# A cohomology for foliated manifolds.

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# A cohomology for foliated manifolds

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## 0. Introduction

In this note we construct a resolution of a sheaf $\Theta$ naturally associated to a foliated manifold $M$. If $F$ is a foliation on $M$, then $\Theta$ is the sheaf over $M$ of germs of non-trivial infinitesimal transformations of $F$. A sheaf quite similar to $\Theta$ was studied by Kodaira and Spencer in their fundamental paper [7], which served as an inspiration and guide for much of what appears here. Where possible we have pointed out the similarities and differences of the two papers.

The cohomology groups $H^*(M; \Theta)$ have been studied by various authors and are of great importance in the theory of deformations of foliations and their associated characteristic classes. Besides [7], these cohomology groups occur in the work of Gel'fand-Fuchs [5] and Kamber-Tondeur [6]. These groups are a special case of a theory of Nijenhuis [14] and we note that this resolution has also been discovered by Vaisman [16].

The resolution presented here is useful in the following construction. The details will appear in [8]. An element of $H^1(M; \Theta)$ is an infinitesimal deformation of the foliation $F$. Denote the tangent bundle of the foliation by $\tau$ and its normal bundle by $\nu$. An arbitrary section of the bundle $\tau^* \otimes \nu$ is an infinitesimal deformation of the plane field $\tau$. The resolution of $\Theta$ allows us to represent any $\alpha \in H^1(M; \Theta)$ by a global section of $\tau^* \otimes \nu$ satisfying certain auxiliary conditions. In [3], Bott and Haefliger construct a natural map, depending only on the foliation $F$, from the cohomology of formal vector fields an $\mathbb{R}^q$ ($q$ is the codimension of $F$) relative to $O_q$, denoted $H^*(WO_q)$ to the real cohomology of $M$, $H^*(M; \mathbb{R})$. In [8], we will show how to construct a pairing

$$H^1(M; \Theta) \times H^*(WO_q) \to H^*(M; \mathbb{R}),$$

naturally associated to the Bott-Haefliger construction. For each $\alpha \in H^1(M; \Theta)$ the induced map $H^*(WO_q) \to H^*(M; \mathbb{R})$ may be viewed as the derivative of the Bott-Haefliger map in the direction of $\alpha$. In this interpretation we are thinking of $\alpha$ as a tangent vector to the point $F$ in the space of foliations on $M$.

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In addition to the above, the resolution has the advantages of simplicity, a very strong intuitive appeal, and computability. In particular, $H^1(M; \Theta)$ can be used to distinguish between the rational and irrational constant slope foliations of the two torus.

The note is divided into four sections. In Section 1 we collect the relevant facts about foliations, flat connections and $\Gamma$ vector fields. Section 2 contains the construction of the resolution. In Section 3 we compute $H^*(T^2; \Theta)$ for the constant slope foliations of the 2 torus. Section 4 discusses the resolution restricted to a leaf.

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## 1. Foliations and $\Gamma$ vector fields

We will consider smooth foliations on real manifolds and complex foliations on complex manifolds. In this section we treat these two cases separately. Most of the material is well known. For specific information, the unfamiliar reader should consult the references given or [10] for an excellent survey of foliations.

*The real case.* All objects are assumed to be smooth, that is $C^\infty$. Let $M$ be a real connected manifold of dimension $n$, $TM$ its tangent bundle and $F$ a foliation of codimension $q$ on $M$. $F$ is given by an open cover of $M$ by coordinate charts $\{U_\alpha\}$ with local coordinates $x_1^\alpha, \ldots, x_n^\alpha$ satisfying

$$\frac{\partial x_i^\alpha}{\partial x_j^\beta} = 0 \quad \text{on} \quad U_\alpha \cap U_\beta \quad \text{for} \quad 1 \leqslant j \leqslant n-q < i \leqslant n.$$

We call such an atlas an *F atlas*, and each $U_\alpha$ is called an *F chart*. The tangent bundle to $F$ is denoted by $\tau$ and $\tau|_{U_\alpha}$ is spanned by $\partial/\partial x_1^\alpha, \ldots, \partial/\partial x_{n-q}^\alpha$. $\tau$ is an involutive sub bundle of $TM$.

The classical Frobenius theorem implies that through each point $p \in M$ there passes a unique maximal connected differentiable sub-manifold $N$ of dimension $n-q$, such that at each point $q \in N$ the tangent bundle of $N$ at $q$, $TN_q$ is the subspace of $TM_q$ tangent to the foliation, i.e. $TN_q = \tau_q$ all $q \in N$. Such a submanifold is called a *leaf* of the foliation.

The quotient bundle $TM/\tau$ is denoted by $\nu$ and is called the normal bundle of $F$. If $Y \in TM$ its equivalence class in $\nu$ is given by $\langle Y \rangle$.

If $E$ is any vector bundle over $M$ we denote the vector space of $C^\infty$ sections of $E$ by $C^\infty(E)$. The space of $C^\infty$ functions on $M$ is denoted by $C^\infty(M)$. A connection on a vector bundle $E$ is a rule $\nabla$ which assigns to each vector field $X \in C^\infty(TM)$ a linear operator

$$\nabla_X : C^\infty(E) \to C^\infty(E)$$

satisfying

(i) $\nabla_X(f\cdot\sigma)=(Xf)\,\sigma+f\,\nabla_X\sigma \quad \sigma\in C^\infty(E), f\in C^\infty(M)$

(ii) $\nabla_{fX+Y}\sigma=f\,\nabla_X\sigma+\nabla_Y\sigma \quad f\in C^\infty(M)$.

If $\nabla$ is a connection on $E$, the curvature $K$ of $\nabla$ assigns to each pair of vector fields $X$, $Y$ the linear operator

$$K(X, Y): C^\infty(E)\to C^\infty(E)$$

given by $K(X, Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$.

THEOREM 1.1. (Bott [2]). *There is a connection* $\nabla$ *on* $v$ *such that if* $X\in C^\infty(\tau)$ *and* $\langle Y\rangle\in C^\infty(v)$ *then*

$$\nabla_X\langle Y\rangle=\langle[X, Y]\rangle.$$

Such a connection is called *basic.*

COROLLARY. *If* $K$ *is the curvature of a basic connection on* $v$ *and* $X$, $Y\in C^\infty(\tau)$, *then* $K(X, Y)\equiv 0$.

This corollary follows directly from the Jacobi identity for the Lie bracket of vector fields and the fact that if $X$, $Y\in C^\infty(\tau)$, $[X, Y]\in C^\infty(\tau)$. The corollary is the basis of the Bott vanishing theorem [2] for the rational characteristic classes of the normal bundle of a real smooth foliation.

Let $\{U_\alpha\}$ be an $F$ atlas.

DEFINITION 1.2. A *$\Gamma$ vector field* on $M$ is an element $\langle Y\rangle\in C^\infty(v)$ such that if

$$Y\,|_{U_\alpha}=\sum_{i=1}^{n} Y_i^\alpha \frac{\partial}{\partial x_i^\alpha}$$

then

$$\frac{\partial Y_i^\alpha}{\partial x_j^\alpha}=0 \quad \text{for} \quad 1\leqslant j\leqslant n-q<i\leqslant n.$$

Note that the space of $\Gamma$ vector fields is the set of projectable vector fields, modulo vector fields tangent to $F$. We denote by $\Theta_{\mathbb{R}}$ the sheaf of germs of local $\Gamma$ vector fields.

A $\Gamma$ vector field is characterized by the fact that if $\varphi_t$, $t\in(-\varepsilon, \varepsilon)\subseteq\mathbb{R}$ is the local 1-parameter family of diffeomorphisms generated by $Y$, then for each $t\in(-\varepsilon, \varepsilon)$ $\varphi_t$ maps leaves of $F$ onto leaves of $F$. This can be seen most easily by noting that on each $U_\alpha$ we have a projection $\pi^\alpha$ onto $\mathbb{R}^q$ given by the coordinates $x_{n-q+1}^\alpha, \ldots, x_n^\alpha$. If $N$ is a leaf of $\tau$ then $\pi^\alpha$ maps each component of $N\cap U_\alpha$ to a distinct point of $\mathbb{R}^q$. A normal vector field $Y$ is a $\Gamma$ vector field provided that for each $U_\alpha$, $\pi^\alpha_*(Y|_{U_\alpha})$ is a well defined vector field on $\mathbb{R}^q$.

