colon BSG \to B(G/\mathrm{TOP})$$
$$B\tau\colon B(G/O) \to B(G/\mathrm{TOP}).$$

The results of the section are all 2-local and we consequently assume all spaces and maps to be taken in the 2-local category.

We start out by reviewing the basic primitive class $\hat{e}_{4n+1}$ in $PH^{4n+1}(B(G/O); \mathbf{Z}_{(2)})$. A more thorough treatment can be found in [23].

We fix a solution of the Adams conjecture $\alpha\colon BSO \to G/O$, that is, a mapping such that the diagram

$$\begin{tikzcd} & G/O \arrow[d, "i"] \\ BSO \arrow[ur, "\alpha"] \arrow[r, "\psi^3-1"] & BSO \end{tikzcd}$$

is homotopy commutative. Here i is the natural infinite loop map and ψ^3-1 the map which represents ψ^3-1 in 2-local real K-theory. There are at least two natural solutions α available – the one constructed by Sullivan [41] and the one constructed in [8] as an application of the Becker-Gottlieb proof of the Adams conjecture. For our purpose, however, it does not matter which map we pick. The only relevant point is that α is well defined in the rational category. This follows since the fibre of i is the space SG whose rational type is that of a point by a famous theorem of Serre. The map ψ^3-1 is an H-map and a rational equivalence α is consequently an H-equivalence in the rational category.

It is well known that $H^*(BSO; \mathbf{Z}_{(2)})$ only has torsion of order 2 and that

$$H_*(BSO; \mathbf{Z}_{(2)})/\mathrm{Tor} = P\{a_1, a_2, \ldots\},$$

where a_n is dual to the n'th power of the first Pontrjagin class. By a slight abuse of notation we also denote by a_n a lifting to $H^*(BSO; \mathbf{Z}_{(2)})$ of the generators above.

The Adams conjecture along with a simple spectral sequence argument leads to

$$H_*(B(G/O); \mathbf{Q}) = E\{\sigma_*\alpha_*(a_1), \sigma_*\alpha_*(a_2), \ldots\}$$

where $E\{\ \}$ is the exterior algebra.

In the previous section we listed the homology with $\mathbf{Z}/2$ coefficients of G/O. It is a polynomial algebra with generators $u_{a,b}(b \leqslant a \leqslant 2b)$ and $u_I(I \in \mathscr{J})$. The Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{H_*(G/O,\mathbf{Z}/2)}(\mathbf{Z}/2, \mathbf{Z}/2) \Rightarrow H_*(B(G/O); \mathbf{Z}/2)$$

then collapses for trivial reasons. In particular, the indecomposable elements of $H_*(B(G/O); \mathbf{Z}/2)$ are contained among the classes $\sigma_*(u_{a,b})$ and $\sigma_*(u_I)$.

In [23] we found that the space $B(G/O)$ is *Henselian*. A primitive $(4n+1)$-dimensional cohomology class (with $\mathbf{Z}_{(2)}$ coefficients) is consequently determined by its $\mathbf{Q}$ and $\mathbf{Z}/2$ reductions.

The "basic" primitive class $\hat{\varepsilon}_{4n+1} \in PH^{4n+1}(B(G/O); \mathbf{Z}_{(2)})$ is uniquely characterized by

(i) $\varrho_0(\hat{\varepsilon}_{4n+1})$ is dual to $\sigma_*\alpha_*(a_n)$
(ii) $\varrho_1(\hat{\varepsilon}_{4n+1})$ annihilates the elements $\sigma_*(u_I)$ and $\sigma_*(u_{a,b})$ $(a \neq b)$ and evaluates non-zero on $\sigma_*(u_{2n,2n})$. (4.1)

The existence of such a class $\hat{\varepsilon}_{4n+1}$ is not completely obvious. It requires checking that the defining conditions (i) and (ii) in 4.1 are compatible. The argument can be found in [23] and we shall not repeat it here.