*The complex case.* Let $M$ be a connected complex analytic manifold of complex dimension $n$, $T_{\mathbb{C}}M = TM \oplus \bar{T}M$ the standard splitting of the complexified tangent bundle of $M$. $T^*_{\mathbb{C}}M = T^*M \oplus \bar{T}^*M$ the splitting of the complexified cotangent bundle. $TM$ is the holomorphic tangent bundle of $M$. $\bar{T}M$ is the antiholomorphic tangent bundle of $M$. An element of $C^\infty(T^*M)$ is a one form of type (1, 0). An element of $C^\infty(\bar{T}^*M)$ is a one form of type (0, 1). Denote by $C^\infty(M)$ the space of $C^\infty$ complex functions on $M$.

A complex analytic foliation $F$ on $M$ of complex codimension $q$ is given by an open cover of $M$ by coordinate charts $\{U_\alpha\}$ with local holomorphic coordinate functions $z^\alpha_1, \ldots, z^\alpha_n$ such that

$$\frac{\partial z^\alpha_i}{\partial z^\beta_j} = 0 \quad \text{on} \quad U_\alpha \cap U_\beta \qquad 1 \leqslant j \leqslant n-q < i \leqslant n.$$

As above we call $\{U_\alpha\}$ an $F$ atlas and the $U_\alpha$ $F$ charts. The tangent bundle of $F$ is denoted by $\tau$ and $\tau|_{U_\alpha}$ is spanned by $\partial/\partial z^\alpha_1, \ldots, \partial/\partial z^\alpha_{n-q}$. $\tau$ is an involutive holomorphic subbundle of $TM$ and the complex Frobenius theorem gives the existence of maximal integral complex submanifolds of complex dimension $n-q$ through each point of $M$. The quotient bundle $TM/\tau$ is again denoted by $\nu$. If $Y \in C^\infty(TM)$ its equivalence class in $\nu$ is given by $\langle Y \rangle$.

As $\nu$ is a holomorphic bundle over $M$ there is the $\bar{\partial}$ operator $\bar{\partial}: C^\infty(\nu) \to \to C^\infty(\bar{T}^* \otimes \nu)$. $\bar{\partial}$ is a local operator and if

$$\sigma \in C^\infty(\nu|_{U_\alpha}), \qquad \sigma = \sum_{j=n-q+1}^{n} f_j \langle \partial/\partial z^\alpha_j \rangle$$

then

$$\bar{\partial}\sigma = \sum_{j=n-q+1}^{n} \bar{\partial} f_j \otimes \langle \partial/\partial z^\alpha_j \rangle .$$

Thus if $\sigma \in C^\infty(\nu|_{U_\alpha})$, is holomorphic if and only if $\bar{\partial}\sigma = 0$.

A connection of type (1, 0) on a holomorphic vector bundle $E$ over $M$ is a rule $\nabla$ which assigns to each $X \in C^\infty(T_{\mathbb{C}}M)$ a linear operator

$$\nabla_X : C^\infty(E) \to C^\infty(E)$$

satisfying

(i) $\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X\sigma \qquad \sigma \in C^\infty(E), f \in C^\infty(M)$

(ii) $\nabla_{fX+Y}\sigma = f\nabla_X\sigma + \nabla_Y\sigma \qquad f \in C^\infty(M)$

(iii) If $X \in C^\infty(\bar{T}M)$ then

$$\nabla_X\sigma = (\bar{\partial}\sigma)(X).$$

The curvature of $\nabla$ is denoted by $K$ and for $X, Y \in C^\infty(T_{\mathbb{C}}M)$ $K(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

THEOREM 1.3. (Bott [1]). *There is a connection* $\nabla$ *on* $v$ *of type* $(1, 0)$ *such that if* $\langle Y\rangle \in C^\infty(v)$ *and* $X \in C^\infty(\tau)$ *then* $\nabla_X\langle Y\rangle = \langle [X, Y]\rangle$.

Such a connection is called *basic.*

COROLLARY 1.4. *If* $K$ *is the curvature of a basic connection on* $v$ *then* $K(X, Y) = 0$ *for* $X, Y \in C^\infty(\tau \oplus \bar{T}M)$.

This implies the Bott vanishing theorem [1] for the rational characteristic classes of the normal bundle of a complex foliation.

Let $\{U_\alpha\}$ be an $F$ atlas for the complex foliation on $M$.

DEFINITION 1.5. A *$\Gamma$ vector field* on $M$ is a vector field $\langle Y\rangle \in C^\infty(v)$ such that if

$$Y\,|_{U_\alpha} = Z_\alpha + \sum_{i=k}^{n} Y_i^\alpha \frac{\partial}{\partial z_i^\alpha}, \qquad k = n - q + 1,$$

where $Z_\alpha \in C^\infty(\tau \oplus \bar{T}M)$, then the $Y_i^\alpha$ are holomorphic and

$$\frac{\partial Y_i^\alpha}{\partial z_j^\alpha} \equiv 0 \quad \text{for} \quad 1 \leqslant j \leqslant n - q < i \leqslant n.$$

The space of complex $\Gamma$ vector fields is the set of $C^\infty$ vector fields on $M$ which are projectable with holomorphic projection, modulo $C^\infty$ vector fields tangent to $F$. If $\langle Y\rangle$ is a $\Gamma$ vector field then the associated real part of $Y$ preserves the foliation in the sense that the local diffeomorphism it generates maps leaves onto leaves. We denote by $\Theta_{\mathbb{C}}$ the sheaf of germs of local $\Gamma$ vector fields.

In [7], Kodaira and Spencer consider sheaves also called sheaves of $\Gamma$ vector fields. Their sheaves are analogous to ours but they are not identical. If $U_\alpha$ is an $F$ chart for a real or complex foliation, a Kodaira-Spencer $\Gamma$ vector field is a smooth or holomorphic vector field $Y$ such that if

$$Y\,|_{U_\alpha} = \sum_{i=1}^{n} Y_i^\alpha \frac{\partial}{\partial w_i^\alpha}$$

then

$$\frac{\partial Y_i^\alpha}{\partial w_j^\alpha} = 0 \quad \text{for} \quad 1 \leqslant j \leqslant n - q < i \leqslant n.$$

For a real foliation $w_j^\alpha = x_j^\alpha$ and for a complex foliation $w_j^\alpha = z_j^\alpha$. We denote the sheaf of germs of local Kodaira-Spencer $\Gamma$ vector fields by $\hat{\Theta}_{\mathbb{R}}$ for a real foliation and $\hat{\Theta}_{\mathbb{C}}$ for a complex foliation. For a real, (respectively complex), foliation denote by $\bar{\Theta}_{\mathbb{R}}$, ($\bar{\Theta}_{\mathbb{C}}$), the sheaf of germs of local smooth, (holomorphic) sections of the tangent bundle to the foliation. Note that $\bar{\Theta}_{\mathbb{R}}$ is a fine sheaf. The various sheaves are related by the exact sequences.

$$0 \to \bar{\Theta}_{\mathbb{R}} \to \hat{\Theta}_{\mathbb{R}} \to \Theta_{\mathbb{R}} \to 0$$
$$0 \to \bar{\Theta}_{\mathbb{C}} \to \hat{\Theta}_{\mathbb{C}} \to \Theta_{\mathbb{C}} \to 0.$$

The sheaves $\hat{\Theta}_{\mathbb{R}}$, $\hat{\Theta}_{\mathbb{C}}$ contain all the information about vector fields tangent to the foliation, which is extraneous to the question of deformations. We have the long exact sequences of cohomology groups

$$\cdots \to H^k(M; \Theta_{\mathbb{R}}) \to H^{k+1}(M; \bar{\Theta}_{\mathbb{R}}) \to H^{k+1}(M; \hat{\Theta}_{\mathbb{R}}) \to H^{k+1}(M; \Theta_{\mathbb{R}}) \to \cdots$$
$$\cdots \to H^k(M; \Theta_{\mathbb{C}}) \to H^{k+1}(H; \bar{\Theta}_{\mathbb{C}}) \to H^{k+1}(M; \hat{\Theta}_{\mathbb{C}}) \to H^{k+1}(M; \Theta_{\mathbb{C}}) \to \cdots.$$

Note that $\bar{\Theta}_{\mathbb{C}}$ is the sheaf of germs of holomorphic sections of the holomorphic bundle $\tau$ over $M$. On $M$ there is the $\bar{\partial}$ complex of $\tau$

$$0 \to C^\infty(\tau) \to C^\infty(\bar{T}^*M \otimes \tau) \to \cdots \to C^\infty(\Lambda^n \bar{T}^*M \otimes \tau) \to 0.$$

By the Dolbeault isomorphism [4] the $k$-th homology group of this complex is $H^k(M; \bar{\Theta}_{\mathbb{C}})$. The complex is elliptic and thus for compact $M$, $H^k(M; \bar{\Theta}_{\mathbb{C}})$ is finite dimensional for all $k$. From [7], p. 87 we have that for compact $M$, $H^k(M; \hat{\Theta}_{\mathbb{C}})$ is finite dimensional for all $k$. These two facts and the long exact cohomology sequence above prove the following.

PROPOSITION 1.6. *If $F$ is a complex analytic foliation on a compact complex analytic manifold $M$, then $H^k(M; \Theta_{\mathbb{C}})$ is finite dimensional for all $k$.*

## 2. Resolutions of the sheaves

We will treat the real and complex cases together. In the real case $F$ is a $C^\infty$ foliation of codimension $q$ on an $n$ dimensional $C^\infty$ manifold $M$. The tangent bundle to $F$ is now denoted by $\xi$ and its dual bundle by $\xi^*$. The bundle $TM/\tau$ is still called $\nu$. In the complex case $F$ is a complex analytic foliation of complex codimension $q$ on a complex analytic manifold $M$ of complex dimension $n$. If $\tau$ is the tangent bundle to the foliation denote the bundle $\tau\oplus\bar{T}M$ by $\xi$ and $\tau^*\oplus\bar{T}^*M$ by $\xi^*$. As above the bundle $TM/\tau$ is called $\nu$. A basic connection for a real or complex foliation is denoted by $\nabla$.

Consider the following complex

$$C^\infty(\nu)\xrightarrow{\hat{d}}C^\infty((\Lambda^1\xi^*)\otimes\nu)\xrightarrow{\hat{d}}C^\infty((\Lambda^2\xi^*)\otimes\nu)\to\cdots.$$

Where if

$$\sigma\in C^\infty((\Lambda^k\xi^*)\otimes\nu) \quad\text{and}\quad X_0,\dots,X_k\in C^\infty(\xi)$$

we define

$$(\hat{d}\sigma)(X_0,\dots,X_k)=\sum_{0\leqslant i\leqslant k}(-1)^i\,\nabla_{X_i}\sigma(X_0,\dots,\hat{X}_i,\dots,X_k)$$
$$+\sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\,\sigma([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k).$$

The ^ over $X_i$ or $X_j$ means that entry is deleted.

LEMMA 2.1. $\hat{d}\circ\hat{d}\equiv 0$.

*Proof.* It is not difficult to see that $\hat{d}$ is a local operator. Thus we need only show that on any open set $U$ of $M$ that $\hat{d}\cdot\hat{d}|_U\equiv 0$.

*Real case.* Let $U$ be an $F$ chart. Since $\sigma\in C^\infty((\Lambda^k\tau^*)\otimes\nu)$, we may write

$$\sigma\,|_U=\sum\sigma_J^i\,dx_J\otimes\left\langle\frac{\partial}{\partial x_i}\right\rangle,$$

where the sum runs over all $i$, $J$ with $n-q<i\leqslant n$, $J=(j_1,\dots,j_k)$ $1\leqslant j_1<j_2<\cdots<j_k\leqslant n-q$, $dx_J=dx_{j_1}\wedge\dots\wedge dx_{j_k}$ and each $\sigma_J^i$ is a $C^\infty$ function on $U$.

Let $d'$ be the operator on functions on $U$ defined by

$$d'f=\sum_{j=1}^{n-q}\frac{\partial f}{\partial x_j}\cdot dx_j.$$

A simple compulation using the definition of $\nabla$ shows

$$\hat{d}\sigma\,|_U=\hat{d}(\sigma\,|_U)=\sum d'\sigma_J^i\wedge dx_J\otimes\left\langle\frac{\partial}{\partial x_i}\right\rangle$$

and we have $\hat{d}^2=0$.

*Complex case.* As $\sigma$ is in $C^\infty(\Lambda^k(\tau^*\oplus\bar{T}^*M)\otimes\nu)$ we have

$$\sigma\,|_U=\sum\sigma_{J,L}^i\,dz_J\wedge d\bar{z}_L\otimes\left\langle\frac{\partial}{\partial z_i}\right\rangle.$$

The sum is taken over all $J$, $L$, $i$ where

$$J=(j_1,\ldots,j_r) \quad 1\leqslant j_1<j_2<\cdots<j_r\leqslant n-q$$
$$L=(l_1,\ldots,l_s) \quad 1\leqslant l_1<l_2<\cdots<l_s\leqslant n, \quad r+s=k$$
$$dz_J=dz_{j_1}\wedge\ldots\wedge dz_{j_r}$$
$$d\bar{z}_L=d\bar{z}_{l_1}\wedge\ldots\wedge d\bar{z}_{l_s}, \quad i=n-q+1,\ldots,n$$

and $\sigma^i_{J,L}$ is a $C^\infty$ complex function on $U$.

Define

$$\partial'\sigma^i_{J,L}=\sum_{j=1}^{n-q}\frac{\partial\sigma^i_{J,L}}{\partial z_j}\,dz_j.$$

Again a simple computation shows that

$$\hat{d}\sigma\,|_U=\hat{d}(\sigma\,|_U)=\sum(\partial'\sigma^i_{J,L}+\hat{\partial}\sigma^i_{J,L})\,dz_J\wedge d\bar{z}_L\otimes\left\langle\frac{\partial}{\partial z_i}\right\rangle.$$

Since $\partial'^2=\hat{\partial}^2=\partial'\hat{\partial}+\hat{\partial}\partial'=0$ we have $\hat{d}^2=0$.

For a real foliation we denote the homology of this complex by $F^*_{\mathbb{R}}(\tau;\nu)$; for a complex foliation by $F^*_{\mathbb{C}}(\tau;\nu)$.

*Comments* 1. In the terminology of Gel'fand-Fuchs [5], $F^*_{\mathbb{R}}(\tau;\nu)$ is the cohomology of the Lie algebra of vector fields tangent to the foliation with coefficients in the normal bundle, with the representation being given by the connection, where we consider only cochains of order zero (i.e. $C^\infty$ linear).

2. These cohomology groups also appear in the work of Kamber-Tondeur [6]. The complex in the notation of [6] is $I_{\mathbf{L}}(\mathbf{Q})$ and $F^*_{\mathbb{R}}(\tau;\nu)=H^*(\Gamma_M I_{\mathbf{L}}(\mathbf{Q}))$.

In general the groups $F^k_{\mathbb{R}}(\tau;\nu)$ are not finitely generated as the complex is not elliptic. However, under suitable restrictions on the foliation one can conclude that certain of the $F^k_{\mathbb{R}}(\tau;\nu)$ are finitely generated. See [11]. We shall show that $F^k_{\mathbb{C}}(\tau;\nu)=$ $=H^k(H;\Theta_{\mathbb{C}})$ and thus if $M$ is compact, $F^k_{\mathbb{C}}(\tau;\nu)$ is finite dimensional for all $k$.

4. The groups $F^k_{\mathbb{R}}(\tau;\nu)$ are modules over the ring $\Omega_F$ of smooth functions on $M$ which are constant on the leaves of $F$.

*Conjecture* 2.2. The groups $F^k_{\mathbb{R}}(\tau;\nu)$ are countably generated as modules over $\Omega_F$. Counter examples can be easily constructed to show that $F^k_{\mathbb{R}}(\tau;\nu)$ need not be finitely generated over $\Omega_F$.

5. In both the real and complex case the kernel of the map

$$\hat{d}: C^\infty(\nu)\to C^\infty((\Lambda^1\xi^*)\otimes\nu)$$

is the set of $\Gamma$ vector fields on $M$.

For a real, (respectively complex), foliation, let $\Phi^k_{\mathbb{R}}$, $(\Phi^k_{\mathbb{C}})$, be the sheaf of germs of local $C^\infty$ sections of the bundle $(\Lambda^k\xi^*)\otimes v$. We then have maps

$$\hat{d}:\Phi^k_{\mathbb{R}}\to\Phi^{k+1}_{\mathbb{R}},\qquad \hat{d}:\Phi^k_{\mathbb{C}}\to\Phi^{k+1}_{\mathbb{C}}$$

induced by $\hat{d}:C^\infty((\Lambda^k\xi^*)\otimes v)\to C^\infty((\Lambda^{k+1}\xi^*)\otimes v)$. Let

$$i:\Theta_{\mathbb{R}}\to\Phi^0_{\mathbb{R}}\quad\text{and}\quad i:\Theta_{\mathbb{C}}\to\Phi^0_{\mathbb{C}}$$

be the injection maps.