The homology suspension from $QH_{2n}(G/O)$ to $QH_{2n+1}(B(G/O))$ is an isomorphism with both $\mathbf{Q}$ and $\mathbf{Z}/2$ coefficients. It follows from this (since $B(G/O)$ is Henselian) that

$$\sigma^*: PH^{2n+1}(B(G/O); \mathbf{Z}_{(2)}) \to PH^{2n}(G/O; \mathbf{Z}_{(2)})$$

is injective. In view of 3.6 it is therefore equivalent to evaluate $\tau: G/O \to G/\mathrm{TOP}$ and $B\tau: B(G/O) \to B(G/\mathrm{TOP})$ in the 2-local category.

Let $\alpha(n)$ be the number of non-zero terms in the dyadic expansion of n, $s_n(p_1, \ldots, p_n) \in H^{4n}(BSO; \mathbf{Z}_{(2)})$ the Newton polynomial in the Pontrjagin classes and $k_{4n} \in H^{4n}(G/\mathrm{TOP}; \mathbf{Z}_{(2)})$ the fundamental class constructed in §3 (3.3).

LEMMA 4.2. *In cohomology with rational coefficients*

$$\alpha^*\tau^*(k_{4n}) = 2^{\alpha(n)-1} u_n \cdot s_n(p_1, \ldots, p_n)$$

where u_n is a unit of $\mathbf{Z}_{(2)}$.

Proof. We consider the exact homotopy sequence of the fibration $\mathrm{PL}/O \to G/O \xrightarrow{\tau} G/\mathrm{PL}$,

$$\cdots \to \pi_{4n}(G/O) \to \pi_{4n}(G/\mathrm{PL}) \xrightarrow{\partial} \pi_{4n-1}(\mathrm{PL}/O) \to \cdots.$$

For $n>1$, $\pi_{4n-1}(\mathrm{PL}/O)$ is the group Γ_{4n-1} of homotopy $4n-1$ spheres. The image of the boundary homomorphism is the subgroup bP_{4n} of homotopy spheres which bound parallelizable manifolds, [16]. The structure of bP_{4n} was determined in [16]; it is cyclic of order Θ_n with

$$\Theta_n = \operatorname{num}(B_n/4n)\, 2^{2n-2}\, a_n (2^{2n-1}-1),$$

where $a_n=1$ for n even, $a_n=2$ for n odd and $\operatorname{num}(B_n/4n)$ is the numerator in the n'th Bernoulli number B_n divided by $4n$ – which is an odd number.

It is a well known consequence of the Pontrjagin character that

$$\langle p_n, h(\iota_{4n})\rangle = a_n(2n-1)!$$

where $p_n \in H^{4n}(BSO; \mathbf{Z})$ is the Pontrjagin class, $\iota_{4n} \in \pi_{4n}(BSO)$ the generator and h the Hurewicz homomorphism. Since the Newton polynomial $s_n(p_1, \ldots, p_n)$ is congruent to np_n modulo decomposable terms

$$\langle s_n(p_1, \ldots, p_n), h(\iota_{4n})\rangle = na_n(2n-1)!$$

Suppose now first that $n>1$. The fundamental class $k_{4n} \in H^{4n}(G/\mathrm{TOP}; \mathbf{Z}_{(2)})$ maps onto a fundamental class of $H^{4n}(G/\mathrm{PL}; \mathbf{Z}_{(2)})$ (cf. §2). On the other hand, $\tau\alpha: BSO \to G/\mathrm{PL}$ is multiplication with Θ_n on homotopy in dimension $4n$ so that

$$\langle \alpha^*\tau^*(k_{4n}), h(\iota_{4n})\rangle = \Theta_n.$$

Since $(2n)! = 2^{2n-\alpha(n)} \cdot u_n$, where u_n is an odd number, we get

$$\alpha^*\tau^*(k_{4n}) = 2^{\alpha(n)-1} u_n s_n(p_1, \ldots, p_n).$$

For $n=1$ we must proceed a little differently. One checks that $H_*(BSO; \mathbf{Z}/2) \simeq H_*(G/O; \mathbf{Z}/2)$ through dimension 5. The orientation map $e: G/O \to BSO$ (Sullivan [41]) splits any solution α, that is, $e \circ \alpha$ is a homotopy equivalence. Thus α induces a monomorphism, hence an isomorphism, on cohomology in dimensions less than 5. It follows that

$$\alpha_*: \pi_4(BSO) \to \pi_4(G/O)$$

is an isomorphism. But, PL/O is 6-connected (Cerf [11]) and $\mathrm{TOP}/\mathrm{PL} = K(\mathbf{Z}/2, 3)$. Therefore we have

$$\pi_4(BSO) \xrightarrow[\simeq]{\alpha_*} \pi_4(G/O) \xrightarrow[\simeq]{\tau_*} \pi_4(G/\mathrm{PL}) \xrightarrow{2\cdot} \pi_4(G/\mathrm{TOP}).$$

The Hurewicz homomorphism for BSO in dimension 4 is multiplication by 2 and we conclude that $\alpha^*\tau^*(k_4) = p_1$. This completes the proof.

In [9] we determined the map $\tau: G/O \rightarrow G/\mathrm{TOP}$ on cohomology with $\mathbf{Z}/2$ coefficients. The result is:

$$\begin{array}{lll}
\tau^*(\varrho_1(k_{4n}))=0 & & \text{if } n\neq 2^i \\
\tau^*(\varrho_1(k_{4n})) & \text{is dual to } u_{2n,2n} & \text{if } n=2^i \\
\tau^*(k_{4n-2})=0 & & \text{if } n\neq 2^i \\
\tau^*(k_{4n-2}) & \text{is dual to } u_{2n-1,2n-1} & \text{if } n=2^i.
\end{array}$$

Here dual means dual with respect to the basis $\{u_{a,b}, u_I\}$ of $QH_*(G/O; \mathbf{Z}/2)$.

Remark 4.4. The result for $\tau^*(k_{4n-2})$ $(n=2^i)$ was formulated somewhat differently in [9]. There we proved ([9], 3.6)

$$\begin{array}{ll}
\langle \tau^*(k_{4n-2}), j_*(e_a \underline{*} e_b)\rangle \neq 0 & (a+b=4n-2, n=2^i) \\
\langle \tau^*(k_{4n-2}), j_*(e_{a_1} \underline{*} \cdots \underline{*} e_{a_k})\rangle = 0 & \text{for } \quad k>2,
\end{array} \qquad (*)$$

where $j: SG \rightarrow G/O$ is the natural map, e_a the unique class of degree a in the image of $RP^\infty \rightarrow SO \rightarrow SG$ and where $\underline{*}$ denotes the loop product in $H_*(SG; \mathbf{Z}/2)$.

Now, $e_a = Q^a[1]*[-1]$ where Q^a denotes the homology operation in $\Omega^\infty S^\infty$ $(SG \subset \Omega^\infty S^\infty)$ associated with the loop structure and $u_{a,b} = j_*(Q^a Q^b [1]*[-3])$. To get from $(*)$ above to 4.3 it suffices to prove in $QH_*(SG; \mathbf{Z}/2)$,

(i) $\langle \tau^*(k_{4n-2}), u_{2n-1,2n-1}\rangle \neq 0$ when $n=2^i$

(ii) $e_a \underline{*} e_b = u_{2n-1,2n-1} +$ other terms when $a+b=4n-2$ and $n=2^i$.

(iii) $e_{a_1} \underline{*} \cdots \underline{*} e_{a_k}$ is a linear combination of the u_I, $I \in \mathscr{J}$ when $k>2$.

The statements (i), (ii) and (iii) are consequences of the various formulas in $H_*(Q(S^0); \mathbf{Z}/2)$ relating the loop structure and the composition structure (see e.g. [22], §§ 3 and 4). We leave this unilluminating and tedious computation to the reader.