THEOREM 2.3. *The complexes*

$$0\to\Theta_{\mathbb{R}}\xrightarrow{i}\Phi^0_{\mathbb{R}}\xrightarrow{\hat{d}}\Phi^1_{\mathbb{R}}\xrightarrow{\hat{d}}\cdots\xrightarrow{\hat{d}}\Phi^{n-k}_{\mathbb{R}}\to 0$$
$$0\to\Theta_{\mathbb{C}}\xrightarrow{i}\Phi^0_{\mathbb{C}}\xrightarrow{\hat{d}}\Phi^1_{\mathbb{C}}\xrightarrow{\hat{d}}\cdots\xrightarrow{\hat{d}}\Phi^{2n-k}_{\mathbb{C}}\to 0$$

*are fine torsionless resolutions of $\Theta_{\mathbb{R}}$ and $\Theta_{\mathbb{C}}$ respectively.*

*Proof.* Each of the sheaves $\Phi^k_{\mathbb{R}}$, $\Phi^k_{\mathbb{C}}$ is obviously fine and torsionless. The fact that $\hat{d}^2=0$ follows from the corresponding statement for the complexes $C^\infty((\Lambda^k\xi^*)\otimes v)$. Comment 5 gives that the complexes are exact at $\Phi^0_{\mathbb{R}}$ and $\Phi^0_{\mathbb{C}}$. To complete the proof we have

LEMMA 2.4. *If $\varphi\in\Phi^k_{\mathbb{R}}$, (respectively $\Phi^k_{\mathbb{C}}$) satisfies $\hat{d}\varphi=0$, then there is an element* $\sigma\in\Phi^{k-1}_{\mathbb{R}}$, $(\Phi^{k-1}_{\mathbb{C}})$ with $\hat{d}\sigma=\varphi$.

*Proof. Real case.* $\varphi$ is the germ at $p\in M$ of a local $C^\infty$ section, also denoted $\varphi$, of $(\Lambda^k\tau^*)\otimes v$. Let $U$ be an $F$ chart with coordinate functions $x_1,\dots,x_n$ which map $U$ onto $\mathbb{R}^n$. Assume $p\in U$ and $\sigma$ is defined on $U$. Thus we may set

$$\varphi=\sum\varphi^i_J\,dx_J\otimes\left\langle\frac{\partial}{\partial x_i}\right\rangle,$$

where the sum is over all $J=(j_1,\dots,j_k)$, $1\leqslant j_1<j_2<\cdots<j_k\leqslant n-q$ and $i=n-q+1,\dots,n$. Each $\varphi^i_J$ is a $C^\infty$ function on $U$ and we may assume without loss of generality that $\varphi^i_J\equiv 0$ for $i<n$. We may view $\varphi$ as a family of $k$ forms on $\mathbb{R}^{n-q}$ indexed by $\mathbb{R}^q$. Here we are identifying $U$ with $\mathbb{R}^n=\mathbb{R}^{n-q}\times\mathbb{R}^q$. More specifically if $x_{n-q+1},\dots,x_n$ are coordinates of a point in $\mathbb{R}^q$ and $x_1,\dots,x_{n-q}$ are coordinates of a point in $\mathbb{R}^{n-q}$, the $k$ form $\varphi(x_{n-q+1},\dots,x_n)$ at the point $(x_1,\dots,x_{n-q})$ is

$$\sum_J\varphi^n_J(x_1,\dots,x_n)\,dx_J.$$

The fact that $\hat{d}\varphi=0$ implies that for each $(x_{n-q+1},\dots,x_n)\in\mathbb{R}^q$ $d(\varphi(x_{n-q+1},\dots,x_n))=0$ where $d$ is the usual exterior derivative. See the proof of Lemma 2.1. Applying the classical Poincaré Lemma (see for example Warner [13], p. 155) to each $\varphi(x_{n-q+1},\dots,x_n)$

we have the existence of a $k-1$ form $\sigma(x_{n-q+1},\dots,x_n)$ on $\mathbb{R}^{n-q}$ satisfying $d\sigma(x_{n-q+1},\dots,x_n)=\varphi(x_{n-q+1},\dots,x_n)$. The $\sigma(x_{n-q+1},\dots,x_n)$ can be chosen so as to depend differentiably, i.e. $C^\infty$, on $x_{n-q+1},\dots,x_n$. Let $\sigma$ be the local $C^\infty$ section of $(\Lambda^{k-1}\tau^*)\otimes\nu$ given by

$$\sigma(x_1,\dots,x_n)=[\sigma(x_{n-q+1},\dots,x_n)(x_1,\dots,x_{n-q})]\otimes\left\langle\frac{\partial}{\partial x_n}\right\rangle.$$

From the local definition of $\hat{d}$ it is trivial to show $\hat{d}\sigma=\varphi$. If we denote also by $\sigma$ the germ of $\sigma$ at the point $p$, then $\sigma\in\Phi_{\mathbb{R}}^{k-1}$ and $\hat{d}\sigma=\varphi$.

*Complex case.* (Proof due to Kodaira-Spencer [7], p. 79). As above we may assume $\sigma$ is a smooth local section of the bundle $\Lambda^k(\tau^*\oplus\bar{T}^*M)\otimes\nu$ and that $\sigma$ is defined on an $F$ chart $U$. We have that $U$ has holomorphic coordinates $z_1,\dots,z_n$ and we may assume that $\sigma=\bar{\sigma}\otimes\langle\partial/\partial z_n\rangle$ with $\bar{\sigma}\in C^\infty(\Lambda^k(\tau^*\oplus\bar{T}^*M)|_U)$. Write $\bar{\sigma}=\sum_{r=0}^k\sigma_r$ where $\sigma_r$ is an $(r,k-r)$ form on $U$, and set

$$\partial'\sigma_r=\sum_{j=1}^{n-q}\frac{\partial\sigma_r}{\partial z_j}dz_j.$$

Letting $\hat{d}\bar{\sigma}=\sum_{r=0}^k\partial'\sigma_r+\bar{\partial}\sigma_r$ and noting that $\hat{d}\sigma=\hat{d}\bar{\sigma}\otimes\langle\partial/\partial z_n\rangle$ we have that $\hat{d}\sigma=0$ implies $\hat{d}\bar{\sigma}=0$. Thus

$$\bar{\partial}\sigma_0=0,\quad \partial'\sigma_0+\bar{\partial}\sigma_1=0,\quad \partial'\sigma_1+\bar{\partial}\sigma_2=0,\dots,\quad \partial'\sigma_{k-1}+\bar{\partial}\sigma_k=0,\quad \partial'\sigma^k=0.$$

By the Poincaré lemma for $\bar{\partial}$ and $(0,k)$ forms there is a $(0,k-1)$ form $\Psi_0$ such that $\bar{\partial}\Psi_0=\sigma_0$. Now

$$\bar{\partial}(\sigma_1-\partial'\Psi_0)=\bar{\partial}\sigma_1+\partial'\bar{\partial}\Psi_0=\bar{\partial}\sigma_1+\partial'\sigma_0=0.$$

By the Poincaré lemma for $\bar{\partial}$ and $(1,k-1)$ forms there is a $(1,k-2)$ form $\Psi_1$ such that $\sigma_1=\partial'\Psi_0+\bar{\partial}\Psi_1$. Now

$$\bar{\partial}(\sigma_2-\partial'\Psi_1)=\bar{\partial}\sigma_2+\partial'\bar{\partial}\Psi_1=\bar{\partial}\sigma_2+\partial'\sigma_1=0.$$

In this way we obtain $\Psi_2,\dots,\Psi_{k-1}$ such that

$$\sigma_2=\partial'\Psi_1+\bar{\partial}\Psi_2,\dots,\qquad \sigma_{k-1}=\partial'\Psi_{k-2}+\bar{\partial}\Psi_{k-1}.$$

Let $\eta=\sigma_k-\partial'\Psi_{k-1}$ a $(k,0)$ form. $\bar{\partial}\eta=\bar{\partial}\sigma_k+\partial'\bar{\partial}\Psi_{k-1}=\bar{\partial}\sigma_k+\partial'\sigma_{k-1}=0$, so $\eta$ is holomorphic. Also $\partial'\eta=\partial'\sigma_k=0$. By extending the Poincaré lemma for holomorphic $k$ forms as was done for real forms we have that there is a holomorphic $k-1$ form $\Psi_k$

such that $\partial'\Psi_k=\eta$. Let $\Psi=(\sum_{r=0}^{k}\Psi_r)\otimes\langle\partial/\partial z_n\rangle$. Then $\hat{d}\Psi=\sigma$. Taking germs to get this same equation in $\hat{d}:\Phi_{\mathbb{C}}^{k-1}\to\Phi_{\mathbb{C}}^{k}$ we are done.