Let $\hat{k}_{4n+2} \in H^{4n+2}(B^2(B/\mathrm{TOP}); \mathbf{Z}_{(2)})$ be a fundamental class satisfying (a) of 3.6. The cohomology suspension maps $\hat{k}_{4n+2}$ to a primitive fundamental class $\hat{k}_{4n+1}$ in $H^{4n+1}(B(G/\mathrm{TOP}); \mathbf{Z}_{(2)})$ whose image in $H^{4n+1}(B(G/O); \mathbf{Z}_{(2)})$ is unambiguously determined (compare 3.6 or Theorem D in §1).

THEOREM 4.5. *The natural map* $B\tau: B(G/O) \rightarrow B(G/\mathrm{TOP})$ *is given as*

$$(B\tau)^*(\hat{k}_{4n+1}) = 2^{\alpha(n)-1} u_n \hat{\varepsilon}_{4n+1}$$

where $\hat{\varepsilon}_{4n+1}$ *is the class defined in* 4.1 *and* u_n *is a unit of* $\mathbf{Z}_{(2)}$.

Proof. According to 4.1 (i) and 4.2 the rational reduction of both sides agree. Since $B(G/O)$ is Henselian a $4n+1$ dimensional primitive cohomology class is determined by its rational reduction and its reduction to $\mathbf{Z}/2$ coefficients. Now,

$$\sigma^*: PH^{2n+1}(B(G/O); \mathbf{Z}/2) \rightarrow PH^{2n}(G/O; \mathbf{Z}/2)$$

is injective (in fact an isomorphism). To complete the proof we need to show that

$$\varrho_1\tau^*(\sigma^*(\hat{k}_{4n+1}))=2^{\alpha(n)-1}\varrho_1\sigma^*(\hat{\varepsilon}_{4n+1}).$$

But this is a consequence of 3.6 (c) and 4.3.

We get as an immediate corollary Theorem B of the introduction.

COROLLARY 4.6. *The composite*

$$G/O\xrightarrow{\tau}G/\mathrm{TOP}\xrightarrow{k_{4n}}K(\mathbf{Z}_{(2)},4n)$$

defines the cohomology class $2^{\alpha(n)-1}u_n\sigma^*(\hat{\varepsilon}_{4n+1})$.

We conclude this section by transferring the results above to an evaluation (2-locally) of the map $B\pi:BSG\to B(G/\mathrm{TOP})$. First recall that the Stiefel-Whitney classes are universally defined as classes of $H^*(BSG;\mathbf{Z}/2)$. The natural map $BSO\to BSG$ therefore induces a surjection in mod. 2 cohomology and

$$H^*(BSG;\mathbf{Z}/2)\simeq H^*(BSO;\mathbf{Z}/2)\otimes H^*(B(G/O);\mathbf{Z}/2).$$

The higher torsion structure of BSG and of the map $i:BSG\to B(G/O)$ was examined in [23]. We give a brief review of the results. The "mod. 2 Pontrjagin classes" $w_{2n}^2\in H^{4n}(BSG;\mathbf{Z}/2)$ lift to classes $p_n\in H^{4n}(BSG;\mathbf{Z}/8)$ and *not* to $H^{4n}(BSG;\mathbf{Z}/16)$. Indeed, in the E_3-term of the Bockstein spectral sequence for BSG,

$$d_3(w_{2n}^2)=e_{4n+1},$$

where $e_{4n+1}=\sum w_{2n-2k}^2 i^*(\hat{\varepsilon}_{4k+1})$. The primitive element $i^*(\hat{\varepsilon}_{4k+1})$ survives to the $E_{3+\nu(k)}$-term ($k=2^{\nu(k)}\cdot$odd) of the Bockstein spectral sequence where it becomes a boundary of the Newton polynomial in the classes $w_2^2, w_4^2, \ldots$. It follows that $i^*(\hat{\varepsilon}_{4k+1})$ is a torsion element of order $2^{\nu(k)+3}$ in $H^{4k+1}(BSG;\mathbf{Z}_{(2)})$.

We finally recall from [23] the behaviour of the cohomology suspension. The sequence

$$0\to\mathbf{Z}/2^{\nu(n)+1}\to PH^{4n+1}(BSG;\mathbf{Z}_{(2)})\xrightarrow{\sigma^*}PH^{4n}(SG;\mathbf{Z}_{(2)})$$

is exact where the cyclic summand is generated by $4i^*(\hat{\varepsilon}_{4n+1})$.

COROLLARY 4.7. *The natural map* $BSG\xrightarrow{B\pi}B(G/\mathrm{TOP})$ *maps* $\hat{k}_{4n+1}$ *to a class of order* $2^{\nu(n)-\alpha(n)+4}$.

5. Topological Reductions of Spherical Fibrations

Stable spherical fibrations, that is, fibre spaces whose fibres are homotopy spheres of high dimension compared with the base space, are classified by BSG. Since the

$\pi_i(BSG)$ are finite for all i the homotopy set $[X, BSG]$ is a finite abelian group when X is a finite complex. In geometric terms, a spherical fibration ξ splits in a sum of its p-primary parts, $\xi=\oplus\xi_{(p)}$ where $p^a\xi_{(p)}$ is trivial for a sufficiently high power of p. On the classifying space level we get

$$BSG \simeq \prod_{p \text{ prime}} BSG_{(p)}$$

where $[X, BSG_{(p)}]=[X, BSG]\otimes\mathbf{Z}_{(p)}$ and $\mathbf{Z}_{(p)}$ denotes the integers localized at p, $\mathbf{Z}_{(p)}=\{r/s\in\mathbf{Q} \mid (s,p)=1\}$.

The question of reducing a spherical fibration to a honest (topological) sphere bundle splits accordingly in its p-primary parts. At odd primes the reduction problem has been extensively explored by Sullivan [41].

Consider (away from the prime 2) the orientation sequence

$$SG \xrightarrow{e} BO^{\otimes} \to BKOG \to BSG \tag{$*$}$$

where $BKOG$ is the classifying space for odd-local spherical fibrations with a $KO(\ )\otimes\mathbf{Z}[\frac{1}{2}]$ orientation and $BO^{\otimes}$ denotes the infinite loop space whose underlying H-structure is induced from tensor product of vector bundles of virtual dimension 1. The sequence $(*)$ can be identified (in the world of odd primes) with the natural sequence

$$SG \to G/\mathrm{TOP} \to B\mathrm{STOP} \to BSG.$$

Thus one gets

THEOREM 5.1. (Sullivan). *An odd-primary stable spherical fibration admits a topological* (PL) *reduction if and only if it is orientable with respect to* $KO(\ )\otimes\mathbf{Z}[\frac{1}{2}]$.

Recently in [28] it has been proved that $(*)$ can be continued to the right as a fibration sequence of infinite loop spaces. In particular we have the fibration

$$BKOG \to BSG \xrightarrow{Be} B(BO^{\otimes}).$$

On the other hand Adams and Priddy [2] have proved that (at each prime separately) there is only one infinite loop space structure on the space BSO. Therefore, at an odd prime p, $B(BO^{\otimes})_{(p)}=B^2O_{(p)}$.

COROLLARY 5.2. *Let ξ be a stable p-primary spherical fibration (p an odd prime) classified by a map $X\to BSG$. Then ξ has a topological* (*hence* PL) *reduction if and only if the composite $X\to BSG\xrightarrow{Be}B^2O$ represents zero in $KO^{-1}(X)$.*

Next, we consider a 2-primary stable spherical fibration ξ over X. The natural fibration

$$B\mathrm{STOP} \to BSG \to B(G/\mathrm{TOP})$$

along with our 2-local splitting results for $B(G/\mathrm{TOP})$ show that the obstruction to reducing ξ to a topological bundle is a graded cohomology class

$$\sigma(\xi)=\sum \sigma_{4n+1}(\xi)+\sum \sigma_{4n-1}(\xi), \tag{5.3}$$

where $\sigma_{4n+1}(\xi)\in H^{4n+1}(X;\mathbf{Z}_{(2)})$ and $\sigma_{4n-1}(\xi)\in H^{4n-1}(X;\mathbf{Z}/2)$.