THEOREM 2.5. *If F is a real foliation on a smooth manifold M then*

$$F_{\mathbb{R}}^{*}(\tau;\nu)=H^{*}(M;\Theta_{\mathbb{R}}).$$

*If F is a complex analytic foliation on a complex analytic manifold M then*

$$F_{\mathbb{C}}^{*}(\tau;\nu)=H^{*}(M;\Theta_{\mathbb{C}}).$$

*Proof.* By definition $H^k(M;\Theta_{\mathbb{R}})$ is given as follows. (See [13]). For each $k$ we may form $C^\infty(\Phi_{\mathbb{R}}^k)$ and we have induced maps $\hat{d}:C^\infty(\Phi_{\mathbb{R}}^k)\to C^\infty(\Phi_{\mathbb{R}}^{k+1})$.

Then

$$H^k(M;\Theta_{\mathbb{R}})=\frac{\ker\{\hat{d}:C^\infty(\Phi_{\mathbb{R}}^k)\to C^\infty(\Phi_{\mathbb{R}}^{k+1})\}}{\text{image}\{\hat{d}:C^\infty(\Phi_{\mathbb{R}}^{k-1})\to C^\infty(\Phi_{\mathbb{R}}^{k})\}}.$$

But $C^\infty(\Phi_{\mathbb{R}}^k)=C^\infty((\Lambda^k\tau^*)\otimes\nu)$ and the $\hat{d}$'s are identical. Equality follows. The proof in the complex case is the same.

Theorem 2.5 permits us to represent any element of $H^k(M;\Theta_{\mathbb{R}})$ or $H^k(M;\Theta_{\mathbb{C}})$ by a global section $\sigma$ of the bundle $(\Lambda^k\xi^*)\otimes\nu$, satisfying $\hat{d}\sigma=0$. This is of special significance in the case $k=1$ for it allows us to give an intuitively pleasing interpretation of $H^1(M;\Theta_{\mathbb{R}})$ or $H^1(M;\Theta_{\mathbb{C}})$ as infinitesimal deformations of the foliation.

By a deformation of a plane field $\tau$ on $M$ we will mean a family $\tau_s$ of plane fields on $M$ depending differentiably on $s\in\mathbb{R}$ such that $\tau_0=\tau$. By a deformation of a foliation $\tau$ on $M$ we will mean a deformation $\tau_s$ of the plane field $\tau$ such that each $\tau_s$ is a foliation. For a real foliation $\tau$, each $\tau_s$ is required to be a smooth foliation and if $\tau$ is a complex foliation we assume each $\tau_s$ is a complex foliation.

Assume now that $\tau$ is a real, (respectively complex), foliation on a ɪeal, (complex) manifold $M$. Let $\tau_s$ be a deformation of the plane field $\tau$. We obtain a section of $\xi^*\otimes\nu$ as follows. For each $s$ we have the quotient bundle $TM/\tau_s$. The reader is reminded that in the complex case $TM$ is the holomorphic tangent bundle of $M$. Choose a Riemannian or Hermitian metric on $TM$, as $M$ is real or complex. The metric gives a splitting $TH=\tau_s\oplus\nu_s$. In both cases $\nu_s$ is canonically isomorphic to $TM/\tau_s$ and we make this identification in what follows. The splittings give natural projection operators

$$\pi_s:TM\to\tau_s,\qquad \pi_s^{\perp}:TM\to\nu_s.$$

In the complex case we can extend $\pi_s$ and $\pi_s^{\perp}$ to $\bar{T}M$ by defining them to be identically zero on this bundle.

LEMMA 2.6.

(i) $\left(\frac{\partial}{\partial s}\pi_s\right)\circ\pi_s+\pi_s\circ\left(\frac{\partial}{\partial s}\pi_s\right)=\frac{\partial}{\partial s}\pi_s$.

(ii) $\frac{\partial}{\partial s}\pi_s+\frac{\partial}{\partial s}\pi_s^{\perp}=0$.

(iii) *If* $X\in\tau_s$ *then* $\frac{\partial}{\partial s}\pi_s(X)\in\nu_s$.

*Proof.* (i) and (ii) follow from the equations $\pi_s\circ\pi_s=\pi_s$ and $\pi_s+\pi_s^{\perp}=\text{identity}$. (iii) follows directly from (i).

DEFINITION 2.7. The *infinitesimal deformation* $\sigma$ associated to $\tau_s$ is the element of $C^{\infty}(\xi^*\otimes\nu)$ given by

$$\sigma(X)=\pi_0^{\perp}\left\{\frac{\partial}{\partial s}\pi_s(X)\,|_{s=0}\right\}$$

for $X\in\xi$.

Recall that for real manifolds $\xi=\tau$ and for complex manifolds $\xi=\tau\oplus\bar{T}M$.

LEMMA 2.8. *The linear map* $\sigma:\xi\to TM/\tau$ *is independent of the choice of metric on M. If* $\tau$ *is a complex analytic foliation then* $\sigma$ *is holomorphic. Thus if U is an F chart we may write*

$$\sigma\,|_U=\sum_{1\leqslant j\leqslant n-q<i\leqslant n} f_j^i\,dz_j\otimes\left\langle\frac{\partial}{\partial z_i}\right\rangle$$

*with the* $f_j^i$ *holomorphic functions on U.*

*Proof.* Let $U$ be an $F$ chart, and choose vector fields $e_1(s),\dots,e_{n-q}(s)$, differentiable in $s$ on $U$, which span $\tau_s$ for each $s$. We have $\pi_s(X)=\sum_{j=1}^{n-q}f_j(X,s)\,e_j(s)$. The functions $f_j(X,s)$ are determined by $X$, $s$ and the metric. Also note that

$$X=\sum_{j=1}^{n-q}f_j(X,0)\,e_j(0).$$

$$\frac{\partial}{\partial s}\pi_s(X)\,|_{s=0}=\sum_{j=1}^{n-q}\frac{\partial f_j}{\partial s}(X,0)\,e_j(0)+f_j(X,0)\frac{\partial e_j}{\partial s}(0).$$

As $e_j(0)\in\tau$ we have

$$\pi_0^{\perp}\left\{\frac{\partial}{\partial s}\pi_s(X)\,|_{s=0}\right\}=\left\langle\sum_{j=1}^{n-q}f_j(X,0)\frac{\partial e_j}{\partial s}(0)\right\rangle$$

and the right hand side is independent of the metric. If $\tau$ is a complex foliation we may assume that the $e_j(s)$ are holomorphic vector fields and $e_j(0)=\partial/\partial z_j$. It follows easily that

$$\left\langle\frac{\partial e_j}{\partial s}(0)\right\rangle=\sum_{i=n-q+1}^{n} f_j^i\left\langle\frac{\partial}{\partial z_i}\right\rangle$$

is holomorphic and so $\sigma$ is holomorphic.

DEFINITION 2.9. For each $s\in\mathbb{R}$ and each pair of tangent vectors $X$, $Y$ to $M$, let

$$A_s(X, Y)=\pi_s^{\perp}[\pi_s X, \pi_s Y]).$$

$A_s$ is called the *integrability tensor* of the deformation $\tau_s$. Note that if $M$ is complex $X$, $Y$ are elements of $TM\oplus\bar{T}M$.

PROPOSITION 2.10.
(i) *$A_s$ is a $v_s$ valued exterior 2 form on $M$.*
(ii) *$\tau_s$ is involutive if and only if $A_s\equiv 0$.*
(iii) *If $\sigma$ is the infinitesimal deformation associated to $\tau_s$ and $X$, $Y\in\xi$ then*

$$\hat{d}\sigma(X, Y)=\frac{\partial}{\partial s}(A_s(X, Y))\big|_{s=0}.$$

*Proof.* The proofs of (i) and (ii) are trivial and are omitted.