More precisely, we find from 4.5 and the discussion preceding 4.7.

THEOREM 5.4. *The 2-primary obstruction* $\sigma(\xi)$ *satisfies*

(i) $\sigma_{4n-1}(\xi)=0$ *unless n is a power of* 2

(ii) $\sigma_{4n+1}(\xi)=2^{\alpha(n)-1}\cdot\varepsilon_{4n+1}(\xi)$,

where $\varepsilon_{4n+1}(\xi)$ *is a characteristic class of order at most* $2^{\nu(n)+3}$. *Moreover, in the* $E_{3+\nu(n)}$*-term of the Bockstein spectral sequence of* X,

$$d_{3+\nu(n)}(s_n(w_2(\xi)^2,\dots,w_{2n}(\xi)^2))=\varepsilon_{4n+1}(\xi)$$

where $w_{2i}(\xi)$ *is the* $2i$'th *Stiefel-Whitney class and* s_n *the Newton polynomial.*

Remark. There is a curious difference between 5.4 and recent results of Brumfiel and Morgan [10]. At the prime 2 they construct a fibration (see also [15] and [36])

$$B\mathrm{STOP}\to BSG\xrightarrow{t}\prod K(\mathbf{Z}_{(2)},4n+1)\times\prod K(\mathbf{Z}/2,4n-1)$$

based on the transversality obstruction in the Poincaré duality category. This leads to an obstruction class

$$t(\xi)=\sum t_{4n-1}(\xi)+\sum t_{4n-1}(\xi)$$

to topological reduction. The class $t_{4n+1}(\xi)$ has order 8 whereas our class $\sigma_{4n+1}(\xi)$ has order $2^{\nu(n)-\alpha(n)+4}$. The explanation seems to be that $\varepsilon_{4n+1}(\xi)$ is an additive characteristic class, in fact, a higher-order Bockstein applied to a "Newton type" polynomial in the mod. 8 Pontrjagin classes of ξ whereas $t_{4n+1}(\xi)$ is a third-order Bockstein in a "Hirzebruch type" polynomial in the mod. 8 Pontrjagin classes. The relationship between $\sigma_{4n+1}(\xi)$ and $t_{4n+1}(\xi)$ is, however, not fully understood at present.

The $4n-1$ dimensional components of $t(\xi)$ and $\sigma(\xi)$ are related by

$$t_{4_*-1}(\xi)=V(\xi)^2\cdot\sigma_{4_*-1}(\xi)$$

where $V(\xi)$ is the total Wu class.

Let X be a simply connected Poincaré duality space of dimension $n\geqslant 5$ and let ξ denote its Spivak normal fibration. Topological (or PL) reductions of ξ and (homotopy) manifold structures on X correspond via the theory of simply connected surgery. In particular, we have the following well known consequences of the plumbing theorem ([7]).

THEOREM 5.5. *There is a topological* (PL) *closed n-manifold in the homotopy type of* X *if and only if* ξ *admits a topological* (PL) *reduction.*

When $2^{\alpha(n)-1}H^*(X; \mathbf{Z}_{(2)})$ is torsion-free then the obstructions $\sigma_{4n+1}(\xi)$ vanish and X has a PL-manifold structure if and only if $\sigma_{2^i-1}(\xi)=0$ and ξ is $KO(\ )\otimes\mathbf{Z}[\frac{1}{2}]$ orientable.