(iii) *Real case.* If $X$, $Y\in\xi=\tau$ then $\nabla_X\sigma(Y)=\pi_0^{\perp}[X, \sigma(Y)]$ where $\nabla$ is a basic connection on $v$. We have

$$\begin{aligned}\hat{d}\sigma(X, Y)&=\nabla_X\sigma(Y)-\nabla_Y\sigma(X)-\sigma([X, Y])\\&=\pi_0^{\perp}([X, \sigma(Y)])-\pi_0^{\perp}([Y, \sigma(X)])-\sigma([X, Y])\\&=\pi_0^{\perp}\left(\left[X, \pi_0^{\perp}\left\{\frac{\partial}{\partial s}\pi_s Y\big|_{s=0}\right\}\right]\right)+\pi_0^{\perp}\left(\left[\pi_0^{\perp}\left\{\frac{\partial}{\partial s}\pi_s X\big|_{s=0}\right\}, Y\right]\right)\\&\quad-\pi_0^{\perp}\left\{\frac{\partial}{\partial s}\pi_s([X, Y])\big|_{s=0}\right\}.\end{aligned}$$

Since $\tau$ is involutive,

$$\begin{aligned}\pi_0^{\perp}[X, \pi_0^{\perp}Z])&=\pi_0^{\perp}[X, \pi_0^{\perp}Z+\pi_0 Z])\\&=\pi_0^{\perp}[X, Z])\quad\text{for all}\quad Z\in TM.\end{aligned}$$

By (iii) of Lemma 2.6, $(\partial/\partial s)\,\pi_s([X, Y])|_{s=0}$ is in $\nu$ and by (ii) of the same lemma

$$\frac{\partial}{\partial s}\pi_s([X, Y])\,|_{s=0} = -\frac{\partial}{\partial s}\pi_s^{\perp}([X, Y])\,|_{s=0}$$

so we have

$$-\pi_0^{\perp}\left\{\frac{\partial}{\partial s}\pi_s([X, Y])\,|_{s=0}\right\} = \frac{\partial}{\partial s}\pi_0^{\perp}([X, Y])\,|_{s=0}.$$

Finally, since $X, Y \in \tau$, $\pi_0 X = X$, $\pi_0 Y = Y$.

Thus

$$\hat{d}\sigma(X, Y) = \left\{\pi_s^{\perp}\left(\left[\frac{\partial}{\partial s}\pi_s(X), \pi_s Y\right]\right) + \pi_s^{\perp}\left[\pi_s X, \frac{\partial}{\partial s}\pi_s(Y)\right]\right.$$
$$\left. + \frac{\partial}{\partial s}\pi_s^{\perp}([\pi_s X, \pi_s Y])\right\}\,|_{s=0} = \frac{\partial}{\partial s}A_s(X, Y)\,|_{s=0}$$

*Complex case.* As $\xi = \tau \oplus \bar{T}M$ we have three cases to consider, (a) $X, Y \in \tau$, (b) $X, Y \in \bar{T}M$, and (c) $X \in \tau$, $Y \in \bar{T}M$.

(a) If $X, Y \in \tau$ the proof in the real case works.

(b) If $X, Y \in \bar{T}M$, $A_s(X, Y) \equiv 0$, and as $\pi_s(X) = \pi_s(Y) = 0$ for all $s$ we have $\sigma(X) = \sigma(Y) = \sigma([X, Y]) = 0$ so $\hat{d}\sigma(X, Y) = 0$.

(c) For $X \in \tau$ and $Y \in \bar{T}M$, again $A_s(X, Y) \equiv 0$. Now as $\sigma(Y) = 0$

$$\hat{d}\sigma(X, Y) = \nabla_X \sigma(Y) - \nabla_Y \sigma(X) - \sigma([X, Y])$$
$$= -\bar{\partial}(\sigma(X))(Y) - \sigma([X, Y]).$$

Since $\sigma$ is the infinitesimal deformation associated to a family of holomorphic plane fields $\bar{\partial}\sigma = 0$. Thus if $U$ is an $F$ chart we may write

$$\sigma\,|_U = \sum_{1 \leq j \leq n-q < i \leq n} f_j^i\, dz_j \otimes \left\langle \frac{\partial}{\partial z_i} \right\rangle$$

with each $f_j^i$ a holomorphic function on $U$.

Let

$$X\,|_U = \sum_{i=1}^{n} X_i \frac{\partial}{\partial z_i} \quad \text{and} \quad Y\,|_U = \sum_{l=1}^{n} Y_l \frac{\partial}{\partial \bar{z}_l}$$

then

$$-\bar{\partial}(\sigma(X))(Y) = -\sum_{i,j,l} f_j^i \frac{\partial X_j}{\partial \bar{z}_l} Y_l \otimes \left\langle \frac{\partial}{\partial z_i} \right\rangle$$

and

$$-\sigma([X, Y])=\sum_{i,j,l} f_j^i \frac{\partial X_j}{\partial \bar{z}_l} Y_l \otimes \left\langle \frac{\partial}{\partial z_i} \right\rangle$$

so

$$\hat{d}\sigma(X, Y)=0.$$

COROLLARY 2.11. *Let $\tau_s$ be a deformation of a real (respectively complex analytic) foliation $\tau$ on a real (complex analytic) manifold $M$. Let $\sigma$ be the associated infinitesimal deformation then $\hat{d}\sigma=0$, and $\sigma$ represents an element of $H^1(M; \Theta_{\mathbb{R}})$, $(H^1(M; \Theta_{\mathbb{C}}))$.*

PROPOSITION 2.12. *Let $F$ be a foliation on a manifold $M$ and let $Y$ be a complete vector field on $M$ with $\lambda=\langle Y\rangle \in C^\infty(\nu)$. Let $\varphi_s$ be the one parameter family of diffeomorphisms generated by $Y$ and let $F_s=\varphi_s^* F$ be the associated deformation of $F$. Then $\hat{d}\lambda$ is the infinitesimal deformation associated to the actual deformation $F_s$.*

*Proof.* By Definition 2.7 we have that the infinitesimal deformation associated to $F_s$ is for $X \in \tau$

$$\sigma(X)=\pi_0^\perp \left\{\frac{\partial}{\partial s} \pi_s(X)\Big|_{s=0}\right\}=\left\langle \frac{\partial}{\partial s} \pi_s(X)\Big|_{s=0}\right\rangle.$$

Let $U$ be an $F$ chart on which $F$ is defined by one forms $\omega_1, \ldots, \omega_q$. The forms $\varphi_s^*\omega_i$ define $F_s$. Let $X_1(s), \ldots, X_q(s)$ be vector fields dual to $\varphi_s^*\omega_1, \ldots, \varphi_s^*\omega_q$ and normal to $\tau_s$. Then for $X \in \tau$

$$\pi_s(X)=X-\sum_{i=1}^q \varphi_s^*\omega_i(X)\, X_i(s).$$

Since $\omega_i(X)=0$ we have

$$\begin{aligned}\frac{\partial}{\partial s} \pi_s(X)\Big|_{s=0} &= -\sum_{i=1}^q \omega_i\left(\frac{\partial}{\partial s} \varphi_{s*}(X)\Big|_{s=0}\right) X_i(0)\\ &= \sum_{i=1}^q \omega_i([X, Y])\, X_i(0).\end{aligned}$$

Thus

$$\sigma(X)=\left\langle \frac{\partial}{\partial s} \pi_s(X)\Big|_{s=0}\right\rangle=\langle [X, Y]\rangle=\hat{d}\lambda(X).$$

In view of Proposition 2.12 we may interpret $H^1(M; \Theta)$ as infinitesimal deformations of the foliation modulo trivial infinitesimal deformations, i.e., those given by vector fields.

## 3. Example of the two torus

The material in this section is related to [15]. Let $T^2 = \mathbb{R}/\mathbb{Z}^2$ be the standard flat two torus. Denote by $\tau_\alpha$ the foliation of $T^2$ given by all straight lines of slope $\alpha$ and let $\Theta_\alpha$ be the corresponding sheaf of germs of $\Gamma$ vector fields.

THEOREM 3.1. *If $\alpha$ is rational then*
(i) $H^0(M; \Theta_\alpha) \cong C^\infty(S^1)$
(ii) $H^1(M; \Theta_\alpha) \cong C^\infty(S^1)$.
Each element of $H^1(M; \Theta_\alpha)$ can be realized as the associated infinitesimal deformation of a differentiable family of foliations.

The last statement of the theorem follows from the fact that any non-zero vector field is a 1 dimensional foliation. The rest of the theorem follows from the more general.

THEOREM 3.2. *Let $M$ be a smooth manifold. The foliation of $M \times S^1$ whose leaves are $\{p\} \times S^1$, $p \in M$, satisfies*
(i) $H^0(M \times S^1; \Theta) \cong C^\infty(TM)$.
(ii) $H^1(M \times S^1; \Theta) \cong C^\infty(TM)$.