We conclude this section with a discussion of the obstruction $\sigma_{2^i-1}(\xi)\in H^*(X; \mathbf{Z}/2)$. Let U be the Thom class in $H^k(T(\xi); \mathbf{Z}/2)$ and let $\psi_{i,i}$ be the secondary operation associated with the relation

$$\mathrm{Sq}^{2^{i-1}}\,\mathrm{Sq}^{2^{i-1}}+\sum_{j=1}^{i-2}\mathrm{Sq}^{2^i-2^j}\,\mathrm{Sq}^{2^j}=0.$$

If the Stiefel-Whitney classes of ξ all vanish then $\psi_{i,i}(U)$ is defined with zero indeterminacy (since $\mathrm{Sq}^{2^i-2^j}(xU)=\mathrm{Sq}^{2^i-2^j}(x)\,U$ and $\mathrm{Sq}^{2^i-2^j}(x)=0$ when $x\in H^{2^j-1}(X;\mathbf{Z}/2)$). We let $\tau_{2^i-1}(\xi)\in H^{2^i-1}(X; \mathbf{Z}/2)$ be the associated characteristic class,

$$\tau_{2^i-1}(\xi)\cdot U=\psi_{i,i}(U).$$

$\tau_{2^i-1}(\xi)$ is an additive characteristic class on spherical fibrations with vanishing Stiefel-Whitney classes, as we see from the Cartian formula

$$\psi_{i,i}(U_\xi\otimes U_\eta)=\psi_{i,i}(U_\xi)\otimes U_\eta+U_\xi\otimes\psi_{i,i}(U_\eta)$$

where U_ξ and U_η are the relevant Thom classes.

In fact we have the following (unpublished) result of Mahowald

THEOREM 5.6 (Mahowald). *The class* $\tau_{2^i-1}(\xi)$ *agrees with* $\sigma_{2^i-1}(\xi)$ *on spherical fibrations with vanishing Stiefel-Whitney classes.* (For a proof see [37]).

We return to the situation where X is a Poincaré duality space with normal fibration ξ. Suppose that X has vanishing Stiefel-Whitney classes and that $\psi_{i,i}$ is defined on all of $H^{n-2^i+1}(X; \mathbf{Z}/2)$ ($n=\dim X$).

COROLLARY 5.7. *With the above assumptions* $\sigma_{2^i-1}(\xi)$ *is the secondary Wu class of* $\psi_{i,i}$,

$$\langle\sigma_{2^i-1}(\xi)\cup x, [X]\rangle=\langle\psi_{i,i}(x), [X]\rangle.$$

Proof. Let $U\in H^k(T(\xi); \mathbf{Z}/2)$ be the Thom class. The Cartian formula for $\psi_{i,i}$ along with 5.5 gives

$$\psi_{i,i}(xU)=\psi_{i,i}(x)\,U+(x\cup\sigma_{2^i-1}(\xi))\,U.$$

But, the top class of $H^*(T(\xi); \mathbf{Z}/2)$ is spherical so that $\psi_{i,i}(xU)=0$.

REFERENCES

[1] ADAMS, J. F., *On the non-existence of elements of Hopf invariant one*, Ann. of Math., *72* (1960), 20–104.

[2] ADAMS, J. F. and PRIDDY, S., (to appear).

[3] BECKER, J. and GOTTLIEB, D., *The transfer map and fibre bundles* (to appear).

[4] BOARDMAN, J. M. and VOGT, R., *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics *347*, Springer Verlag 1973.

[5] BROWDER, W., *Torsion in H-spaces*, Ann. of Math. *74* (1961), 24–51.

[6] ——, *On differential Hopf algebras*, Trans. Amer. Math. Soc. *107* (1963).

[7] ——, *Surgery on simply connected manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band *65*, Springer Verlag, Berlin 1972.

[8] BRUMFIEL, G. and MADSEN, I., (to appear).

[9] BRUMFIEL, G., MADSEN, I., and MILGRAM, R. J., *PL-characteristic classes and cobordism*, Ann. of Math., *97* (1973), 82–159.

[10] BRUMFIEL, G. and MORGAN, J., *Homotopy theoretic consequences of N. Levitt's obstruction theory to transversality for spherical fibrations.*

[11] CERF, J., *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Mathematics *53*, Springer Verlag, 1968.

[12] CLARK, A., *Homotopy commutativity and the Moore spectral sequence*, Pacific J. Math. *15* (1965), 65–74.

[13] DYER, E. and LASHOF, R., *Homology of iterated loop spaces*, Amer. J. Math. *84* (1962), 35–88.

[14] GUGENHEIM, V. K. A. M. and MAY, J. P., *On the theory and applications of differential torsion products*, Mem. Amer. Math. Soc. *142* (1974).

[15] JONES, L., *Patch Spaces: A geometric representation for Poincaré spaces*, Ann. of Math. *97* (1973), 276–306.

[16] KERVAIRE, M. and MILNOR, J., *On the groups of homotopy spheres*, Ann. of Math. *77* (1963), 504–537.

[17] KIRBY, R. and SIEBENMANN, L., *Some theorems on topological manifolds*, Manifolds, Amsterdam 1970, Lecture Notes in Mathematics *197*, Springer Verlag, 1971.

[18] KRISTENSEN, L., *On the cohomology of spaces with two non-vanishing homotopy groups*, Math. Scand. *12* (1963), 83–105.

[19] LEVITT, N., *Generalized Thom spectra and transversality for spherical fibrations*, Bull. Amer. Math. Soc. *76* (1970), 727–731.

[20] LEVITT, N. and MORGAN, J., *Transversality structures and PL structures on spherical fibrations*, Bull. Amer. Math. Soc. *78* (1972), 1064–1068.

[21] LIGAARD, H. and MADSEN, I., *Homology operations in the Eilenberg-Moore spectral sequence*, Preprint Series, Aarhus University, 1974.

[22] MADSEN, I., *On the action of the Dyer-Lashof algebra in* $H_*(G)$, Pacific J. Math. (to appear).

[23] ——, *Higher torsion in SG and BSG*, Preprint Series, Aarhus University.

[24] ——, *Homology operations in* G/TOP (to appear).

[25] MADSEN, I. and MILGRAM, R. J., *On spherical fibre bundles and their PL reduction*, Recent developments in topology, Oxford 1973.

[26] ——, *The oriented topological and PL cobordism groups*, Bull. Amer. Math. Soc. (to appear).

[27] MAY, J. P., *The algebraic Eilenberg-Moore spectral sequence*, Preprint, University of Chicago.

[28] MAY, J. P., QUINN, F., and RAY, N., *E_∞-Ring spectra* (to appear).

[29] MILGRAM, R. J., *The bar construction and abelian H-spaces*, Ill. J. Math. *11* (1967), 242–250.

[30] ——, *Steenrod squares and higher Massey products*, Bull. Soc. Math. Mex. (1968), 32–51.

[31] ——, *The mod2 spherical characteristic classes*, Ann. of Math. *92* (1970), 238–261.

[32] ——, *Surgery with coefficients*, Ann. of Math., *100* (1974), 194–248.

[33] ——, *The structure over the Steenrod algebra of some 2-stage Postnikov systems*, Quart. J. Math., Oxford *20* (1962), 161–169.

[34] MILNOR, J. and MOORE, J., *On the structure of Hopf algebras*, Ann. of Math. *81* (1965), 211–264.

[35] MORGAN, J. and SULLIVAN, D., *The transversalitycharacteristic class and linking cycles in surgery theory*, Ann. of Math. *99* (1974), 384–463.

[36] QUINN, F., *Surgery on Poincaré and normal spaces*, (to appear).

[37] RAVENEL, D. C., *A definition of exotic characteristic classes of spherical fibrations*, Comment. Math. Helv. *47* (1972), 421–436.

[38] ROURKE, C. and SULLIVAN, D., *On the Kervaire obstruction*, Ann. of Math. *94* (1971), 397–413.

[39] SEGAL, G., *Classifying spaces and spectral sequences*, Publ. Math. I.H.E.S., *34* (1968), 105–112.

[40] SULLIVAN, D., *Geometric topology*, Seminar notes, Princeton University (1967).

[41] ——, *Geometric topology, part I: Localization, Periodicity and Galois symmetry*, M.I.T. (1970).

[42] THOMAS, E., *The generalized Pontrjagin cohomology operations and rings with divided powers*, Mem. Amer. Math. Soc. *25* (1957).

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