*Proof.* Let $\pi: M \times S^1 \to M$ be the projection onto the first factor. The normal bundle $\nu$ of the foliation is canonically isomorphic with the pull-back bundle $\pi^*(TM)$, and we make this identification. If $X \in C^\infty(TM)$ the induced vector field in $\nu$ is denoted $\pi^! X$. We are interested in the complex

$$0 \to C^\infty(\nu) \xrightarrow{\hat{d}} C^\infty(\tau^* \otimes \nu) \to 0.$$

Proof of (i). Let $U$ be a coordinate chart on $M$ with coordinates $x_1, \ldots, x_n$. On $\pi^{-1}(U) = U \times S^1$ we have coordinates $x_1, \ldots, x_n, \theta$. If $\sigma \in C^\infty(\nu)$ then

$$\sigma\,|_U = \sum_{i=1}^n f_i(x_1, \ldots, x_n, \theta) \frac{\partial}{\partial x_i}$$

and

$$\hat{d}\sigma\,|_U(\partial/\partial\theta) = \nabla_{\partial/\partial\theta}\sigma\,|_U = \sum_{i=1}^n \frac{\partial f_i}{\partial\theta} \frac{\partial}{\partial x_i}.$$

Thus $\hat{d}\sigma|_U \equiv 0$ if and only if $\partial f_i/\partial\theta = 0$ for all $i$. So we have $\sigma = \pi^! X$ foı some $X \in C^\infty(TM)$. Conversely if $\sigma = \pi^! X$ for $X \in C^\infty(TM)$ then $\hat{d}\sigma = 0$.

Proof of (ii). We define an operator

$$\int : C^\infty(\tau^* \otimes v) \to C^\infty(TM).$$

If $\sigma \in C^\infty(\tau^* \otimes v)$, we can write

$$\sigma = \sum_j f_j \, d\theta \otimes \pi^! X_j$$

for some finite set $\{X_j\} \subset C^\infty(TM)$, and $f_j \in C^\infty(M \times S^1)$.

We set

$$\int \sigma = \sum_j \left( \int_{S^1} f_j \, d\theta \right) X_j.$$

If $f$ is a $C^\infty$ function on $M \times S^1$ then $\int_{S^1} f_j \, d\theta$ is a $C^\infty$ function on $M$. Note also that $\int$ is a homomorphism from $C^\infty(\tau^* \otimes v)$ to $C^\infty(TM)$. Setting $\sigma_X = (1/2\pi)\, \pi^! X$ for $X \in C^\infty(TM)$ we have that $\int$ is onto.

LEMMA 3.3. $\ker \int = \hat{d}(C^\infty(v))$.

*Proof.* If $\sigma \in C^\infty(v)$ write

$$\sigma = \sum_j f_j \pi^! X_j.$$

As

$$\hat{d}\sigma = \sum_j \frac{\partial f_j}{\partial \theta} \, d\theta \otimes \pi^! X_j$$

we have $\int \hat{d}\sigma = 0$.

Now suppose $\sigma \in C^\infty(\tau^* \otimes v)$ satisfies $\int \sigma = 0$. Write

$$\sigma = \sum_j f_j \, d\theta \otimes \pi^! X_j.$$

We construct an element $\gamma$ in $C^\infty(v)$ such that $\hat{d}\gamma = \sigma$.

Let $(p, t)$ be a point of $M \times S^1$. Define $\gamma(p, t) \in v_{(p,t)}$ by

$$\gamma(p, t) = \sum_j \left[ \int_1^t f_j(p, \theta) \, d\theta \right] (\pi^! X_j)_{(p,t)}.$$

The integral is taken from $1\in S^1$ to $t\in S^1$ in the positive direction. $\gamma$ is well defined as

$$\sum_j \left[\int_{S^1} f_j(p,\theta)\,d\theta\right] \pi^! X_j = 0$$

and $\gamma$ is obviously a $C^\infty$ section of $\nu$. It is immediate from the definitions that $\hat{d}\gamma=\sigma$.

Theorem 3.1 now follows as each constant rational slope foliation of $T^2$ gives a splitting of $T^2$ as $S^1\times S^1$ with the leaves of the foliation given by $\{p\}\times S^1$. $C^\infty(TS^1)\cong C^\infty(S^1)$.

DEFINITION 3.4. An irrational real number $\alpha$ is *not a Liouville number* provided there is a positive integer $p$ and $\varepsilon>0$ such that

$$\left|\alpha-\frac{n}{m}\right|>\varepsilon(|m|+|n|)^{-p}$$

for $n$ and $m$ sufficiently large.

THEOREM 3.5. *If $\alpha$ is an irrational real number then*

$$H^0(T^2;\Theta_\alpha)\cong\mathbb{R}.$$

*If $\alpha$ is not a Liouville number then*

$$H^1(T^2;\Theta_\alpha)\cong\mathbb{R}.$$

*Each element of $H^1(T^2;\Theta_\alpha)$ can be realized as the associated infinitesimal deformation of a differential family of foliations.*

*Proof.* Let $\langle\ ,\ \rangle$ be the flat Riemannian structure which $T^2$ inherets from $\mathbb{R}^2$. Let $X_1, X_2$ be vector fields on $T^2$ with $\langle X_i, X_j\rangle=\delta^i_j$ and $X_1$ tangent to the foliation. Each $\sigma\in C^\infty(\nu)$ may be written as $f\cdot\langle X_2\rangle$ where $f\in C^\infty(T^2)$. Since $[X_1, X_2]=0$ we have $\hat{d}\sigma(X_1)=(X_1\cdot f)\langle X_2\rangle$. Thus $\hat{d}\sigma=0$ is equivalent to $X_1 f=0$ and so $f$ must be constant on the leaves of the foliation. As each of the leaves is dense on $T^2$ $f$ is a constant function. This proves the first statement.

Each element $\sigma\in C^\infty(\tau^*\otimes\nu)$ may be represented as $\sigma=fX_1^*\otimes\langle X_2\rangle$ where $f\in C^\infty(T^2)$ and $X_1^*(X_j)=\delta^1_j$. Thus

$$C^\infty(\tau^*\otimes\nu)\cong C^\infty(T^2).$$

Let $x_1$ and $x_2$ be the natural coordinates on $T^2$ inhereted from $\mathbb{R}^2$. If $f\in C^\infty(T^2)$

the Fourier expansion of $f$ is

$$f(x_1, x_2)=\sum f_{m,n} \exp\{2\pi i(mx_1+nx_2)\}.$$

Define $\int : C^\infty(\tau^*\otimes v)\to\mathbb{R}$ by $\int\sigma=f_{0,0}$, where $\sigma=fX_1^*\otimes\langle X_2\rangle$. $\int$ is an epimorphism. To finish the proof we have

LEMMA 3.6. (See [9]) *If $\alpha$ is not a Liouville number*

$$\ker\int=\hat{d}(C^\infty(v)).$$

*Proof.* Suppose $\gamma\in C^\infty(v)$, $\gamma=g\cdot\langle X_2\rangle$. Then if $X_1=a\,\partial/\partial x_1+b\,\partial/\partial x_2$, $a/b=\alpha$

$$\begin{aligned}\hat{d}\gamma&=\left(a\frac{\partial g}{\partial x_1}+b\frac{\partial g}{\partial x_2}\right)X_1^*\otimes\langle X_2\rangle.\\&=(X_1\cdot g)\,X_1^*\otimes\langle X_2\rangle.\end{aligned}$$

Using the Fourier expansion of $g$ we see that if $f=(X_1 g)$ then $f_{0,0}=0$.

Suppose now that $f\in C^\infty(T^2)$ satisfies $f_{0,0}=0$. We must find $g\in C^\infty(T^2)$ with $a\,\partial g/\partial x_1+b\,\partial g/\partial x_2=f$. Suppose we had such a $g$ and that its Fourier expansion was given by

$$g=\sum g_{m,n}\exp\{2\pi i(mx_1+nx_2)\}.$$

Then for each $m, n\neq 0, 0$ we must have

$$g_{m,n}=2\pi i(ma+nb)^{-1}f_{m,n}. \tag{3.7}$$

If for some positive integer $p$ and some $\varepsilon>0$

$$|\alpha-n/m|>\varepsilon(|m|+|n|)^{-p}$$

then the $g_{m,n}$ given by (3.7) are the Fourier coefficients of a $C^\infty$ function $g$ on $T^2$. Setting $\gamma=g\cdot\langle X_1\rangle$ we have $\hat{d}\gamma=\sigma$ and the theorem.

*Note*: If $\alpha$ is a Liouville number we have Image $\hat{d}\subset\ker\int$ but one can show that equality does not hold. See [9].

## 4. The complex restricted to a leaf

Let $L$ be a leaf of a real foliation $\tau$ on a smooth manifold $M$. Let $v_L$ be the normal bundle of $\tau$ restricted to $L$. Note that in general $C^\infty(v)|_L\neq C^\infty(v|_L)$, as an element of

$C^\infty(\nu)|_L$ must extend to a neighborhood of $L$ in $M$ while an element of $C^\infty(\nu|_L)$ need not.

Consider the complex

$$C^\infty(\nu_L) \xrightarrow{\hat{d}} C^\infty(\Lambda^1 T^* L \otimes \nu_L) \xrightarrow{\hat{d}} C^\infty(\Lambda^2 T^* L \otimes \nu) \xrightarrow{\hat{d}} \cdots,$$

where $\hat{d}$ is defined as in Section 2. Again $\hat{d}^2 = 0$ and we denote the resulting cohomology groups by $H^*(L; \nu_L)$.

A basic connection on $\nu$ induces a canonical connection on $\nu_L$ given by the normal projection of the Lie bracket. The Jacobi identity implies that this connection is flat. We also note that the linear holonomy of the foliation is the holonomy of the canonical connection on $\nu_L$, ([12] p. 91.).

The following theorem is well-known.

THEOREM 4.1. *$H^*(L; \nu_L)$ is isomorphic to $H^*(L; \mathbb{R}^q)$, the de Rham cohomology of $L$ with coefficients in the flat bundle $\nu_L$.*

*Proof.* The groups $H^k(L; \mathbb{R}^q)$ are the de Rham cohomology groups of $L$ with coefficients in $\mathbb{R}^q$ twisted over the holonomy of the flat connection. As such they are the homology groups of the complex $\{A_{\pi_1(L)}(\tilde{L}; \mathbb{R}^q); d\}$. This is the de Rham complex of $\mathbb{R}^q$ valued forms on the simply connected covering space $\tilde{L}$ of $L$ which satisfy

$$(\sigma^*\omega)(Y_1, \ldots, Y_k) = h(\sigma^{-1})(\omega(Y_1, \ldots, Y_k)).$$

Here $\sigma$ is an element of the fundamental group of $L$, $\pi_1(L)$, and acts on $\tilde{L}$ by deck transformations. $\omega$ is an $\mathbb{R}^q$ valued $k$ form on $\tilde{L}$ and $Y_1, \ldots, Y_k \in C^\infty(T\tilde{L})$. The map $h: \pi_1(L) \to GL(q; \mathbb{R})$ is the holonomy representation of the connection.

Let $\varrho: \tilde{L} \to L$ be the natural map. The complex $\{C^\infty(\Lambda^k T^* L \otimes \nu_L); \hat{d}\}$ over $L$ induces a complex $\{C^\infty(\Lambda^k T^* \tilde{L} \otimes \tilde{\nu}_L); \tilde{d}\}$ over $\tilde{L}$. $\tilde{\nu}_L$ is the pull back by $\varrho$ of $\nu_L$. Let $\tilde{\nabla}$ be the pull back to $\tilde{\nu}_L$ of the cannonical connection $\nabla$ on $\nu_L$. Define $\tilde{d}$ as $\hat{d}$ was using $\tilde{\nabla}$ in place of $\nabla$. Since $\tilde{\nabla}$ is a flat connection on a bundle over a simply connected manifold, that bundle is trivial and has global flat framings.

LEMMA 4.2. *The complex $\{C^\infty(\Lambda^k T^* \tilde{L} \otimes \tilde{\nu}_L); \tilde{d}\}$ is isomorphic to the complex $\{C^\infty(\Lambda^k T^* \tilde{L} \otimes \mathbb{R}^q); d \otimes 1\}$ the de Rham complex of $\mathbb{R}^q$ valued forms on $\tilde{L}$.*

*Proof.* Let $X_1, \ldots, X_q$ be a global flat frame field for $\tilde{\nu}_L$. For all $Y \in T\tilde{L}$, $\nabla_Y X_j = 0$. Let $\omega \in C^\infty(\Lambda^k T^* \tilde{L} \otimes \tilde{\nu}_L)$ and write $\omega = \sum_{j=1}^q \omega_j \otimes X_j$ where each $\omega_j \in C^\infty(\Lambda^k T^* \tilde{L})$. Define

$$\theta: C^\infty(\Lambda^k T^* \tilde{L} \otimes \tilde{\nu}_L) \to C^\infty(\Lambda^k T^* \tilde{L} \otimes \mathbb{R}^q)$$

by $\theta(\omega)=(\omega_1,\dots,\omega_q)$. $\theta$ is a linear isomorphism. If $Y_1,\dots,Y_{k+1}\in C^\infty(T\tilde{L})$ then

$$\tilde{d}\omega(Y_1,\dots,Y_{k+1})=\sum_{j=1}^{q}\left[\sum_{i=1}^{k+1}(-1)^{i+1}\nabla_{Y_i}(\omega_j(Y_1,\dots,\hat{Y}_i,\dots,Y_{k+1})\cdot X_j)\right.$$
$$\left.+\sum_{1\leqslant i<l\leqslant k+1}(-1)^{i+l}\omega_j([Y_i,Y_l],Y_1,\dots,\hat{Y}_i,\dots,\hat{Y}_l,\dots,Y_{k+1})X_j\right]$$
$$=\sum_{j=1}^{q}\left[\sum_{i=1}^{k+1}(-1)^{i+1}Y_i(\omega_j(Y_1,\dots,\hat{Y}_i,\dots,Y_{k+1})\right.$$
$$\left.+\sum_{1\leqslant i<l\leqslant k+1}(-1)^{i+l}\omega_j([Y_i,Y_l],Y_1,\dots,\hat{Y}_i,\dots,\hat{Y}_l,\dots,Y_{k+1})\right]X_j$$
$$=\sum_{j=1}^{q}(d\omega_j\otimes X_j)(Y_1,\dots,Y_{k+1}).$$

Thus $\theta\circ\tilde{d}=d\circ\theta$ and the lemma is established.

To complete the proof of the theorem we observe that $\varrho^*$ maps $C^\infty(\Lambda^kT^*L\otimes\nu_L)$ onto the subset of elements $\omega$ of $C^\infty(\Lambda^kT^*\tilde{L}\otimes\tilde{\nu}_L)$ such that if

$$\omega=\sum_{j=1}^{q}\omega_j\otimes X_j$$

and we denote by $\bar{\omega}$ the $\mathbb{R}^q$ valued form $(\omega_1,\dots,\omega_q)$ then

$$\overline{(\sigma^*\omega)}(Y_1,\dots,Y_k)=h(\sigma^{-1})(\bar{\omega}(Y_1,\dots,Y_k)\qquad(4.3)$$

for each $\sigma\in\pi_1(L)$. To see this we need three facts. We denote by $\varrho^*$ the two maps

$$\varrho_*:\tilde{\nu}_L\to\nu_L\quad\text{and}\quad\varrho_*:T\tilde{L}\to TL$$

induced by $\varrho$.

1. For each $X_j$ as in Lemma 4.2, $x\in\tilde{L}$ and $\sigma\in\pi_1(L)$,

$$\varrho_*((X_j)_{\sigma(x)})=h(\sigma)(\varrho_*(X_j)_x),$$

i.e. $\varrho_*((X_j)_{\sigma(x)})$ is the parallel transport of $\varrho_*((X_j)_x)$ around a loop in $L$ based at $\varrho(x)$ which represents $\sigma$.

2. For each $\Psi\in C^\infty(\Lambda^kT^*L\otimes\nu_L)$ we have

$$\sigma^*\varrho^*\Psi=(\varrho\circ\sigma)^*\Psi=\varrho^*\Psi.$$

3. If $\varrho^*\Psi=\sum_{j=1}^q\Psi_j\otimes X_j$, with $\Psi_j\in C^\infty(\Lambda^kT^*\tilde{L})$, then the $\Psi_j(Y_1,\dots,Y_k)$ are the coordinates of $\Psi_{\varrho(x)}(\varrho_*Y_1,\dots,\varrho_*Y_k)$ with respect to the basis

$$\varrho_*((X_j)_x)\quad\text{of}\quad(\nu_L)_{\varrho(x)}.$$

Equation 4.3 follows easily from the rule for change of basis from linear algebra. Thus $\theta \circ \varrho^*$ gives the desired isomorphism from the complex $\{C^\infty(\Lambda^k T^* L \otimes \nu_L); \hat{d}\}$ to the complex $\{A_{\pi_1(L)}(\tilde{L}, \mathbb{R}^q); d\}$.

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